THE SCHOOL FOR EXCELLENCE (TSFX)
UNITS 3 & 4 MATHEMATICAL METHODS 2018
WRITTEN EXAMINATION 1 – SOLUTIONS

Errors and updates relating to this examination paper will be posted at
www.tsfx.com.au/examupdates

Marking Legend:

- $\left(\text{A}\,\frac{1}{2}\times 4\downarrow\right)$ means four answer half-marks rounded **down** to the next integer. Rounding occurs at the end of a part of a question.
- **M**1 = 1 **M**ethod mark.
- **A**1 = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H**1 = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip, arithmetic slip preventing an **A** mark).

QUESTION 1

a. $f'(x) = -3e^{-3x}\cos(2x) - 2e^{-3x}\sin(2x)$ **M1 Product Rule + A1**

$= -e^{-3x}\left(3\cos 2x + 2\sin 2x\right)$

b. $f'\left(\frac{\pi}{3}\right) = -e^{-\pi}\left(3\cos\left(\frac{2\pi}{3}\right) + 2\sin\left(\frac{2\pi}{3}\right)\right)$

$= -e^{-\pi}\left(3\left(-\frac{1}{2}\right) + 2\times\frac{\sqrt{3}}{2}\right)$ **A1 for 1 correct trig ratios**

$= -e^{-\pi}\left(-\frac{3}{2} + \sqrt{3}\right)$

$= e^{-\pi}\left(\frac{3}{2} - \sqrt{3}\right)$

Hence $a = \frac{3}{2}$ and $b = -1$ **A1 (both)**

QUESTION 2

a. $\int \frac{1}{(3x+2)^4} - e^{1-2x}\, dx = \int (3x+2)^{-4} - e^{1-2x}\, dx$

Now $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \quad n \neq -1$

$\therefore \quad \frac{1}{3(-4+1)}(3x+2)^{-3} - \frac{e^{1-2x}}{-2} + c$

Answer: $-\frac{1}{9}(3x+2)^{-3} + \frac{e^{1-2x}}{2} + c$ **A2 (must have +*c*)**

b. $\int_0^{\frac{\pi}{6}} \left(\sin\left(\frac{3\pi}{2}+x\right) + \cos\left(\frac{\pi}{2}-x\right) \right) dx = \int_0^{\frac{\pi}{6}} \left(-\cos(x) + \sin(x)\right) dx$

$= [-\sin x - \cos x]_0^{\pi/6}$ **M1**

$= \left(-\sin\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right) \right) - \left(-\sin(0) - \cos(0)\right)$

$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} \right) - (0-1)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}$ **A1**

Alternatively:

$\int_0^{\frac{\pi}{6}} \left(\sin\left(\frac{3\pi}{2}+x\right) + \cos\left(\frac{\pi}{2}-x\right) \right) dx = \left[-\cos\left(\frac{3\pi}{2}+x\right) - \sin\left(\frac{\pi}{2}-x\right) \right]_0^{\frac{\pi}{6}}$

$= \left(-\cos\left(-\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \right) - \left(-\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \right)$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} - (0-1)$

$= \frac{1-\sqrt{3}}{2}$

QUESTION 3

a. $g(x) = x^2 - 1$

Let $y = x^2 - 1$. Interchange x and y:

$x = y^2 - 1$

$y^2 = x + 1$

$y = \pm\sqrt{x+1}$

As $d_g = R^+ = r_{g^{-1}}$ $\therefore$ $g^{-1}(x) = \sqrt{x+1}$

$g^{-1} : (-1, \infty) \to R,\ g^{-1}(x) = \sqrt{x+1}$ **A1 for domain, A1 for rule**

b. For $g^{-1}\left(f(x)\right)$ to exist, $r_f \subseteq d_{g^{-1}}$ **M1**

$R \not\subset (-1, \infty)$ therefore, $g^{-1}\left(f(x)\right)$ does not exist. **M1**

c. For $g^{-1}\left(f(x)\right)$ to exist, $r_f = d_{g^{-1}}$

$r_f = (-1, \infty)$ **M1**

Find x when $f(x) = -1$

$\log_e x = -1$

$x = e^{-1} = \dfrac{1}{e}$

$\therefore k = \dfrac{1}{e}$ **A1**

d. $g^{-1}\left(f(x)\right) = \sqrt{\log_e x + 1},\ x \in \left(\dfrac{1}{e}, \infty\right)$ **A1 for domain, A1 for rule**

QUESTION 4

a. $M_A = 500e^{-0.05t}$

$M_B = 100e^{-kt}$

Let $M_A = M_B$

$100e^{-kt} = 500e^{-0.05t}$ **M1**

$\frac{e^{-kt}}{e^{-0.05t}} = \frac{500}{100}$

$e^{-kt+0.05t} = 5$

$\log_e e^{-kt+0.05t} = \log_e 5$

$-kt + 0.05t = \log_e 5$

$t(0.05 - k) = \log_e 5$

$t = \frac{\log_e 5}{0.05 - k}$ **M1**

b. $M(t) = M_0 e^{-kt}$

When half the mass disappears, $\frac{M}{M_0} = \frac{1}{2}$ and $t = 1590$.

Therefore, $\frac{M}{M_0} = e^{-kt}$

$\frac{1}{2} = e^{-1590k}$ **M1**

$\log_e\left(\frac{1}{2}\right) = \log_e e^{-1590k}$

$\log_e\left(\frac{1}{2}\right) = -1590k$

$-\log_e(2) = -1590k$

$k = \frac{\log_e 2}{1590}$ **A1**

QUESTION 5

$3f'(x)+f(x)=\pi$

$3\cos\left(\frac{x}{3}\right)+3\sin\left(\frac{x}{3}\right)+\pi=\pi$

$3\sin\left(\frac{x}{3}\right)=-3\cos\left(\frac{x}{3}\right)$ **H1**

Dividing both sides by $3\cos\left(\frac{x}{3}\right)$ gives $\tan\left(\frac{x}{3}\right)=-1$

If $0\le x\le 9\pi$ then $0\le\frac{\mathrm{x}}{3}\le 3\pi$

Tan is negative in the 2nd and 4th quadrants: $\frac{x}{3}=\frac{3\pi}{4},\frac{7\pi}{4},\ 2\pi+\frac{3\pi}{4}$

Answer: $x=\frac{9\pi}{4},\frac{21\pi}{4},\frac{33\pi}{4}$

A1(1 correct) A2(all correct) (any extra solutions- minus 1)

QUESTION 6

a. $f(x)=4x-x^2$

Reflect in the X axis:

$$\begin{aligned} f(x)&=-4x-x^2\\ &=-\left(x^2+4x\right)\\ &=-\left(x^2+4x+4-4\right)\\ &=-\left((x+2)^2-4\right)\\ &=-(x+2)^2+4 \end{aligned}$$

$$\begin{aligned} g(x)&=9-6x-3x^2\\ &=-3\left(x^2+2x-3\right)\\ &=-3\left(x^2+2x+1-1-3\right)\\ &=-3\left((x+1)^2-4\right)\\ &=-3(x+1)^2+12 \end{aligned}$$

Reflection in the *y*-axis. **(none – given)**
Dilation by a factor of 3 from the *x*-axis. **A1**
Translation 1 unit to the right. **A1**
Order must have translation last. **A1**

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **A2**

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-x \\ 3y \end{bmatrix}$

$x' = 1 - x$ $\quad\quad$ $y' = 3y$ **M1**

$\therefore x = 1 - x'$ $\quad\quad$ $\therefore y = \frac{y'}{3}$

$y = e^{x+1}$

$\frac{y'}{3} = e^{(1-x')+1}$

$y' = 3e^{(2-x')}$

$y = 3e^{(2-x)}$ **A1**

QUESTION 7

a. Values of $\hat{p}$ = $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ **A1**

b. If the probability of the proportion of red marbles is greater than 0.5 then there must be either 3 red marbles or 4 red marbles in the sample of 4. **M1**

Pr(3 red) = Pr(RRRB or RRBR or RBRR or BRRR) = $4\times\left(\frac{4}{12}\times\frac{3}{11}\times\frac{2}{10}\times\frac{8}{9}\right) = \frac{64}{990}$

Pr(4 red) = Pr(RRRR) = $\frac{4}{12}\times\frac{3}{11}\times\frac{2}{10}\times\frac{1}{9} = \frac{2}{990}$ **M1** For 1 correct probability

So this makes a probability of $\frac{66}{990} = \frac{1}{15}$.

Answer: $\frac{1}{15}$ **A1**

QUESTION 8

a. (a, b) lies on the curve $f(x) = \log_e\left(\frac{x}{3}\right)$

$\therefore b = \log_e\left(\frac{a}{3}\right)$

$e^b = \frac{a}{3}$ and therefore, $a = 3e^b$ **A1**

b. If $f(x) = \log_e\left(\frac{x}{3}\right)$ then $f'(x) = \frac{1}{x}$ **A1**

As (a, b) lies on the tangent:

$y - b = \frac{1}{a}(x - a)$ **A1**

$y = \frac{x}{a} + b - 1$

$ay - ab + a = x$

As tangent passes through the origin:

$-ab + a = 0$

$b = \frac{a}{a} = 1$

Therefore, $ay - ab + a = x$

$ay - a + a = x$

$x = ay$ **M1**

QUESTION 9

$$Area = \int_{-6}^{-3} (x+5--1)\,dx + \int_{-3}^{-0} (2--1)\,dx + \int_{0}^{4} (2-\sqrt{x})\,dx + \int_{0}^{1} (-\sqrt{x}--1)\,dx$$

$$Area = \int_{-6}^{-3} (x+6)\,dx + \int_{-3}^{-0} (3)\,dx + \int_{0}^{4} (2-\sqrt{x})\,dx + \int_{0}^{1} (1-\sqrt{x})\,dx$$

OR

$$Area = Trapezium + \int_{0}^{4} (2-\sqrt{x})\,dx + \int_{0}^{1} (1-\sqrt{x})\,dx$$

$$Area = \frac{1}{2}(3+6)3 + \int_{0}^{4} (2-\sqrt{x})\,dx + \int_{0}^{1} (1-\sqrt{x})\,dx$$

$$= \frac{27}{2} + \left[2x - \frac{2x^{3/2}}{3}\right]_0^4 + \left[x - \frac{2x^{3/2}}{3}\right]_0^1$$

$$= \frac{27}{2} + \left(2(4) - \frac{2(4)^{3/2}}{3}\right) + \left(1 - \frac{2(1)^{3/2}}{3}\right)$$

$$= \frac{27}{2} + 8 - \frac{16}{3} + \frac{1}{3}$$

$$= \frac{81+48-32+2}{6}$$

$$= \frac{99}{6}$$

$$= \frac{33}{2}\ units^2$$