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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2021

Question 1 (3 marks)

a.

$$y = \log_e(x^2+1)$$

$$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

(1 mark)

b.

$$g(x) = \frac{\tan(x)}{x^2+x}$$

$$g'(x) = \frac{(x^2+x)\times\sec^2(x) - (2x+1)\times\tan(x)}{(x^2+x)^2}$$ **(1 mark)**

$$g'(\pi) = \frac{(\pi^2+\pi)\times 1 - (2\pi+1)\times 0}{(\pi^2+\pi)^2}$$

$$= \frac{1}{\pi^2+\pi}$$ **(1 mark)**

Note that $\sec^2(x) = \dfrac{1}{\cos^2(x)}$ and $\cos(\pi) = -1$, so $\cos^2(\pi) = 1$ and $\sec^2(x) = 1$.

Question 2 (3 marks)

a.

$$f(x) = \frac{1}{x+2}$$

Let $y = \dfrac{1}{x+2}$

Swap *x* and *y* for inverse.

$$x = \frac{1}{y+2}$$

$$x(y+2) = 1$$

$$y+2 = \frac{1}{x}$$

$$y = \frac{1}{x} - 2$$

So $f^{-1}(x) = \dfrac{1}{x} - 2$

(1 mark)

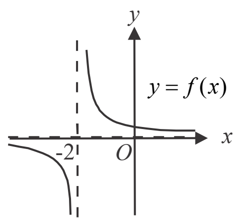
b. Do a quick sketch.

$d_f = R\backslash\{-2\}$

$r_f = R\backslash\{0\}$

So $d_{f^{-1}} = R\backslash\{0\}$

(1 mark)



c. Show $f^{-1}(f(x)) = x$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x+2}\right)$$

$$= \frac{1}{1/(x+2)} - 2$$

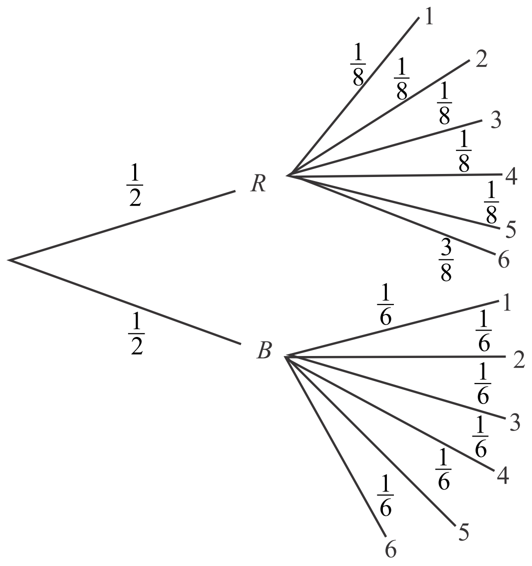
$$= x+2-2$$

$$= x$$

Have shown **(1 mark)**

Question 3 (3 marks)

a. Draw a tree diagram.



The probability of throwing a 1, 2, 3, 4 or 5 with the red die is $\frac{5}{8} \div 5 = \frac{1}{8}$.

$$\Pr(\text{six}) = \frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{1}{6}$$ **(1 mark)**

$$= \frac{3}{16} + \frac{1}{12}$$

$$= \frac{9+4}{48}$$

$$= \frac{13}{48}$$ **(1 mark)**

b.

$$\Pr(B \mid 6) = \frac{\Pr(B \cap 6)}{\Pr(6)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{6}}{\frac{13}{48}}$$

$$= \frac{1}{12} \div \frac{13}{48}$$

$$= \frac{1}{12} \times \frac{48}{13}$$

$$= \frac{4}{13}$$

(1 mark)

Question 4 (4 marks)

a. average rate of change

$$=\frac{f\left(\frac{2\pi}{3}\right)-f\left(\frac{\pi}{2}\right)}{\frac{2\pi}{3}-\frac{\pi}{2}}$$ **(1 mark)**

$$=\frac{\tan\left(\frac{\pi}{3}\right)-1-0}{\frac{4\pi-3\pi}{6}}$$ (Note $f\left(\frac{\pi}{2}\right)=0$ from the graph.)

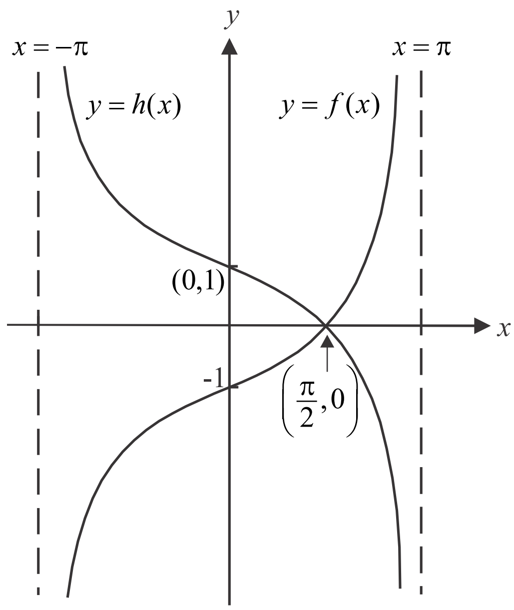
$$=\left(\sqrt{3}-1\right)\div\frac{\pi}{6}$$

$$=\left(\sqrt{3}-1\right)\times\frac{6}{\pi}$$

$$=\frac{6\left(\sqrt{3}-1\right)}{\pi}$$

(1 mark)

b. The transformation involves a reflection in the x-axis.



(1 mark) – correct axis intercepts
(1 mark) – correct shape

Question 5 (5 marks)

a. $\frac{1}{10}$

(1 mark)

b. $X \sim \text{Bi}\left(30, \frac{1}{10}\right)$

$$\Pr(X \geq 2) = 1 - (\Pr(X = 0) + \Pr(X = 1))$$

$$= 1 - \left({}^{30}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{30} + {}^{30}C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{29}\right)$$ **(1 mark)**

$$= 1 - \left(\left(\frac{9}{10}\right)^{30} + 30 \times \frac{1}{10}\left(\frac{9}{10}\right)^{29}\right)$$

$$= 1 - \left(\frac{9}{10}\left(\frac{9}{10}\right)^{29} + 3\left(\frac{9}{10}\right)^{29}\right)$$

$$= 1 - \left(\frac{9}{10}\right)^{29}\left(\frac{9}{10} + 3\right)$$

$$= 1 - \frac{39}{10}\left(\frac{9}{10}\right)^{29}$$

So $a = 1,\ b = \frac{39}{10},\ c = \frac{9}{10}$ and $n = 29$.

(1 mark)

c. approximate confidence interval $= \left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ (formula sheet)

where $\hat{p} = 0.1,\ z = \frac{49}{25}$ and $n = 100$.

Note that $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.1 \times 0.9}{100}}$

$$= \sqrt{\frac{0.09}{100}}$$

$$= \frac{0.3}{10}$$

$$= 0.03$$ **(1 mark)**

So $z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{49}{25} \times 0.03$

$$= \frac{49}{25} \times \frac{3}{100}$$

$$= \frac{196}{100} \times \frac{3}{100}$$

$$= \frac{588}{10\,000}$$

$$= 0.0588$$

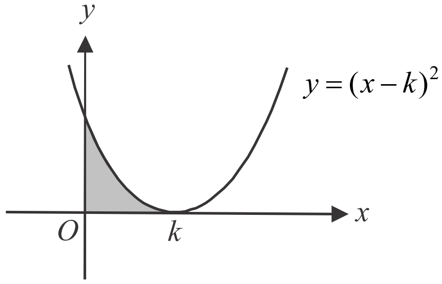
Confidence interval $= (0.1 - 0.0588,\ 0.1 + 0.0588)$

$$= (0.0412,\ 0.1588)$$

(1 mark)

Question 6 (3 marks)

Do a quick sketch.



From the diagram,

$$\int_0^k (x-k)^2 dx = \frac{8}{3}$$ **(1 mark)**

$$\left[\frac{(x-k)^3}{3}\right]_0^k = \frac{8}{3}$$

$$(0)-\left(\frac{(-k)^3}{3}\right) = \frac{8}{3}$$ **(1 mark)**

$$\frac{k^3}{3} = \frac{8}{3}$$

$$k^3 = 8$$

$$k = 2$$ **(1 mark)**

Question 7 (4 marks)

a. From the graph, the minimum value of *f* occurs at the minimum turning point. This occurs when

$$f'(x) = 0$$

$$2e^{2x} - 1 = 0$$

$$e^{2x} = \frac{1}{2}$$

$$\log_e\left(\frac{1}{2}\right) = 2x$$

$$x = \frac{1}{2}\log_e\left(\frac{1}{2}\right)$$ **(1 mark)**

$$f\left(\frac{1}{2}\log_e\left(\frac{1}{2}\right)\right) = e^{2\times\frac{1}{2}\log_e\left(\frac{1}{2}\right)} - \frac{1}{2}\log_e\left(\frac{1}{2}\right)$$

$$= e^{\log_e\left(\frac{1}{2}\right)} - \frac{1}{2}\log_e\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\log_e\left(\frac{1}{2}\right)$$

The minimum value of *f* is $\frac{1}{2} - \frac{1}{2}\log_e\left(\frac{1}{2}\right)$.

(1 mark)

b. average value

$$= \frac{1}{1-0}\int_0^1 f(x)\,dx$$ **(1 mark)**

$$= \int_0^1 (e^{2x} - x)\,dx$$

$$= \left[\frac{1}{2}e^{2x} - \frac{x^2}{2}\right]_0^1$$

$$= \left(\frac{1}{2}e^2 - \frac{1}{2}\right) - \left(\frac{1}{2}e^0 - 0\right)$$

$$= \frac{1}{2}e^2 - \frac{1}{2} - \frac{1}{2}$$

$$= \frac{1}{2}e^2 - 1$$

(1 mark)

Question 8 (7 marks)

a. The graph of f is a semi-circle.

$$\text{area of trapezium} = \frac{1}{2}(a+b)\,h \quad \text{(formula sheet)}$$
$$= \frac{1}{2}(PS + QR) \times f(x)$$
$$= \frac{1}{2}(2x + 2a) \times \sqrt{a^2 - x^2}$$
$$= (a+x)\sqrt{a^2 - x^2}$$

as required

(1 mark)

b. A is a maximum when $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = \sqrt{a^2 - x^2} + (a+x) \times \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x \qquad \text{(product rule)}$$
$$= \sqrt{a^2 - x^2} - \frac{x(a+x)}{\sqrt{a^2 - x^2}}$$
$$= \frac{a^2 - x^2 - ax - x^2}{\sqrt{a^2 - x^2}}$$
$$= \frac{-2x^2 - ax + a^2}{\sqrt{a^2 - x^2}}$$

(1 mark)

$\frac{dA}{dx} = 0$ when

$$-2x^2 - ax + a^2 = 0$$
$$2x^2 + ax - a^2 = 0$$
$$(2x - a)(x + a) = 0$$
$$x = \frac{a}{2} \quad \text{or} \quad x = -a$$

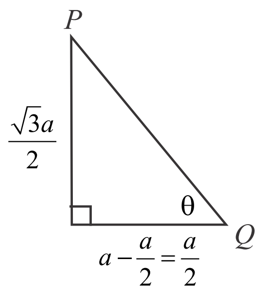
P is in the first quadrant so $x = \frac{a}{2}$. **(1 mark)**

$$f\left(\frac{a}{2}\right) = \sqrt{a^2 - \frac{a^2}{4}}$$
$$= \sqrt{\frac{3a^2}{4}}$$
$$= \frac{\sqrt{3}a}{2}$$

P is the point $\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ when A is a maximum.

(1 mark)

c. From part **b.**, A is a maximum when P is $\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$



$$\tan(\theta) = \frac{\sqrt{3}a}{2} \div \frac{a}{2}$$
$$= \sqrt{3}$$
$$\theta = \frac{\pi}{3}$$

So angle PQR is $\frac{\pi}{3}$ when A is a maximum.

(1 mark)

d. gradient of $PQ = -\sqrt{3}$ (using part **c.**)

gradient of $f = f'(x)$

$$= \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x$$
$$= \frac{-x}{\sqrt{a^2 - x^2}}$$

The tangent to the graph of f that is parallel to the line segment PQ when A is a maximum occurs when

$$-\sqrt{3} = \frac{-x}{\sqrt{a^2 - x^2}}$$

(1 mark)

$$\sqrt{3(a^2 - x^2)} = x$$
$$3a^2 - 3x^2 = x^2$$
$$3a^2 = 4x^2$$
$$x = \frac{\sqrt{3}a}{2} \quad \text{(for first quadrant)}$$
$$f\left(\frac{\sqrt{3}a}{2}\right) = \sqrt{a^2 - \frac{3a^2}{4}}$$
$$= \frac{a}{2}$$

Point of tangency is $\left(\frac{\sqrt{3}a}{2}, \frac{a}{2}\right)$.

Equation of tangent is $y - \frac{a}{2} = -\sqrt{3}\left(x - \frac{\sqrt{3}a}{2}\right)$.

$$y = -\sqrt{3}x + \frac{3a}{2} + \frac{a}{2}$$
$$y = -\sqrt{3}x + 2a$$

(1 mark)

Question 9 (8 marks)

a. $g(x)=\sin^2(x)\cos(x)$

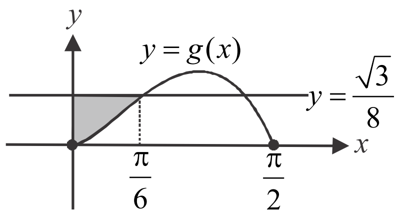
$$g\left(\frac{\pi}{6}\right)=\sin^2\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$
$$=\left(\frac{1}{2}\right)^2\times\frac{\sqrt{3}}{2}$$
$$=\frac{\sqrt{3}}{8}$$

So the point $\left(\frac{\pi}{6},\frac{\sqrt{3}}{8}\right)$ lies on the graph of g as required. **(1 mark)**

b. $\frac{d}{dx}(\sin^3(x))=3\sin^2(x)\times\cos(x)$ (chain rule)

$=3g(x)$ **(1 mark)**

c. Do a quick sketch, it doesn't have to be too detailed, the important thing to note is that for $x\in\left[0,\frac{\pi}{2}\right]$, $g(x)\ge 0$, i.e. sine and cosine are both positive in the first quadrant.



Note that g passes through the point $\left(\frac{\pi}{6},\frac{\sqrt{3}}{8}\right)$.

<u>Method 1</u>

$$\text{area}=\int_0^{\frac{\pi}{6}}\left(\frac{\sqrt{3}}{8}-g(x)\right)dx$$ **(1 mark)**

From part **b.**, $\frac{d}{dx}(\sin^3(x))=3g(x)$

$$\int 3g(x)\,dx=\int\frac{d}{dx}(\sin^3(x))\,dx$$
$$\int g(x)\,dx=\frac{1}{3}\sin^3(x)+\frac{c}{3}$$

$$\text{So area}=\int_0^{\frac{\pi}{6}}\left(\frac{\sqrt{3}}{8}-g(x)\right)dx$$

$$=\left[\frac{\sqrt{3}}{8}x-\frac{1}{3}\sin^3(x)\right]_0^{\frac{\pi}{6}}$$ **(1 mark)**

$$=\left(\frac{\sqrt{3}}{8}\times\frac{\pi}{6}-\frac{1}{3}\times\left(\frac{1}{2}\right)^3\right)-(0-0)$$

$$=\frac{\sqrt{3}\pi}{48}-\frac{1}{24}\text{ square units}$$ **(1 mark)**

Method 2

$$\text{area} = \frac{\pi}{6} \times \frac{\sqrt{3}}{8} - \int_0^{\frac{\pi}{6}} g(x)\,dx$$ **(1 mark)**

$$= \frac{\sqrt{3}\pi}{48} - \left[\frac{1}{3}\sin^3(x)\right]_0^{\frac{\pi}{6}} \text{ \_\_\_\_(*)}$$ **(1 mark)**

$$= \frac{\sqrt{3}\pi}{48} - \left(\left(\frac{1}{3} \times \frac{1}{8}\right) - 0\right)$$

$$= \frac{\sqrt{3}\pi}{48} - \frac{1}{24} \text{ square units}$$ **(1 mark)**

(*) From part **b.**,

$$\frac{d}{dx}(\sin^3(x)) = 3g(x)$$

$$\int 3g(x)\,dx = \int \frac{d}{dx}(\sin^3(x))\,dx$$

$$\int g(x)\,dx = \frac{1}{3}\sin^3(x) + \frac{c}{3}$$

d. **i.** $g(x) = \sin^2(x)\cos(x)$

Let $y = \sin^2(x)\cos(x)$

After a dilation by a factor of a from the y-axis, replace x with $\frac{x}{a}$.

$$y = \sin^2\left(\frac{x}{a}\right)\cos\left(\frac{x}{a}\right)$$

After a reflection in the y-axis, replace x with $-x$.

$$y = \sin^2\left(\frac{-x}{a}\right)\cos\left(\frac{-x}{a}\right)$$

So $h(x) = \sin^2\left(\frac{-x}{a}\right)\cos\left(\frac{-x}{a}\right)$.

(1 mark)

ii. The domain of g is $x \in \left[0, \frac{\pi}{2}\right]$.

After the dilation, it becomes $x \in \left[0, \frac{\pi a}{2}\right]$.

After the reflection, it becomes $x \in \left[-\frac{\pi a}{2}, 0\right]$.

So $d_h = \left[-\frac{\pi a}{2}, 0\right]$.

(1 mark)

e. $p = -\frac{\pi a}{2}$ from part **d. ii.**

Since $p > -\frac{\pi}{4}$, then $-\frac{\pi a}{2} > -\frac{\pi}{4}$

$$a < \frac{1}{2}$$

Also $a > 0$ (given in question).

So $0 < a < \frac{1}{2}$. **(1 mark)**

(Note this means that the dilation was a 'compression' not a 'stretch' which makes sense geometrically if $p > -\frac{\pi}{4}$.)