YEAR 12 *Trial Exam Paper*

2020

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1a.i.

Worked solution

$$f(x) = (1-3x)^{\frac{1}{2}}$$

$$f'(x) = (-3) \times \frac{1}{2}(1-3x)^{-\frac{1}{2}}$$

$$= -\frac{3}{2\sqrt{1-3x}}$$

Mark allocation: 1 mark

- 1 mark for the correct expression

Tip

- *You should expect the first question in the exam to ask you to calculate a derivative. This will require you to apply the chain, product or quotient rules.*

Question 1a.ii.

Worked solution

$$f(x) = (1-3x)^{\frac{1}{2}}$$

$$\int f(x)\,dx = \frac{(1-3x)^{\frac{3}{2}}}{(-3) \times \frac{3}{2}} + c$$

$$= -\frac{2}{9}(1-3x)^{\frac{3}{2}} + c$$

Mark allocation: 1 mark

- 1 mark for $-\frac{2}{9}(1-3x)^{\frac{3}{2}}$ with any constant of integration

Tip

- *When a question asks for* ***an*** *antiderivative, you can use any value for the constant of integration, including zero. When a question asks for* ***the*** *antiderivative, you must provide an expression describing a family of functions and leave the constant of integration as a parameter.*

Question 1b.

Worked solution

$\frac{d}{dx}\sin(x)=\cos(x)$ and $\frac{d}{dx}(\cos(x)+1)=-\sin(x)$.

$$g'(x)=\frac{\cos(x)\times(\cos(x)+1)-\sin(x)\times(-\sin(x))}{(\cos(x)+1)^2}$$

$$=\frac{\cos^2(x)+\cos(x)+\sin^2(x)}{(\cos(x)+1)^2}$$

$$=\frac{\cos(x)+1}{(\cos(x)+1)^2}$$

$$=\frac{1}{\cos(x)+1}$$

$$g'\left(\frac{\pi}{2}\right)=\frac{1}{\cos\left(\frac{\pi}{2}\right)+1}$$

$$=\frac{1}{0+1}=1$$

Alternatively,

$$g'(x)=\frac{\cos(x)\times(\cos(x)+1)-\sin(x)\times(-\sin(x))}{(\cos(x)+1)^2}$$

When $\cos\left(\frac{\pi}{2}\right)=0$ and $\sin\left(\frac{\pi}{2}\right)=1$, then

$$g'\left(\frac{\pi}{2}\right)=\frac{0\times(0+1)-1\times(-1)}{(0+1)^2}$$

$$=\frac{1}{1}=1$$

Mark allocation: 2 marks

- 1 answer mark for calculating the correct derivative $g'(x)=\frac{1}{\cos(x)+1}$ or equivalent
- 1 answer mark for the correct answer $g'\left(\frac{\pi}{2}\right)=1$

Tip

- *An angle such as $\frac{\pi}{2}$ evaluates to values that are easy to work with. So you may find this problem faster to solve if you don't simplify the expression you get from applying the quotient rule and, instead, evaluate the sin and cos functions immediately.*

Question 2a.

Worked solution

Let $y = \sqrt{1-3x}$.

To find the rule for the inverse, write a new equation with the x and y interchanged.

$$x = \sqrt{1-3y}$$

$$1-3y = x^2$$

$$y = -\frac{1}{3}\left(x^2-1\right) = -\frac{1}{3}x^2 + \frac{1}{3}$$

Therefore, the rule for the inverse is

$$f^{-1}(x) = -\frac{1}{3}x^2 + \frac{1}{3}$$

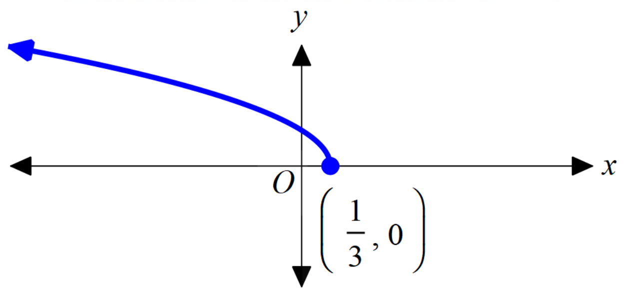
Mark allocation: 2 marks

- 1 method mark for writing an equation with x and y interchanged
- 1 answer mark for $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{1}{3}$ or equivalent

Question 2b.

Worked solution

The range of f is $(0, \infty)$.



Therefore, the domain of f^{-1} is $(0, \infty)$.

Mark allocation: 1 mark

- 1 answer mark for $(0, \infty)$

Tip

- *When asked for the domain of an inverse function, you may find it helpful to do a quick sketch of the original function so that you can determine its range.*

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Question 3a.

Worked solution

$\log_e(x) = 1 - 2\log_e(x)$

$3\log_e(x) = 1$

$\log_e(x) = \frac{1}{3}$

$x = e^{\frac{1}{3}}$

Mark allocation: 1 mark

- 1 answer mark for $x = e^{\frac{1}{3}}$

Question 3b.

Worked solution

$g(x) = 1 - 2\log_e(x)$

There is a vertical asymptote at $x = 0$

The x-intercept can be found by solving $g(x) = 0$.

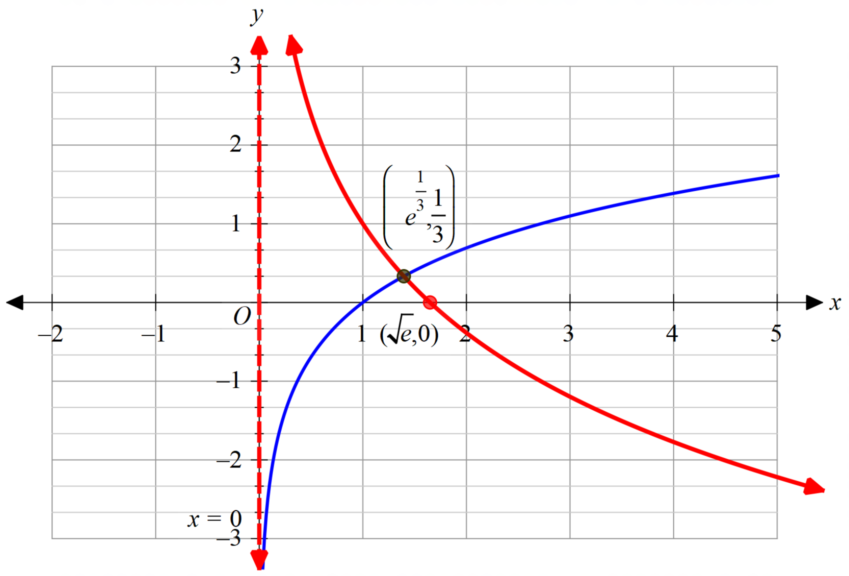
$1 - 2\log_e(x) = 0$

$\log_e(x) = \frac{1}{2}$

$x = e^{\frac{1}{2}} = \sqrt{e}$

From **part a.** the point of intersection occurs at $x = e^{\frac{1}{3}}$.

$y = \log_e(e^{\frac{1}{3}}) = \frac{1}{3}$



Mark allocation: 3 marks

- 3 mark responses will have a correctly shaped curve with asymptote marked and labelled, intersection labelled, and x-intercept labelled, with the point of intersection located on the horizontal gridline at $y=\frac{1}{3}$
- 2 mark responses will have a strictly decreasing curve that does not cross the y-axis, with at least one of the x-intercept or the point of intersection correctly located and labelled
- 1 mark responses will have a strictly decreasing curve that does not cross the y-axis

Tip

- *Writing the function in the form* $g(x)=-2\times\log_e(x)+1$ *might help you recognise the transformations that have been performed to the function* $f(x)=\log_e(x)$. *It has been reflected in the x-axis, dilated by a factor of 2 from the x-axis, and translated up by 1 unit. You can use these transformations to help you sketch the correct graph.*

Question 4a.

Worked solution

$1-\Pr(T>15)-\Pr(T<9)=1-0.05-0.05=0.9$

Mark allocation: 1 mark

- 1 answer mark for 0.9

Question 4b.

Worked solution

$$\Pr(T>15\,|\,T>12)=\frac{\Pr(T>15)}{\Pr(T>12)}=\frac{0.05}{0.5}=\frac{1}{10}$$

Mark allocation: 2 marks

- 1 method mark for $\Pr(T>15\,|\,T>12)=\dfrac{\Pr(T>15)}{\Pr(T>12)}$ or an equivalent expression
- 1 answer mark for $\dfrac{1}{10}$ or 0.1

Question 5a.

Worked solution

$$f\left(\frac{3}{2}\right)=\frac{4}{\left(2\times\frac{3}{2}\right)-1}-1$$

$$=\frac{4}{3-1}-1$$

$$=\frac{4}{2}-1$$

$$=2-1$$

$$=1$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5b.

Worked solution

First, find $f'(x)$.

$$f(x)=4(2x-1)^{-1}-1$$

$$f'(x)=4\times(-1)\times(2)\times(2x-1)^{-2}=-\frac{8}{(2x-1)^2}$$

$$f'\left(\frac{3}{2}\right)=-\frac{8}{\left(2\times\frac{3}{2}-1\right)^2}=-\frac{8}{2^2}=-2$$

Tangent is $y-1=-2\left(x-\frac{3}{2}\right)$, which simplifies to $y=-2x+4$.

Mark allocation: 2 marks

- 1 method mark for $f'\left(\frac{3}{2}\right)=-2$
- 1 answer mark for $y=-2x+4$

Question 5c.

Worked solution

When endpoint and y-axis intercept at $x=0$:

$$f(0)=\frac{4}{2\times 0-1}-1$$
$$=\frac{4}{-1}-1$$
$$=-4-1$$
$$=-5$$

When x-axis intercept at $y=0$:

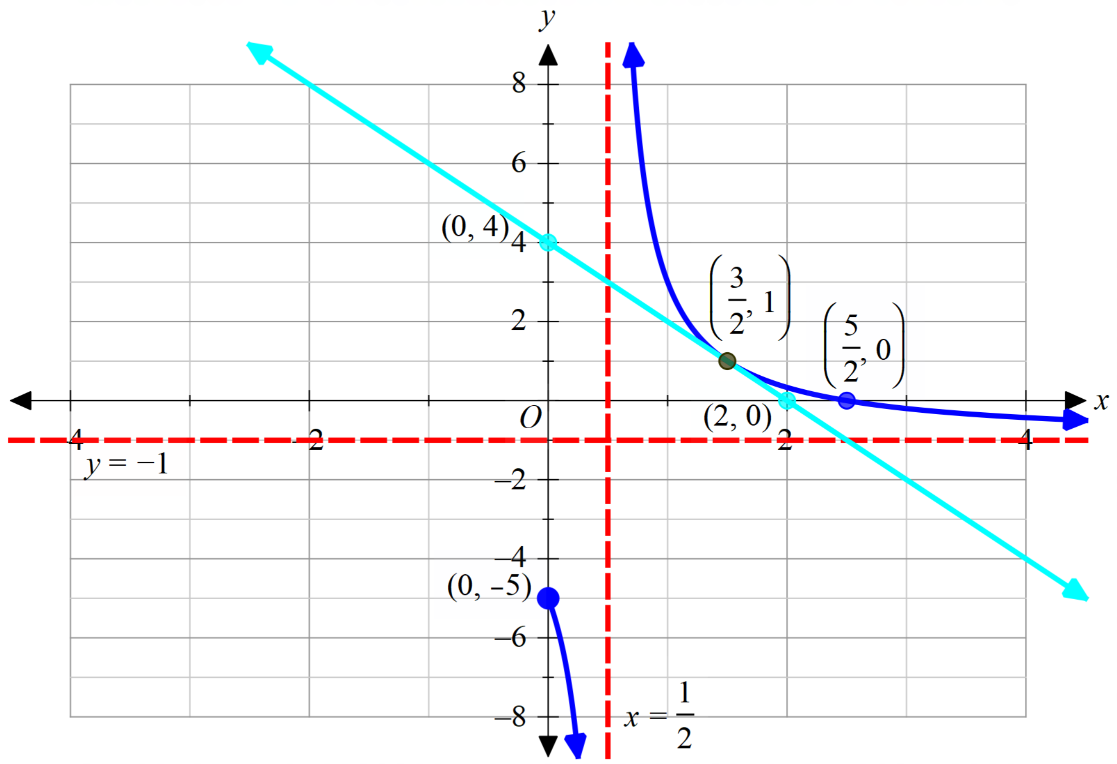
$$0=\frac{4}{2x-1}-1$$

$$\frac{4}{2x-1}=1$$

$$2x-1=4$$

$$2x=5$$

$$x=\frac{5}{2}$$



Mark allocation: 3 marks

- 2 marks for sketching $y = f(x)$
  - 2 mark responses will have both asymptotes clearly labelled, the points $(0, -5)$ and $\left(\frac{5}{2}, 0\right)$ labelled, and the curve correctly drawn over the appropriate domain
  - 1 mark responses will have a correct hyperbola shape, the horizontal asymptote correctly labelled, and at least one of the points $(0, -5)$ and $\left(\frac{5}{2}, 0\right)$ labelled
- 1 mark for the sketch of the tangent drawn through $(0, 4)$, $\left(\frac{3}{2}, 1\right)$ and $(2, 0)$, with $\left(\frac{3}{2}, 1\right)$ labelled.

Note: No labels are required for the axes intercepts of the tangent.

Tip

- *When sketching hyperbolas, the first thing that you should sketch are the asymptotes.*

Question 6a.

Worked solution

$$\frac{3}{8}\times 10=\frac{15}{4}=3.75$$

Mark allocation: 1 mark

- 1 answer mark for $\frac{15}{4}$ or 3.75

Question 6b.

Worked solution

$$\frac{3}{8}\times\frac{5}{8}\times 10=\frac{75}{32}$$

Mark allocation: 1 mark

- 1 answer mark for $\frac{75}{32}$ or 2.34375

Question 7a.

Worked solution

The gradient of the tangent is the gradient of e^x at P and is given by $\frac{d}{dx}e^x = e^x$ at $x = p$, giving e^p.

The tangent passes through the point $\left(p, e^p\right)$.

Therefore, the equation of the tangent is given by

$$y - e^p = e^p(x - p)$$

$$y = e^p(x - p) + e^p$$

$$y = e^p x + e^p(1 - p)$$

Mark allocation: 1 mark

- 1 method mark for evaluating the derivative, for substituting the derivative and point to determine the equation of the tangent, and for rearranging the equation into the given form

Tip

- *The results of a 'show that' question are usually required to answer the question.*

Question 7b.

Worked solution

Enclosed area is given by

$$A = \int_0^1 e^x - \left(e^p x + e^p(1-p)\right)dx$$

$$= \int_0^1 e^x - e^p x + e^p(p-1)dx$$

$$= \left[e^x - \frac{e^p}{2}x^2 + e^p(p-1)x\right]_0^1$$

$$= \left[e - \frac{e^p}{2} + e^p(p-1)\right] - [1]$$

$$= e^p\left(p - \frac{3}{2}\right) + e - 1$$

Area is minimised when $\frac{dA}{dp} = 0$.

$$\frac{dA}{dp} = e^p \times \frac{d}{dp}\left(p - \frac{3}{2}\right) + \frac{d}{dp}\left(e^p\right) \times \left(p - \frac{3}{2}\right)$$

$$= e^p \times 1 + e^p \times \left(p - \frac{3}{2}\right)$$

$$= e^p\left(p - \frac{1}{2}\right)$$

$\frac{dA}{dp} = 0 = e^p\left(p - \frac{1}{2}\right)$, where $e^p > 0$.

Therefore,

$$p - \frac{1}{2} = 0$$

$$p = \frac{1}{2}$$

Mark allocation: 3 marks

- 1 method mark for the calculation of the rule for the area in terms of p:
  $A = e^p\left(p - \frac{3}{2}\right) + e - 1$
- 1 method mark for the calculation of the derivative of the rule for the area in terms of p: $\frac{dA}{dp} = e^p\left(p - \frac{1}{2}\right)$
- 1 answer mark for $p = \frac{1}{2}$

Tip

- *You need to be careful about which variable you are integrating or differentiating with respect to each step of this question.*

Question 8a.i.

Worked solution

The limits of g's range are $a\times -1+a=0$ and $a\times 1+a=2a$.

Hence, when $a\geq 0$, the range of g is $[0, 2a]$.

Therefore, if $a\geq 0$, then $g\left(f(x)\right)\geq 0$.

Mark allocation: 1 mark

- 1 mark for $a\geq 0$

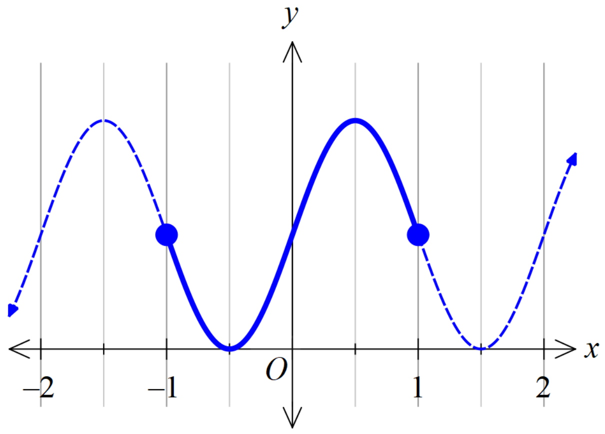
Tip

- *As no graphs for the functions are provided, you could do a quick sketch of a relevant portion of each function to help you visualise the problem.*

Question 8a.ii.

Worked solution

$f(x)$ has a range of $[-1, 1]$, so first consider when $g(x)$ is at its maximum on this interval.



$g(x)$ is at its maximum when $x=\frac{1}{2}$.

Therefore, we need to solve $f(x)=\frac{1}{2}$ on the interval $x\in[0, \pi]$.

$$\sin\left(\frac{x}{2}\right)=\frac{1}{2}$$

$$\frac{x}{2}=\frac{\pi}{6}\quad\left(\text{solutions on the interval } \frac{x}{2}\in\left[0, \frac{\pi}{2}\right]\right)$$

$$x=\frac{\pi}{3}$$

Alternatively, the maximum value of $g\left(f(x)\right)$ is $2a$. Therefore, solve $g\left(f(x)\right)=2a$.

$$g\left(f(x)\right)=2a$$

$$a\sin\left(\pi f(x)\right)+a=2a$$

$$\sin\left(\pi f(x)\right)=1$$

$$\pi f(x)=\frac{\pi}{2}$$

$$f(x)=\frac{1}{2}$$

$$\sin\left(\frac{x}{2}\right)=\frac{1}{2}$$

$\frac{x}{2}=\frac{\pi}{6}$ (solutions on the interval $\frac{x}{2}\in\left[0,\frac{\pi}{2}\right]$)

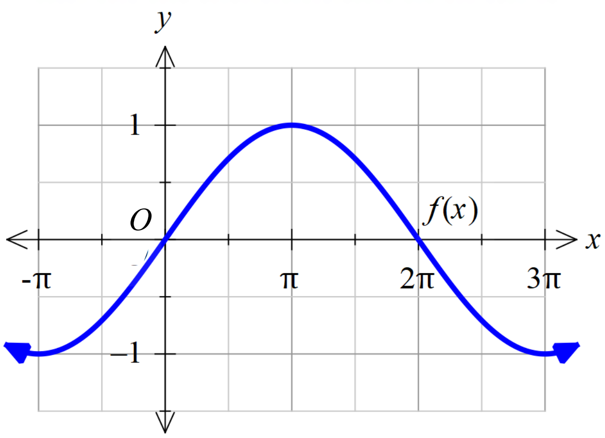
$$x=\frac{\pi}{3}$$

Mark allocation: 3 marks

- 1 method mark for identifying that $g(x)$ is at its maximum at $x=\frac{1}{2}$ or that $g\left(f(x)\right)=2a$
- 1 method mark for $\sin\left(\frac{x}{2}\right)=\frac{1}{2}$
- 1 answer mark for $x=\frac{\pi}{3}$

Question 8b.

Worked solution



$f(x) \geq 0$ when $x \in [0, 2\pi]$. Hence, the range of $g(x)$ must fall within this interval.

The range of g is $[0, 2a]$ when $a \geq 0$ and is $[2a, 0]$ when $a < 0$.

Therefore, we need $2a = 2\pi$, hence, $a = \pi$.

Therefore, for $f\left(g(x)\right) \geq 0$ we must have $a \in \left[0, \pi\right]$.

Alternatively, $f\left(g(x)\right) = \sin\left(\frac{1}{2}\left(a\sin(\pi x) + a\right)\right)$.

For $f\left(g(x)\right) \geq 0$ we must have $0 \leq \frac{1}{2}\left(a\sin(\pi x) + a\right) \leq \pi$.

$$0 \leq \frac{1}{2}\left(a\sin(\pi x) + a\right) \leq \pi$$

$$0 \leq a\sin(\pi x) + a \leq 2\pi$$

$a\sin(\pi x) + a$ has values on the interval $[0, 2a]$, hence, $2a \leq 2\pi$ and $a \geq 0$.

Hence, $a \in \left[0, \pi\right]$.

Alternatively, $f(x) \geq 0$ when $x \in [0, 2\pi]$. Hence, the range of $g(x)$ must fall within this interval. $g(x)$ is a sinusoidal function, so for its range to be in the interval $[0, 2\pi]$, its median position must be greater than zero and less than π. From the equation for $g(x)$, its median position is at a; hence, $a \in \left[0, \pi\right]$.

Mark allocation: 2 marks

- 1 method mark for identifying that $2a = 2\pi$ or that $f(x) \geq 0$ when $x \in [0, 2\pi]$ or that $0 \leq \frac{1}{2}\left(a\sin(\pi x) + a\right) \leq \pi$
- 1 answer mark for the answer $a \in \left[0, \pi\right]$ or an equivalent expression

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Question 9a.

Worked solution

$$\text{Area} = \int_0^2 -2x^2 + 4x\,dx$$

$$= \left[-\frac{2}{3}x^3 + 2x^2\right]_0^2$$

$$= -\frac{16}{3} + 8$$

$$= \frac{8}{3}$$

Mark allocation: 1 mark

- 1 mark for the calculation of the antiderivative $-\frac{2}{3}x^3 + 2x^2$ and substitution of $x = 2$ to evaluate $\frac{8}{3}$

Question 9b.

Worked solution

The transformation *T* dilates *f* by a factor of *a* from the *y*-axis. The domain of *f* is $[0, 2]$; hence, the domain of *g* is $[0, 2a]$.

Mark allocation: 1 mark

- 1 mark for $[0, 2a]$

Question 9c.

Worked solution

The area enclosed by *g* is composed of a copy of *f* dilated by a factor of *a* and a rectangle with dimensions 1 and $2a$.

$$\frac{8}{3}a + 2a = \frac{8}{3}$$

$$\frac{14}{3}a = \frac{8}{3}$$

$$a = \frac{8}{14} = \frac{4}{7}$$

Alternatively, solving $\int_0^{2a} -2\left(\frac{x}{a}\right)^2 + 4\left(\frac{x}{a}\right) + 1\,dx = \frac{8}{3}$ will give the same solution.

Mark allocation: 2 marks

- 1 method mark for $\frac{8}{3}a + 2a = \frac{8}{3}$ or $\int_0^{2a} -2\left(\frac{x}{a}\right)^2 + 4\left(\frac{x}{a}\right) + 1\,dx = \frac{8}{3}$
- 1 answer mark for $a = \frac{4}{7}$

Question 9d.i.

Worked solution

$$-2x^2+4x=1$$
$$2x^2-4x+1=0$$
$$x=\frac{4\pm\sqrt{4^2-4\times2\times1}}{2\times2}$$
$$=\frac{4\pm\sqrt{8}}{4}$$
$$=1\pm\frac{\sqrt{2}}{2}$$

Mark allocation: 1 mark

- 1 answer mark for $x=1\pm\frac{\sqrt{2}}{2}$

Question 9d.ii.

Worked solution

When $a = 3$:

$$-2x^2+4x=-\frac{2}{9}x^2+\frac{4}{3}x+1$$
$$\frac{16}{9}x^2-\frac{8}{3}x+1=0$$

This will have a unique solution when $\Delta=0$.

$$\Delta=\left(\frac{8}{3}\right)^2-4\times\frac{16}{9}\times1$$
$$=\frac{64}{9}-\frac{64}{9}$$
$$=0$$

as required.

Mark allocation: 1 mark

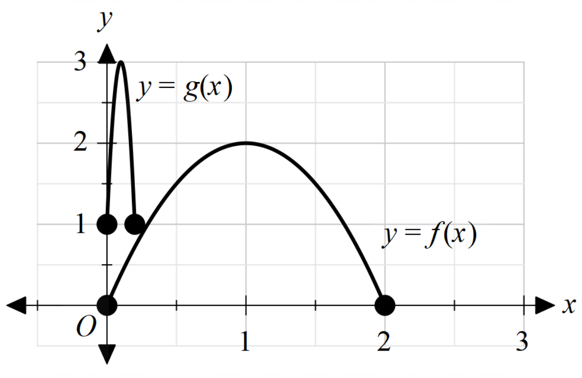
- 1 method mark for showing that the discriminant is equal to zero

Question 9d.iii.

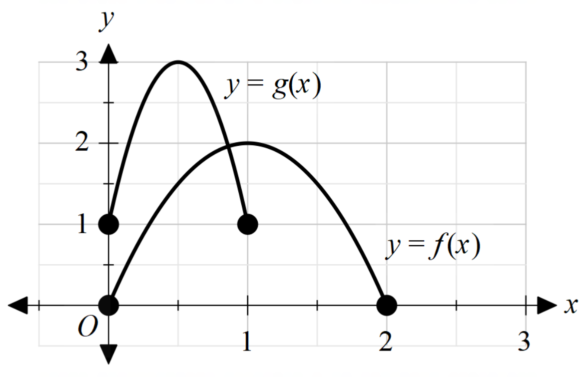
Worked solution

As a increases, the following progression of relationships between $f(x)$ and $g(x)$ is seen.

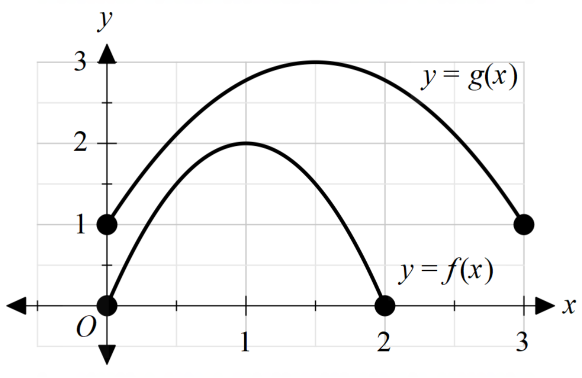
$g(x) > f(x)$



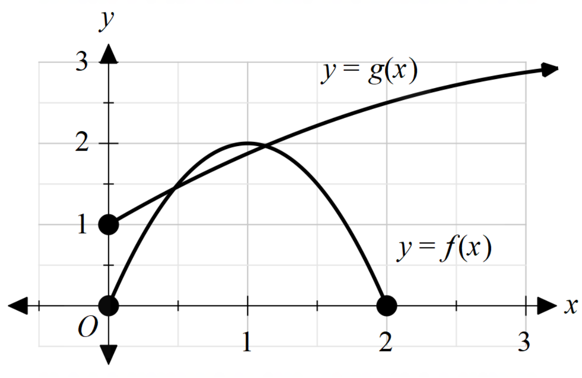
$g(x) \ngtr f(x)$



$g(x) > f(x)$

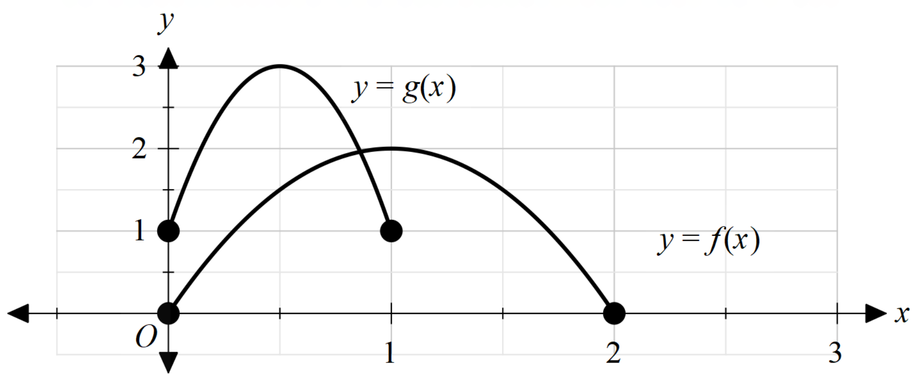


$g(x) \ngtr f(x)$

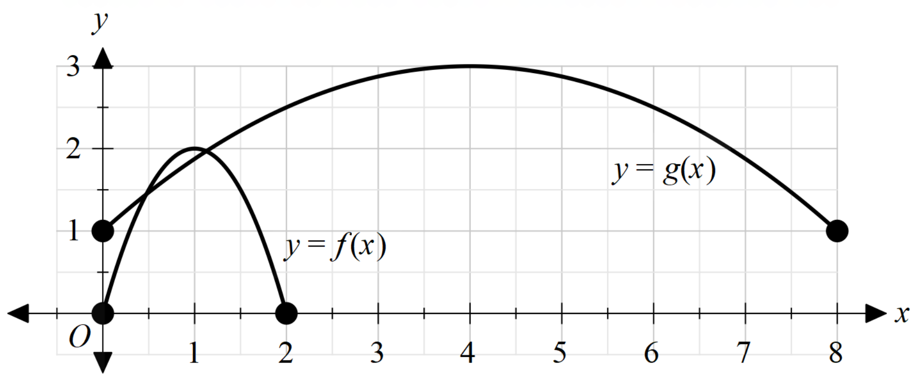


The condition that $g(x) > f(x)$ is violated in two situations.

Firstly, this occurs when the right endpoint is between $1-\frac{\sqrt{2}}{2} \le x \le 1+\frac{\sqrt{2}}{2}$. This occurs when $\frac{1}{2}-\frac{\sqrt{2}}{4} \le a \le \frac{1}{2}+\frac{\sqrt{2}}{4}$.



Secondly, it occurs when $a \ge 3$ and $g(x)$ crosses below $f(x)$.



Therefore, $g(x) > f(x)$ when $0 < a \le \frac{1}{2}-\frac{\sqrt{2}}{4}$ or $\frac{1}{2}+\frac{\sqrt{2}}{4} < a < 3$.

Mark allocation: 2 marks

- 1 answer mark for $0 < a \le 2-\sqrt{2}$ or $2+\sqrt{2} < a < 3$
- 1 answer mark for the union of both intervals

END OF WORKED SOLUTIONS