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Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

Let $f: \left(\frac{1}{3}, \infty\right) \to R,\ f(x) = \frac{1}{3x-1}$.

a. i. Find $f'(x)$. 1 mark

$$f(x) = (3x-1)^{-1}$$

$$f'(x) = -(3x-1)^{-2} \cdot 3$$

$$= \frac{-3}{(3x-1)^2}$$

ii. Find an antiderivative of $f(x)$. 1 mark

$$\int f(x)\,dx = \frac{\log_e|3x-1|}{3}$$

b. Let $g: R \setminus \{-1\} \to R,\ g(x) = \frac{\sin(\pi x)}{x+1}$.

Evaluate $g'(1)$. 2 marks

$$g'(x) = \frac{\pi\cos(\pi x)(x+1) - \sin(\pi x)\cdot 1}{(x+1)^2}$$

$$g'(1) = \frac{\pi\cos(\pi)\cdot 2 - \sin(\pi)}{4}$$

$$= -\frac{\pi}{2}$$

TURN OVER

Question 2 (4 marks)

a. Let $f: R \setminus \left\{\frac{1}{3}\right\} \to R, f(x) = \frac{1}{3x-1}$.

Find the rule of f^{-1}.

2 marks

$f(x): \quad y = \frac{1}{3x-1}$

$f^{-1}(x): \quad x = \frac{1}{3y-1}$

$3y - 1 = \frac{1}{x}$

$3y = \frac{1}{x} + 1$

$y = \frac{1}{3x} + \frac{1}{3}$

b. State the domain of f^{-1}.

1 mark

$\mathbb{R} \setminus \{0\}$

c. Let g be the function obtained by applying the transformation T to the function f, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

$f(x) \xrightarrow{T} g(x)$

and $c, d \in R$.

Find the values of c and d given that $g = f^{-1}$.

1 mark

$\begin{cases} y = \frac{1}{3x-1} = \frac{1}{3(x-\frac{1}{3})} \\ y' - \frac{1}{3} = \frac{1}{3x'} \end{cases}$

$y' - \frac{1}{3} = y$

$\begin{cases} y' = y + \frac{1}{3} \\ x' = x - \frac{1}{3} \end{cases}$

$\begin{cases} c = -\frac{1}{3} \\ d = \frac{1}{3} \end{cases}$

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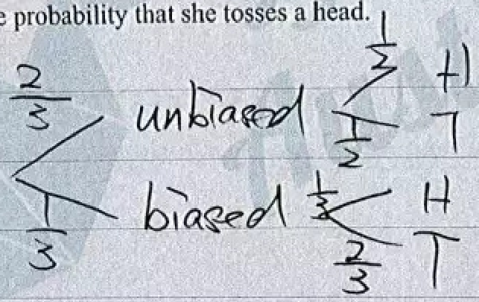
Question 3 (3 marks)

The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail.

Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$.

Jo randomly selects a coin from her pocket and tosses it.

a. Find the probability that she tosses a head. 2 marks



$Pr(Head) = \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{3} = \frac{4}{9}$

b. Find the probability that she selected an unbiased coin, given that she tossed a head. 1 mark

$Pr(unbiased \mid head)$

$= \frac{Pr(unbiased \cap head)}{Pr(head)} = \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{4}{9}} = \frac{3}{4}$

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Question 4 (4 marks)

a. Solve $1-\cos\left(\frac{x}{2}\right)=\cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$.

2 marks

$1 = 2\cos\left(\frac{x}{2}\right)$

$\cos\left(\frac{x}{2}\right) = \frac{1}{2}$

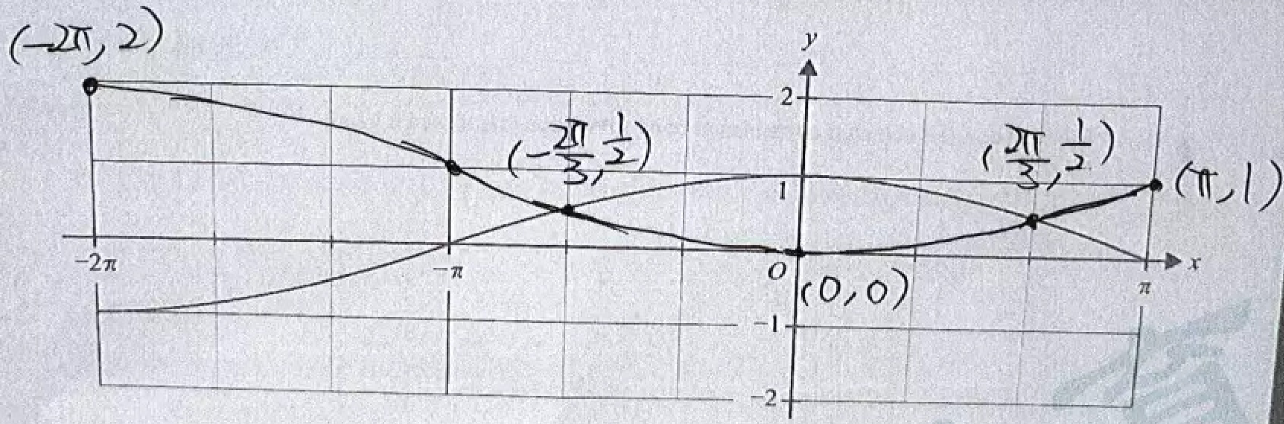
$k \in \mathbb{Z}$

$\frac{x}{2} = \frac{\pi}{3}$ or $\frac{5\pi}{3} + 2k\pi$

$x = \frac{2\pi}{3}$ or $\frac{10\pi}{3} + 4k\pi$

$x = \frac{2\pi}{3}$ or $\frac{-2\pi}{3}$

b. The function $f: [-2\pi, \pi] \rightarrow R$, $f(x)=\cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g: [-2\pi, \pi] \rightarrow R$, $g(x) = 1 - f(x)$.

Sketch the graph of g on the axes above. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates.

2 marks

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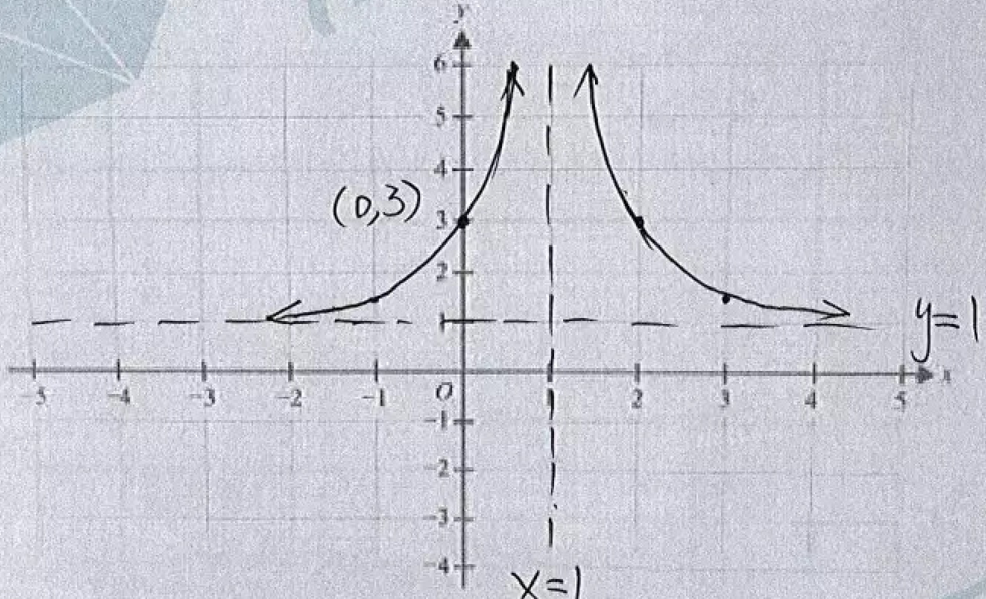
Question 5 (5 marks)

Let $f: R\setminus\{1\} \to R,\ f(x) = \dfrac{2}{(x-1)^2} + 1$.

a. i. Evaluate $f(-1)$. 1 mark

$f(-1) = \frac{2}{4} + 1 = \frac{3}{2}$

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations. 2 marks



b. Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$. 2 marks

$\text{Area} = \int_{-1}^{0} f(x)\,dx$

$= \int_{-1}^{0} 2(x-1)^{-2} + 1\,dx$

$= \left[-2(x-1)^{-1} + x\right]_{-1}^{0}$

$= (2) - (0)$

$= 2 \text{ unit}^2$

Question 6 (3 marks)

Fred owns a company that produces thousands of pegs each day. He randomly selects 41 pegs that are produced on one day and finds eight faulty pegs.

a. What is the proportion of faulty pegs in this sample? 1 mark

$\hat{p} = \frac{8}{41}$

b. Pegs are packed each day in boxes. Each box holds 12 pegs. Let $\hat{P}$ be the random variable that represents the proportion of faulty pegs in a box.

The actual proportion of faulty pegs produced by the company each day is $\frac{1}{6}$.

Find $\Pr\left(\hat{P} < \frac{1}{6}\right)$. Express your answer in the form $a(b)^n$, where a and b are positive rational numbers and n is a positive integer. 2 marks

$n = 12 \quad p = \frac{1}{6}$

$\Pr(\hat{p} < \frac{1}{6})$

$= \Pr(X < 2)$

$= \Pr(X=1) + \Pr(X=0)$

$= \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11} + \binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12}$

$= \left(\frac{17}{6}\right)\left(\frac{5}{6}\right)^{11}$

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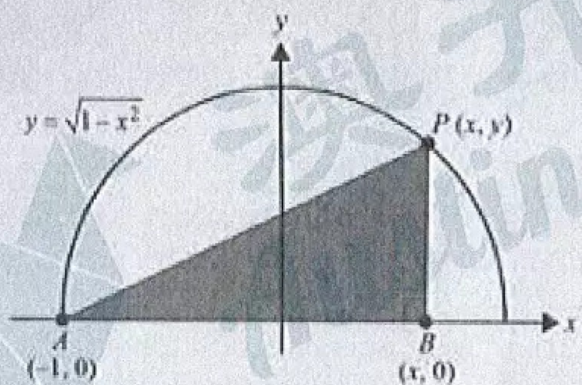
Question 7 (4 marks)

The graph of the relation $y=\sqrt{1-x^2}$ is shown on the axes below. P is a point on the graph of this relation, A is the point $(-1, 0)$ and B is the point $(x, 0)$.



a. Find an expression for the length PB in terms of x only. 1 mark

$L_{PB}=\sqrt{1-x^2}$

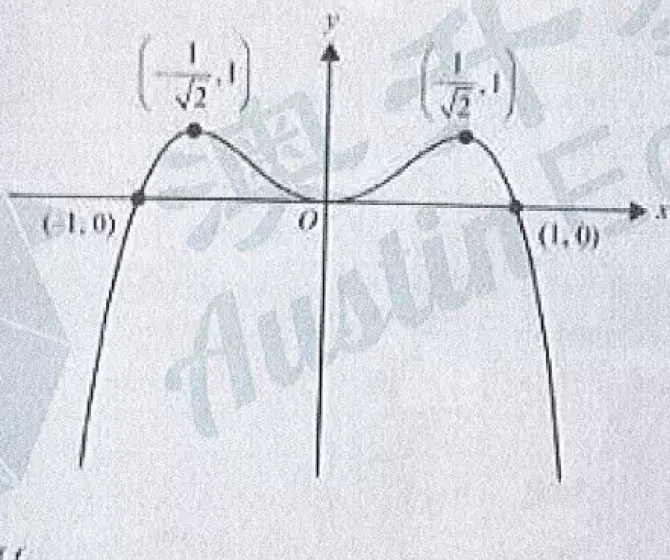
b. Find the maximum area of the triangle ABP. 3 marks

$$\text{Area}=A(x)=\frac{1}{2}(x-(-1))\sqrt{1-x^2}$$
$$=\frac{(x+1)}{2}\sqrt{1-x^2}$$
$$A'(x)=\frac{1}{2}\left(1\cdot\sqrt{1-x^2}+(x+1)\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right)$$
$$=\frac{1}{2}\left(\frac{1-x^2-x^2-x}{\sqrt{1-x^2}}\right)=0$$
$$-2x^2-x+1=0$$
$$2x^2+x-1=0$$
$$(2x-1)(x+1)=0$$
$$\cancel{x=-1} \text{ or } x=\frac{1}{2}$$
$$-1<x<1$$

$$A\left(\frac{1}{2}\right)=\frac{3}{4}\sqrt{\frac{3}{4}}$$
$$=\frac{3\sqrt{3}}{8}\text{ units}^2$$

Question 8 (4 marks)

The function $f : R \to R$, $f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown below. The graph of f touches the x-axis at the origin.



a. Find the rule of f. 1 mark

$f(x) = a(x+1)(x-1)x^2$

$\because f\left(\frac{1}{\sqrt{2}}\right) = 1$

$1 = a\left(\frac{1}{\sqrt{2}}+1\right)\left(\frac{1}{\sqrt{2}}-1\right)\left(\frac{1}{\sqrt{2}}\right)^2$

$1 = a\left(-\frac{1}{2}\right) \cdot \frac{1}{2}$

$a = -4$

$\therefore f(x) = -4(x+1)(x-1)x^2$

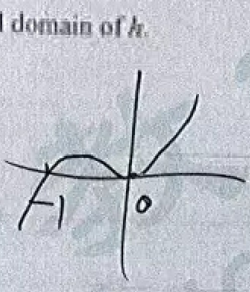
Let g be a function with the same rule as f.

Let $h : D \to R$, $h(x) = \log_e(g(x)) - \log_e(x^3 + x^2)$, where D is the maximal domain of h.

b. State D. 1 mark

$g(x) > 0 \cap x^3 + x^2 > 0$

$(-1, 1)\backslash\{0\} \cap (-1, \infty)\backslash\{0\}$



$\Downarrow$

$d_D \in (-1, 1)\backslash\{0\}$

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c. State the range of h. **2 marks**

$$h(x) = \log_e\left(\frac{-4\cancel{(x+1)}(x-1)\cancel{x^2}}{\cancel{x^3+x^2}}\right)$$

$$= \log_e(-4(x-1))$$

$$d_h \in (-1, 1) \setminus \{0\}$$

$$\log_e(8) \quad (-\infty) \quad \log_e(4)$$

$$r_h \in (-\infty, \log_e(8)) \setminus \{\log_e(4)\}$$

Question 9 (9 marks)

Consider the functions $f: R \to R, f(x) = 3 + 2x - x^2$ and $g: R \to R, g(x) = e^x$.

a. State the rule of $g(f(x))$. 1 mark

$$g(f(x)) = e^{3+2x-x^2}$$

b. Find the values of x for which the derivative of $g(f(x))$ is negative. 2 marks

$$\frac{d}{dx}(g(f(x)) = (2-2x)e^{3+2x-x^2}$$

$$(2-2x) < 0$$

$$-2x < -2$$

$$x > 1$$

c. State the rule of $f(g(x))$. 1 mark

$$f(g(x)) = 3+2e^x - e^{2x}$$

d. Solve $f(g(x)) = 0$. 2 marks

$f(g(x))=0$ let $a=e^x$

$$f(g(x)) = 3+2a-a^2$$

$$= -(a-3)(a+1)$$

$$e^x = 3 \text{ or } \cancel{e^x = -1}$$

$$x = \log_e(3)$$

e. Find the coordinates of the stationary point of the graph of $f(g(x))$. 2 marks

$$\frac{d}{dx}(f(g(x))) = 2e^{x} - 2e^{2x}$$

$$= 2e^{x}(1-e^{x}) = 0$$

$\therefore 1-e^{x}=0$ | $f(g(0)) = 4$

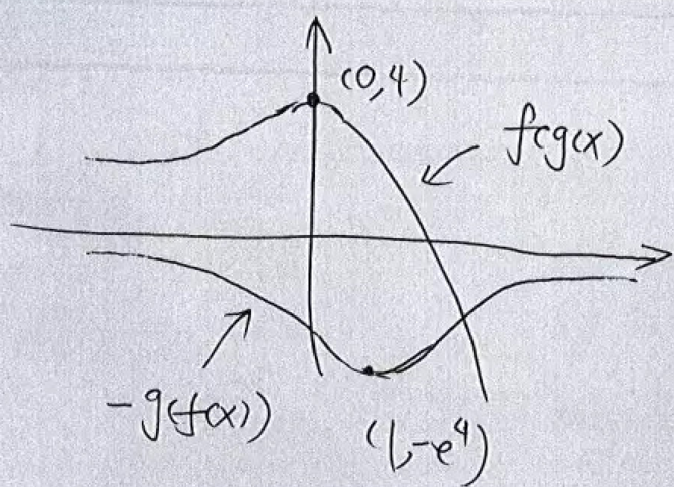
$e^{x}=1$

$x=0$ | $\therefore (0,4)$

f. State the number of solutions to $g(f(x)) + f(g(x)) = 0$. 1 mark

$-g(f(x)) = f(g(x))$

using (b) & (e)



$\Rightarrow$ 1 solution

END OF QUESTION AND ANSWER BOOK

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