MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2

TSSM™
Creating VCE Success

2019 Trial Examination

SOLUTIONS

SECTION A: Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation:
Period: $\frac{2\pi}{\frac{1}{4}} = 2\pi \times 4 = 8\pi$
Range: $[c - a, c + a] = [-1 - 2, -1 + 2] = [-3, 1]$

Question 2

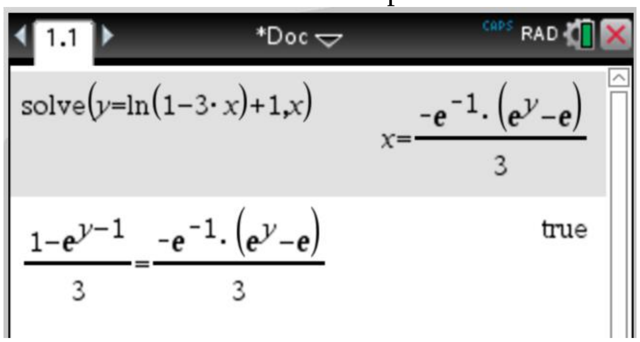
Answer: B

Explanation:
negative quartic graph
x-intercepts at $x = a, x = b$ and $x = c$ which gives brackets of $(x - a)(x - b)(x - c)$
repeated factor at $x = c$, means $(x - c)^2$

Question 3

Answer: C

Explanation:
Solving by hand gives C, or use solve function on CAS and equate each option to the answer until the CAS offers a true response



Question 4

Answer: E

Explanation:
For inverse swap x and y and solve for y on CAS, take the equation of $y^{-1} = 3 - \frac{1}{\sqrt{x}}$ and domain of $(0, \infty)$ as $dom\ f^{-1} = ran\ f$

Question 5

Answer: C

Explanation:

	$f(x)$	$g(x)$
Domain	$[0, 10]$	R
Range	$[0, 10]$	$(0, \infty)$

Firstly for $g(f(x))$ to exist $range\ f \subseteq domain\ g$
The $range\ g(f(x)) = range\ f$

Question 6

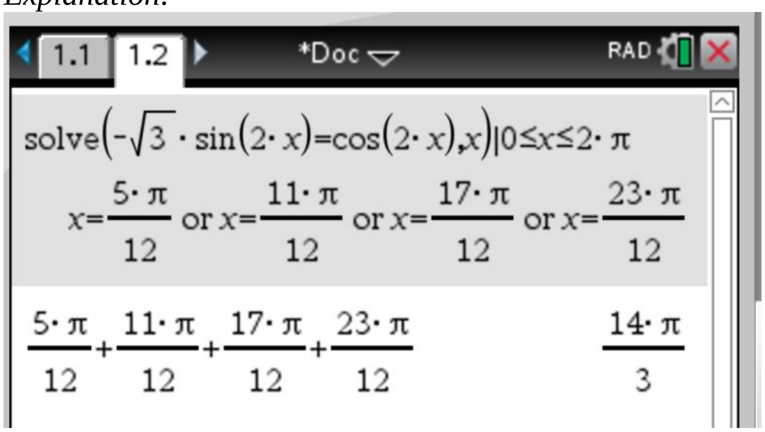
Answer: D

Explanation:
$\begin{bmatrix} m & 5 \\ 1 & m+4 \end{bmatrix}$
$m^2 + 4m - 5 = 0$
$m = -5, m = 1$
For infinitely many solutions same gradient and same y-intercept
Sub in $m = -5, m = 1$ into both equations to find the answer

Question 7

Answer: B

Explanation:



Question 8

Answer: D

Explanation:
Independent events $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.3 \times 0.45 = 0.135$
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.3 + 0.45 - 0.135 = 0.615$

Question 9

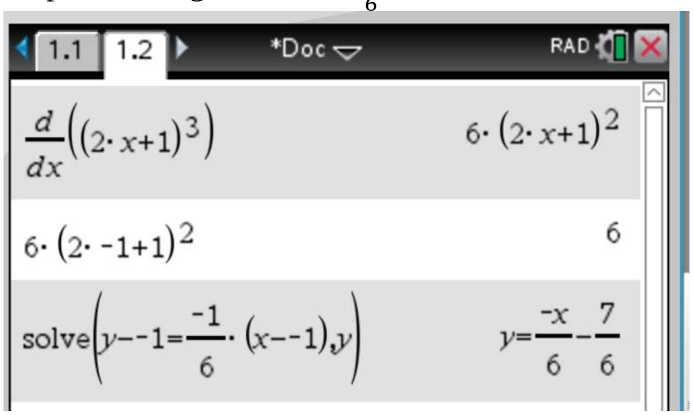
Answer: E

Explanation:
$f(x) \to -f(x)$ vertical reflection
$-f(x) \to -f(-x)$ horizontal reflection
$-f(x) \to -f(-x) + 1$ vertical translation

Question 10

Answer: C

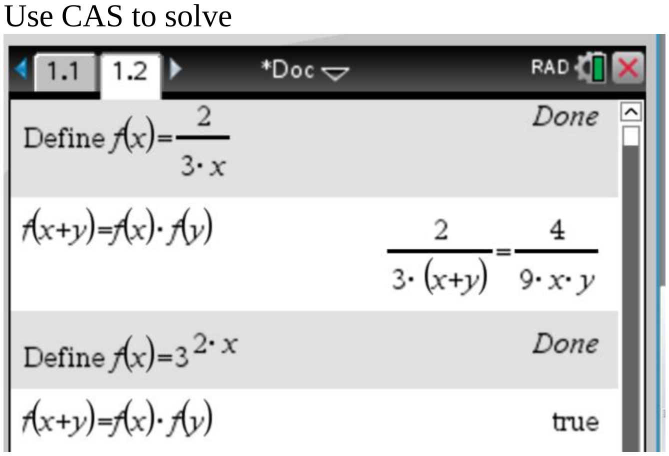
Explanation:
Perpendicular gradient $= -\frac{1}{6}$



Question 11

Answer: E

Explanation:
Use CAS to solve



Question 12

Answer: A

Explanation:
$X \sim Bi\left(10, \frac{1}{6}\right)$
$\Pr(X \leq 4) =$

Question 13

Answer: C

Explanation:
$x' = x - 3 \rightarrow x = x' + 3$
$y' = -2y + 4 \rightarrow y = 2 - \frac{y'}{2}$
$y = \frac{1}{x^2}$
$2 - \frac{y'}{2} = \frac{1}{(x'+3)^2}$
$-\frac{y'}{2} = \frac{1}{(x'+3)^2} - 2$
$\frac{y'}{2} = \frac{-1}{(x'+3)^2} + 2$
$y' = \frac{-2}{(x'+3)^2} + 4$

Question 14

Answer: A

Explanation:
$p + 2p + p + \frac{p}{2} + \frac{3p}{2} = 1$
$6p = 1$
$p = \frac{1}{6}$
$E(X) = 1 \times \frac{1}{6} + 2 \times 2\left(\frac{1}{6}\right) + 3 \times \frac{1}{6} + 4 \times \frac{\frac{1}{6}}{2} + 5 \times \frac{3 \times \frac{1}{6}}{2}$
$E(X) = \frac{1}{6} + \frac{2}{3} + \frac{1}{2} + \frac{1}{3} + \frac{5}{4}$
$E(X) = \frac{35}{12}$

Question 15

Answer: E

Explanation:
$p = 1$ two intersections
$p = 4$ two intersections
Between $p = 1$ and $p = 4$ three intersections therefore $1 < p < 4$

Question 16

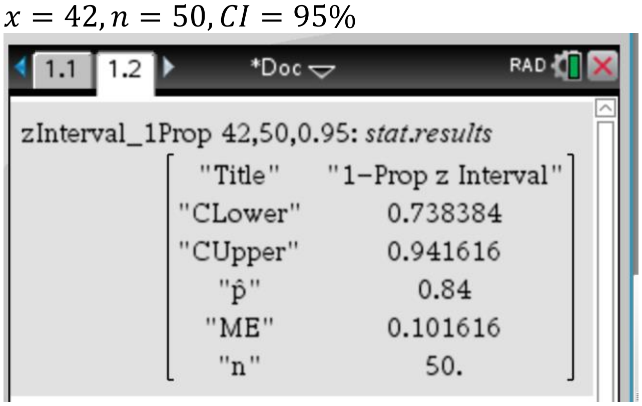
Answer: D

Explanation:
integration by recognition
$\int \frac{2x}{x^2+2} dx = \log_e(x^2 + 2) + c$
$2\int \frac{x}{x^2+2} dx = \log_e(x^2 + 2) + c$
$\int \frac{x}{x^2+2} dx = \frac{1}{2}\log_e(x^2 + 2) + c$
$\int \frac{3x}{x^2+2} dx = \frac{3}{2}\log_e(x^2 + 2) + c$

Question 17

Answer: A

Explanation:
$x = 42, n = 50, CI = 95\%$



Question 18

Answer: B

Explanation:
$y_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$
$0.4 = \frac{1}{a--a}\int_{-a}^{a} \sin^2(x)\,dx$ solve on CAS for a

Question 19

Answer: C

Explanation:
400 pens initially, 10 run out in first 6 weeks
$\Pr(X < 6) = \frac{10}{400} = 0.025$
$\Pr(Z < z) = 0.025 \quad invNorm(0.025,0,1) = -1.9599639$ ....
$z = \frac{x-\mu}{\sigma} \qquad -1.9599 = \frac{6-14}{\sigma}$
$\sigma = 4.08 \approx 4$ weeks

Question 20

Answer: E

Explanation:
Each area between the two graphs is the same
$g(x)$ starts above $f(x)$ hence E is correct.

SECTION B: Extended response questions

Question 1 (9 marks)

a. $B(23) = 2A$

$2A = Ae^{23k}$ (1M)

$2 = e^{23k}$

$\log_e(2) = 23k$ (1M)

$k = \frac{1}{23}\log_e(2)$

$k = 0.030137 \ldots$ (1M)

$k = 0.03$ as required

3 marks

b. $15000 = Ae^{0.03\times 60}$ (1M)

$15000 = Ae^{1.8}$

$A = 2479.48$

$A = 2479$ bacteria (1A)

2 marks

c. $B(90) = 36887$ bacteria

1 mark

d. 3900 bacteria

1 mark

e. $P(t) = B(t)$

$3900e^{0.02t} = 2479e^{0.03t}$ (1M)

$t = 45.3121$

$t = 45$ mins and 19 seconds (1A)

2 marks

Total 9 marks

Question 2 (15 marks)

a. $f(x) = 0$

$(-1,0)$, $(0,0)$ and $\left(\frac{3}{2}, 0\right)$ (1A)

1 mark

b. $f'(x) = 0$ (1M)

$8x^3 - 3x^2 - 6x = 0$

$x = \frac{-\sqrt{201}+3}{16}, x = 0, x = \frac{\sqrt{201}+3}{16}$

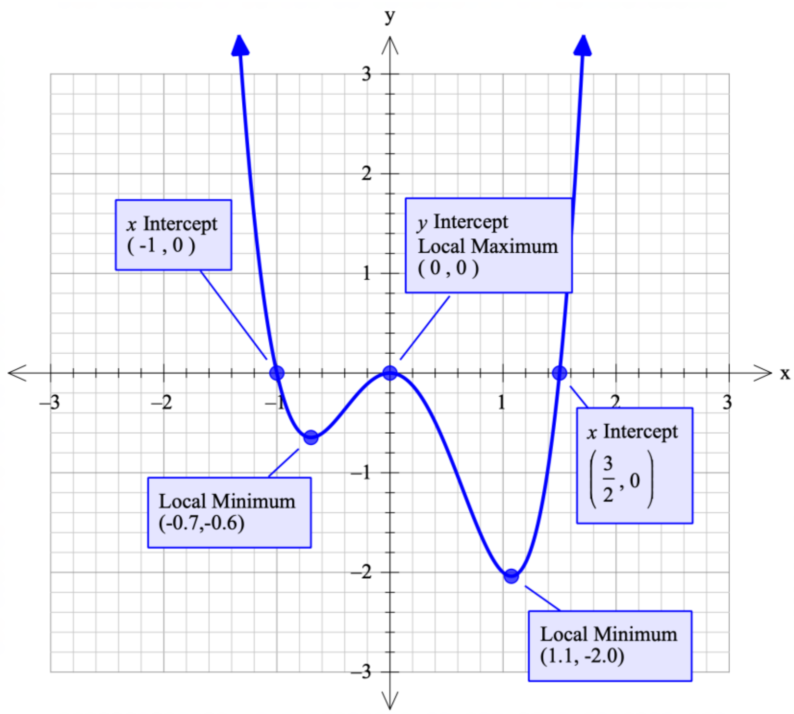
$(-0.7, -0.6)$, $(0,0)$ and $(1.1, -2.0)$

x	-1	-0.7	-0.2	0	1	1.1	2
sign	$-$	0	$+$	0	$-$	0	$+$
slope	\	$-$	/	$-$	\	$-$	/

Local Minimum at $(-0.7, -0.6)$ (1A)
Local Maximum at $(0,0)$ and (1A)
Local Minimum at $(1.1, -2.0)$ (1A)

4 marks

c.



1M – correct shape
1M – correct intercepts
1M – correct turning points
3 marks

d. $c > 2$ (1A)

1 mark

e. $f'(x) = 8x^3 - 3x^2 - 6x$
$f'(1) = -1$ (1M)
$f(1) = -2$
$y - -2 = -1(x - 1)$
$y = -x - 1$ (1A)

2 marks

f. $-x - 1 = 2x^4 - x^3 - 3x^2$ (1M)
$x = -1,\ x = -\frac{1}{2}, x = 1$
$(-1,0), \left(-\frac{1}{2}, -\frac{1}{2}\right), (1,-2)$ (1A)

2 marks

g. $\int_{-1}^{-\frac{1}{2}}\left((-x-1)-f(x)\right)dx+\int_{-\frac{1}{2}}^{1}\left(f(x)-(-x-1)\right)dx$ (1M)

$\frac{41}{320}+\frac{297}{320}=\frac{169}{160}\ units^2$ (1A)

2 marks
Total 15 marks

Question 3 (12 marks)

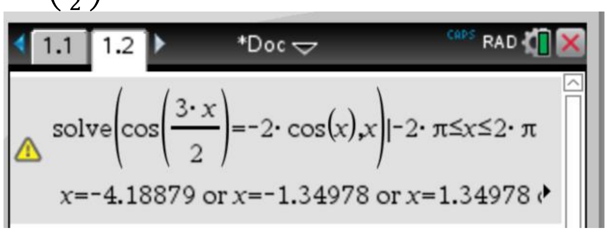
a.

i. Period $\frac{2\pi}{\frac{3}{2}}=2\pi\times\frac{2}{3}=\frac{4\pi}{3}$ (1A)
Amplitude 1 (1A)

ii. Period 2π (1A)
Amplitude 2 (1A)

2 + 2 = 4 marks

b. $\cos\left(\frac{3x}{2}\right)=-2\cos(x)$ (1M)



$x=-4.18879, x=-1.34978, x=1.34978, x=4.18879$

$(-4.189, 1.000), (-1.350, -0.438), (1.350, -0.438), (4.189, 1.000)$ (1/2A each coordinate)

3 marks

c. $[-4.189, 4.189]$

1 marks

d.

i. $[-1, 1]$ (1A)

ii. $[-2, 2]$ (1A)

1 + 1 = 2 marks

e. $2\int_{-4.189}^{-1.350}(g(x)-f(x))\,dx+\int_{-1.350}^{1.350}(f(x)-g(x))\,dx$ (1M)

$2\times 4.28257+5.10104=13.6662$

$14m^2$ (1A)

2 marks
Total 12 marks

Question 4 (15 marks)

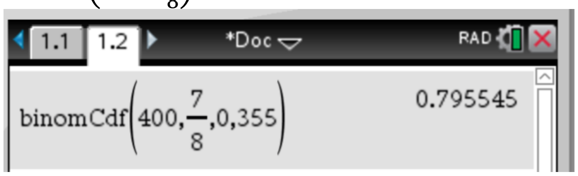
a. $n = 1000, p = \frac{7}{8}$

$E(X) = np = 1000 \times \frac{7}{8}$ (1M)

$E(X) = 875$ (1A)

2 marks

b. $X \sim Bi\left(400, \frac{7}{8}\right)$ (1M)



$\Pr(X \leq 355) = 0.796$ (1A)

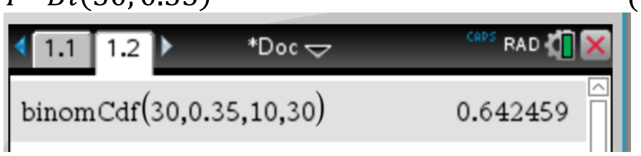
2 marks

c. $\Pr(X < 350 | X > 100) = \frac{\Pr(X<350 \cap X>100)}{\Pr(X>100)}$ (1M)

$\frac{Ps(101 \leq X \leq 349)}{Pr(X \geq 101)} = 0.462$ (1A)

2 marks

d. $Y \sim Bi(30, 0.35)$ (1M)

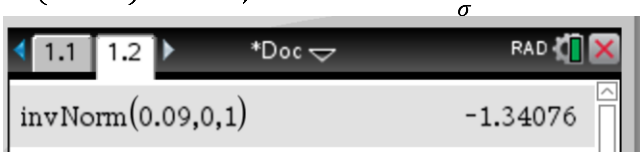


$\Pr(Y \geq 10) = 0.642$ (1A)

2 marks

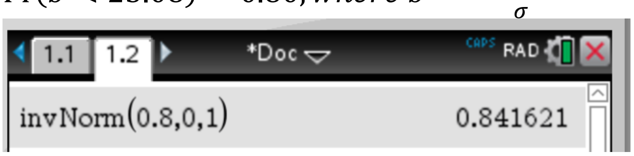
e. Using $z = \frac{x-\mu}{\sigma}$ and simultaneous equations

$\Pr(Z < a) = 0.09, where\ a = \frac{16.57-\mu}{\sigma}$



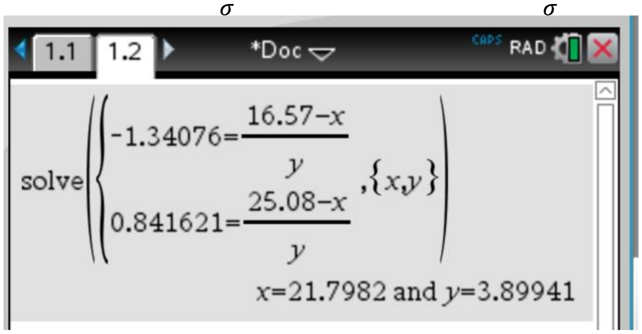
$a = -1.34076\ldots$ (1M)

$\Pr(b < 25.08) = 0.80, where\ b = \frac{25.08-\mu}{\sigma}$



$b = 0.841621\ldots$ (1M)

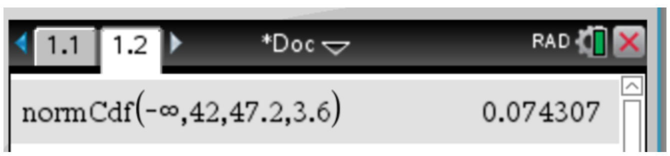
$-1.34076 = \frac{16.57-\mu}{\sigma}$ and $0.841621 = \frac{25.08-\mu}{\sigma}$



$\mu = 21.80°\text{C}$ and $\sigma = 3.90°\text{C}$ (1A)

3 marks

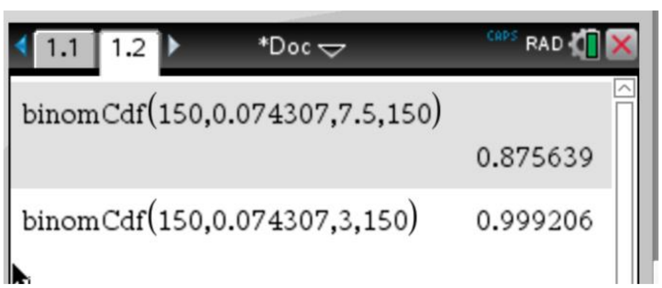
f. $X \sim N(47.2, 3.6^2)$
$\Pr(X < 42) = 0.074307$ (1M)



$\Pr(\hat{P} \geq 0.05 | \hat{P} \geq 0.02) = \Pr(X \geq 150 \times 0.05 | X \geq 150 \times 0.02)$

$\Pr(X \geq 7.5 | X \geq 3) = \frac{\Pr(X \geq 7.5 \cap X \geq 3)}{\Pr(X \geq 3)}$ (1M)

$\Pr(X \geq 7.5 | X \geq 3) = \frac{\Pr(X \geq 7.5)}{\Pr(X \geq 3)}$



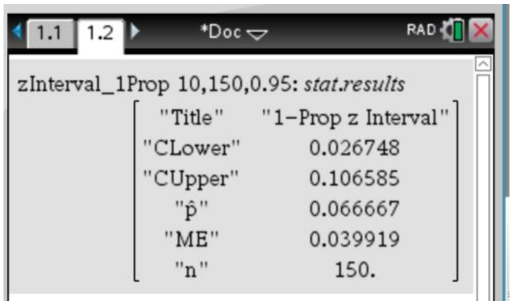
$\Pr(X \geq 7.5 | X \geq 3) = 0.8763$ (1A)

3 marks

g. $\hat{p} = \frac{1}{15}$

$\left(\frac{1}{15} - 1.96\sqrt{\frac{\frac{1}{15}\times\frac{14}{15}}{150}}, \frac{1}{15} + 1.96\sqrt{\frac{\frac{1}{15}\times\frac{14}{15}}{150}}\right)$

OR use CAS



$(0.0267, 0.1066)$

1 mark

Total 15 marks

Question 5 (9 marks)

a. $f'(x) = 0$

$0 = -2x(2x^2 - n^2)$

$\left(\frac{n\sqrt{2}}{2}, \frac{n^4}{4}\right), \left(-\frac{n\sqrt{2}}{2}, \frac{n^4}{4}\right), (0,0)$ (1A)

1 mark

b. $\int_{-n}^{n} f(x)\,dx = \frac{4}{15}$

$\frac{4n^5}{15} = \frac{4}{15}$

$n = 1$ (1A)

$x = 1, x = 0, x = -1$ (1A)

2 marks

c. $0 = -mx^2 + m$

$-m = -mx^2$

$1 = x^2$

$x = \pm 1$ as required

1 mark

d. $Area\ = 2\int_{-1}^{-\sqrt{m}}\big(f(x) - g(x)\big)\,dx + \int_{-\sqrt{m}}^{\sqrt{m}}\big(g(x) - f(x)\big)\,dx$ (1M)

$= \frac{32\,m^2 - 20m + 4}{15}$ (1A)

2 marks

e.

i. $f(b) = -b^2(b-n)(b+n)$
$f'(b) = -2b(2b^2 - n^2)$ (1A)

ii. $y - f(b) = f'(b)(x-b)$ (1M)
$y - \left(-b^2(b-n)(b+n)\right) = \left(-2b(2b^2 - n^2)\right)(x-b)$
$y = -2b(2(2b^2 - n^2)x + 3b^4 - b^2n^2$ (1A)

1 + 2 = 3 marks
Total 9 marks