2020 Mathematical Methods Trial Exam 1 Solutions

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Q1a $y=-x^2+4$

Q1b $y=b$, $x^2=4-b$, $x=\pm\sqrt{4-b}$

Area $A=\frac{1}{2}b\left(4+2\sqrt{4-b}\right)=b\left(2+\sqrt{4-b}\right)$

Q1c Let $\frac{dA}{db}=\left(2+\sqrt{4-b}\right)+b\left(\frac{-1}{2\sqrt{4-b}}\right)=0$

$4\sqrt{4-b}+2(4-b)-b=0$, $4\sqrt{4-b}=3b-8$,

$16(4-b)=9b^2-48b+64$, $9b^2-32b=0$, $b=\frac{32}{9}$

Q2ai

$$
\begin{array}{r}
x^4-4x^2+16 \\
x^2+4\;\overline{)\;x^6+0x^4+0x^2+64} \\
\underline{x^6+4x^4\qquad\qquad\quad} \\
-4x^4+0x^2\qquad \\
\underline{-4x^4-16x^2\qquad} \\
16x^2+64 \\
\underline{4x^2+64} \\
0
\end{array}
$$

$Q(x)=x^4-x^2+4$, Remainder $=0$

Q2aii $P(x)=\left(x^2+4\right)\left(x^4-4x^2+16\right)$

$=\left(x^2+4\right)\left(x^4+8x^2+16-12x^2\right)=\left(x^2+4\right)\left(\left(x^2+4\right)^2-\left(2\sqrt{3}\,x\right)^2\right)$

$=\left(x^2+4\right)\left(x^2-2\sqrt{3}\,x+4\right)\left(x^2+2\sqrt{3}\,x+4\right)$

Q2b The turning points are $(0,4)$, $\left(\sqrt{3},1\right)$ and $\left(-\sqrt{3},1\right)$.

$\tan\theta=\pm\frac{4-1}{0-\sqrt{3}}=\mp\sqrt{3}$, $\theta=\frac{2\pi}{3}$ or $\frac{\pi}{3}$, .: equilateral

Q3 $x'=-y$ and $y'=-x$

$\begin{bmatrix}X\\Y\end{bmatrix}=\frac{1}{2}\left(\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-y\\-x\end{bmatrix}\right)+\begin{bmatrix}-1\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}x-y\\y-x\end{bmatrix}+\begin{bmatrix}-1\\1\end{bmatrix}$

$=\begin{bmatrix}\frac{x-y}{2}-1\\\frac{y-x}{2}+1\end{bmatrix}=\begin{bmatrix}\frac{x-y-2}{2}\\\frac{y-x+2}{2}\end{bmatrix}$

.: $X=\frac{x-y-2}{2}$, $Y=\frac{y-x+2}{2}$, .: $X+Y=0$

Q4a $5e^{-4x}+2e^{-2x}-3=0$, $\left(5e^{-2x}-3\right)\left(5e^{-2x}+1\right)=0$

.: $5e^{-2x}-3=0$, $e^{2x}=\frac{5}{3}$, $x=\frac{1}{2}\log_e\frac{5}{3}$

Q4b $\log_4 x=\log_3 3-\log_3 5=\log_3\frac{3}{5}=\frac{\log_{10}\frac{3}{5}}{\log_{10}3}$

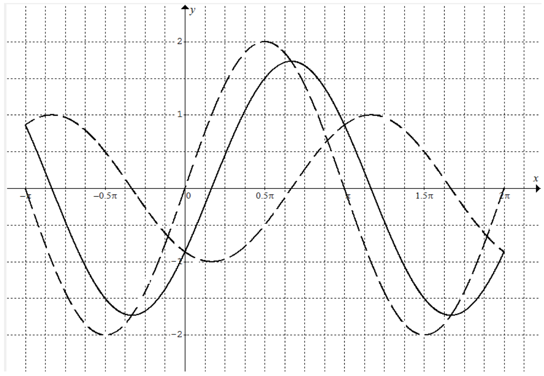
$\log_e x=\frac{\log_4 x}{\log_4 e}=\log_4 x\times\frac{\log_{10}4}{\log_{10}e}=\frac{\left(\log_{10}\frac{3}{5}\right)\left(\log_{10}4\right)}{\left(\log_{10}3\right)\left(\log_{10}e\right)}$

Q4c $y=Ae^{kx}$, $e=Ae^{ke}$ and $e^2=Ae^{ke^2}$

$\frac{e^2}{e}=\frac{e^{ke^2}}{e^{ke}}$, $e=e^{ke^2-ke}$, $ke^2-ke=1$, $k=\frac{1}{e^2-e}$

$A=\frac{e}{e^{ke}}=e^{1-ke}=e^{1-\frac{1}{e-1}}=e^{\frac{e-2}{e-1}}$

Q5a The two dotted curves



Q5b By addition of ordinates, sketch the solid curve $y=2\sin x+\cos\left(x+\frac{5\pi}{6}\right)$.

From graph, $2\sin x+\cos\left(x+\frac{5\pi}{6}\right)=0$ at $x=-\frac{5\pi}{6},\frac{\pi}{6},\frac{7\tau}{6}$

Q5c $x=\frac{\pi}{6}+n\pi$ where n is an integer.

Q6a $f'(x)=\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}=-\frac{1}{\sin^2 x}$

Q6b $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos^2 x}{\sin^2 x}dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1-\sin^2 x}{\sin^2 x}dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{\sin^2 x}-1\right)dx$

$=\left[-\frac{\cos x}{\sin x}-x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\left(-\frac{\pi}{2}\right)-\left(-1-\frac{\pi}{4}\right)=1-\frac{\pi}{4}$

Q7a $\Pr(A\cap B)=\Pr(A)-\Pr(A\cap B')=\frac{3}{5}-\frac{1}{4}=\frac{7}{20}$

Q7b $\Pr(A\mid B)=\frac{3}{5}$, $\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{3}{5}$, .: $\Pr(B)=\frac{7}{12}$

.: $\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)=\frac{5}{6}$

Q7c $\Pr(A'\cap B')=\Pr(B')-\Pr(A\cap B')=\frac{5}{12}-\frac{1}{4}=\frac{1}{6}$

$\Pr(A')\Pr(B')=\frac{2}{5}\times\frac{5}{12}=\frac{1}{6}$, .: A' and B' are independent.

Q8a A: $E\left(\hat{P}\right)\approx 0.36$, $sd\left(\hat{P}\right)\approx\sqrt{\frac{0.36\times 0.64}{4}}=0.24$,

Approx. 95% confidence interval $(0, 0.84)$

B: $E\left(\hat{P}\right)\approx 0.64$, $sd\left(\hat{P}\right)\approx\sqrt{\frac{0.64\times 0.36}{144}}=0.04$

Approx. 95% confidence interval $(0.56, 0.72)$

Better to choose A, because the chance of waiting longer than 5 minutes can be close to zero.

Q8b $\Pr(\text{waiting}>5\text{ min})\approx\frac{1}{2}\times 0.36+\frac{1}{3}\times 0.64+\frac{1}{6}\times 0\approx 0.4$

Q9a Given n is a positive odd integer, .: $n+2$ is also a positive odd integer, .: both $x^{\frac{1}{n}}$ and $x^{\frac{1}{n+2}}$ are odd functions

Given m is a positive even integer, .: $m+2$ is also a positive even integer, $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$, $x^{\frac{m+2}{n+2}} = \left(x^{\frac{1}{n+2}}\right)^{m+2}$,

.: both $x^{\frac{m}{n}}$ and $x^{\frac{m+2}{n+2}}$ are even functions $f(-x) = f(x)$

Given $m > n$, $2m > 2n$, $mn + 2m > mn + 2n$,

$m(n+2) > n(m+2)$, .: $\frac{m}{n} > \frac{m+2}{n+2}$

.: for $x \in (-1, 0)$ or $(0, 1)$, $x^{\frac{m}{n}} < x^{\frac{m+2}{n+2}}$, .: $x^{\frac{m+2}{n+2}} > x^{\frac{m}{n}}$

Q9b $y = x^{\frac{m}{n}}$ and $y = x^{\frac{m+2}{n+2}}$ intersect at $x = -1, 0, 1$

$$A = \int_{-1}^{0}\left(x^{\frac{m+2}{n+2}} - x^{\frac{m}{n}}\right)dx + \int_{0}^{1}\left(x^{\frac{m+2}{n+2}} - x^{\frac{m}{n}}\right)dx = 2\int_{0}^{1}\left(x^{\frac{m+2}{n+2}} - x^{\frac{m}{n}}\right)dx$$

$$= 2\left[\frac{x^{\frac{m+2}{n+2}+1}}{\frac{m+2}{n+2}+1} - \frac{x^{\frac{m}{n}+1}}{\frac{m}{n}+1}\right]_0^1 = 2\left(\frac{1}{\frac{m+2}{n+2}+1} - \frac{1}{\frac{m}{n}+1}\right) = 2\left(\frac{n+2}{m+n+4} - \frac{n}{m+n}\right)$$

$$= \frac{4(m-n)}{(m+n+4)(m+n)}$$

Q9c $A = \frac{4(m-n)}{(m+n+4)(m+n)} < \frac{4(m+n)}{(m+n+4)(m+n)} = \frac{4}{m+n+4}$

i.e. $0 < A < \frac{4}{m+n+4}$

As $n \to \infty$, $m \to \infty$ (since $m > n$), .: $\frac{4}{m+n+4} \to 0$, .: $A \to 0$

Please inform mathline@itute.com re conceptual and/or mathematical errors

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