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MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2021

SECTION A – Multiple-choice answers

1.	B	9.	A	17.	B
2.	D	10.	D	18.	B
3.	E	11.	E	19.	E
4.	C	12.	C	20.	E
5.	B	13.	B		
6.	D	14.	C		
7.	B	15.	A		
8.	A	16.	C		

SECTION A – Multiple-choice solutions

Question 1

Define $p(x) = x^4 - 2x^3 - k$ on your CAS.
Solve $p(3) = 15$ for k (ie using the remainder theorem)

$k = 12$

The answer is B.

Question 2

For option A, maximal domain is $x \in R \setminus \{-1\}$.
For option B, maximal domain is $x + 1 > 0$ i.e. $x \in (-1, \infty)$.
For option C, maximal domain is $x + 1 > 0$ i.e. $x \in (-1, \infty)$.
For option D, maximal domain is $x + 1 \geq 0$ i.e. $x \in [-1, \infty)$.
For option E, maximal domain is $x \in (-\infty, \infty)$ i.e. $x \in R$.
The answer is D.

Question 3

<u>Method 1</u> – using CAS
Solve $2\sin\left(3x + \frac{\pi}{6}\right) - \sqrt{3} = 0$ for x.

$x = \frac{(12k+1)\pi}{18}$ or $x = \frac{(4k+1)\pi}{6}, \quad k \in Z$

$= \frac{(12k+3)\pi}{18}$

The answer is E.

Question 3 (cont'd)

Method 2 – by hand.

$$2\sin\left(3x+\frac{\pi}{6}\right)-\sqrt{3}=0$$

$$\sin\left(3x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$3x+\frac{\pi}{6}=2k\pi+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad \text{or} \quad 3x+\frac{\pi}{6}=(2k+1)\pi-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right),\quad k\in Z$$

$$=2k\pi+\frac{\pi}{3} \qquad\qquad =2k\pi+\pi-\frac{\pi}{3}$$

$$3x=2k\pi+\frac{\pi}{6} \qquad\qquad 3x=2k\pi+\frac{\pi}{2}$$

$$=\frac{12k\pi+\pi}{6} \qquad\qquad =\frac{4k\pi+\pi}{2}$$

$$x=\frac{\pi(12k+1)}{18} \qquad\qquad x=\frac{\pi(4k+1)}{6}$$

$$x=\frac{\pi(12k+3)}{18}$$

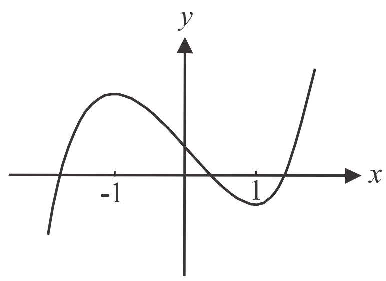
The answer is E.

Question 4

For option A, $h(x)=\frac{1}{3}f(x+2)$, we have a dilation by a factor of $\frac{1}{3}$ from the x-axis so $(1,-2)$ becomes $\left(1,-\frac{2}{3}\right)$ then a translation 2 units to the left, which gives $\left(-1,-\frac{2}{3}\right)$ which is not point Q. Reject option A.
For option B, $h(x)=2f(x-3)$, we have a dilation by a factor of 2 from the x-axis so $(1,-2)$ becomes $(1,-4)$ then a translation 3 units to the right which gives $(4,-4)$. Reject option B.
For option C, $h(x)=3f(x-2)$, we have a dilation by a factor of 3 from the x-axis so $(1,-2)$ becomes $(1,-6)$ then a translation 2 units to the right which gives $(3,-6)$ which is the image point Q that is required.
Option D gives an image point of $(1,-6)$ and option E gives an image point of $(-1,-6)$.
The answer is C.

Question 5

Do a quick sketch of $y=h(x)$.



For h^{-1} to exist, then h has to be $1:1$.
Since the local maximum and local minimum of the graph of $y=4x^3-12x+5$ occurs at $x=-1$ and at $x=1$ respectively, then h will be $1:1$ for $D=[-1,0]$.
The answer is B.

Question 6

The average rate of change of *f* between $x=0$ and $x=4$ is given by

$$\frac{f(4)-f(0)}{4-0}$$
$$=\frac{16-0}{4}$$
$$=4$$

The instantaneous rate of change of *f* at $x=a$ is given by $f'(a)$.

Solve $f'(a)=4$ for a

$$2a=4$$
$$a=2$$

The answer is D.

Question 7

The graph of $y=g(x)$ must have

- a negative gradient for $x<0$
- a zero gradient for $x=0$
- a positive gradient for $x>0$

Option A has a positive gradient for $x<0$.
Reject option A.
Option B has a negative gradient for $x<0$, a zero gradient at $x=0$ and a positive gradient for $x>0$.
Also, as $x\to-\infty$, gradient $\to 0$ and as $x\to\infty$, gradient $\to 0$.
Option B could represent $y=g(x)$.
Option C can't because it has a sharp point at $x=0$ and therefore the gradient at $x=0$ is not defined.
Option D does not have a zero gradient at $x=0$.
Option E has a positive gradient for $x<0$ and a negative gradient for $x>0$.
The answer is B.

Question 8

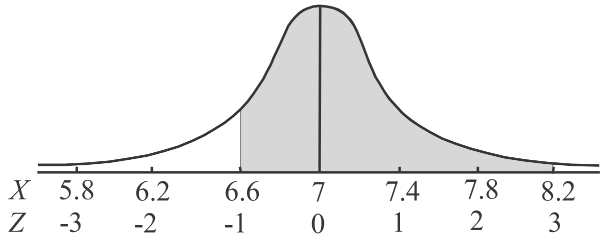
Since *A* and *B* are independent,

$$\Pr(A)\times\Pr(B)=\Pr(A\cap B)$$
$$p\times\Pr(B)=p-q$$
$$\Pr(B)=\frac{p-q}{p}$$

The answer is A.

Question 9

Draw a diagram.
$\Pr(6.6<X<8.2)=\Pr(-1<Z<3)$
Since this answer is not offered, look for an equivalent area under the bell curve, remembering that it is symmetrical about the line $Z=0$.
So $\Pr(-1<Z<3)=\Pr(-3<Z<1)$.
This is offered in option A.
The answer is A.



Question 10

We are sampling from a small population of 10 balls.
If $\hat{p} = 0.25$, then we are interested in one of the four balls in the sample being white.

$$\text{So } \Pr(\hat{P} = 0.25) = \frac{{}^{2}C_1 \times {}^{8}C_3}{{}^{10}C_4}$$

$$= \frac{8}{15}$$

The answer is D.

Question 11

For $f(x) = x + 1$,

$$f(x^2) = x^2 + 1 \text{ and } 2f(x) = 2x + 2$$

Reject option A.

For $f(x) = \frac{1}{\sqrt{x}}$,

$$f(x^2) = \frac{1}{x} \text{ and } 2f(x) = \frac{2}{\sqrt{x}}$$

Reject option B.

For $f(x) = e^{\frac{x}{2}}$,

$$f(x^2) = e^{\frac{x^2}{2}} \text{ and } 2f(x) = 2e^{\frac{x}{2}}$$

Reject option C.

For $f(x) = \frac{1}{x-2}$,

$$f(x^2) = \frac{1}{x^2 - 2} \text{ and } 2f(x) = \frac{2}{x-2}$$

Reject option D.

For $f(x) = \log_e(x)$,

$$f(x^2) = \log_e(x^2) \text{ and } 2f(x) = 2\log_e(x)$$

$$= \log_e(x^2)$$

So $f(x^2) = 2f(x)$

The answer is E.

Question 12

$$\int_2^7 g(x)dx = \int_2^4 g(x)dx + \int_4^7 g(x)dx$$

$$6 = -1 + \int_4^7 g(x)dx$$

$$7 = \int_4^7 g(x)dx$$

So $\int_4^7 (g(x)-1)dx = \int_4^7 g(x)dx - \int_4^7 1dx$

$$= 7 - [x]_4^7$$
$$= 7 - (7-4)$$
$$= 4$$

The answer is C.

Question 13

Note that $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (f(x)-g(x))dx = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (g(x)-f(x))dx$ and also $\int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (f(x)-g(x))dx = \int_{\frac{3\pi}{2}}^{\frac{13\pi}{6}} (g(x)-f(x))dx$

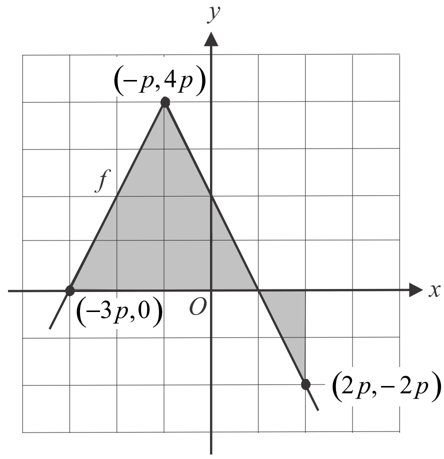
So total area required

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (f(x)-g(x))dx + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (g(x)-f(x))dx + \int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (f(x)-g(x))dx + \int_{\frac{3\pi}{2}}^{\frac{13\pi}{6}} (g(x)-f(x))dx$$

$$= 2\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (g(x)-f(x))dx + 2\int_{\frac{3\pi}{2}}^{\frac{13\pi}{6}} (g(x)-f(x))dx$$

The answer is B.

Question 14



Method 1 – using area of triangles

$$\text{signed area} = \frac{1}{2}\times 16p^2 - \frac{1}{2}\times 2p^2$$
$$= 7p^2$$
$$\text{average value} = \frac{7p^2}{5p}$$
$$= \frac{7p}{5}$$

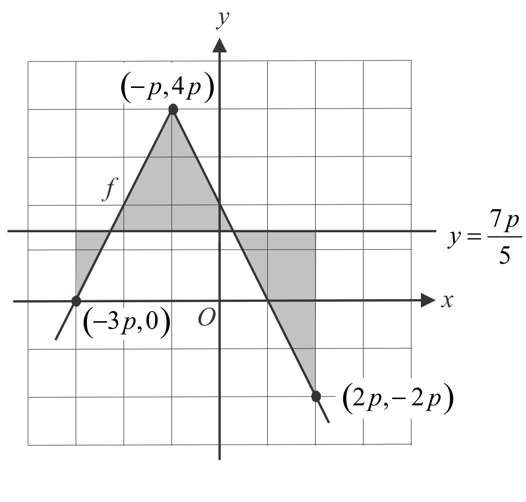
The answer is C.

Method 2 – using formula

$$\text{average value} = \frac{1}{2p - -3p}\int_{-3p}^{2p} f(x)dx$$
$$= \frac{1}{5p}\left(\int_{-3p}^{-p}(2x+6p)dx + \int_{-p}^{2p}(-2x+2p)dx\right)$$
$$= \frac{7p}{5}$$

Note that the left hand branch of the graph has the rule $y = 2x + 6p$ and the right hand branch has the rule $y = -2x + 2p$. (Use the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ and the straight line formula $y - y_1 = m(x - x_1)$ to find these.)
The answer is C.

Just out of interest, what the answer means graphically is that the area enclosed above the line $y = \frac{7p}{5}$ between function *f* and between $x = -3p$ and $x = 2p$ is equal to the area enclosed below the line $y = \frac{7p}{5}$ between the function *f* and between $x = -3p$ and $x = 2p$.
In other words in the diagram below, the big triangle above the line $y = \frac{7p}{5}$ is equal in area to the sum of the areas of the two smaller triangles below the line $= \frac{7p}{5}$.



Question 15

$$\mathrm{E}(\hat{P}) = p \qquad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

$$= 0.4 \qquad = \sqrt{\frac{0.4 \times 0.6}{24}}$$

$$= 0.1$$

So mean $= 0.4$ and standard deviation $= 0.1$.
The answer is A.

Question 16

Since m is the median, $\int_0^m \frac{3}{28}\left(1+\sqrt{x}\right)dx = 0.5$

$$\int_0^m \frac{3}{28}\left(1+\sqrt{x}\right)dx = \frac{m(2\sqrt{m}+3)}{28} \qquad \text{using CAS}$$

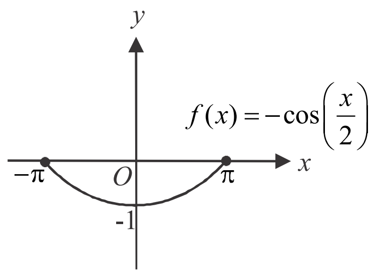
So $\frac{m(2\sqrt{m}+3)}{28} = 0.5$

$3m + 2\sqrt{m^3} - 14 = 0$

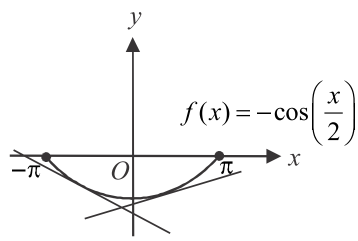
The answer is C.

Question 17

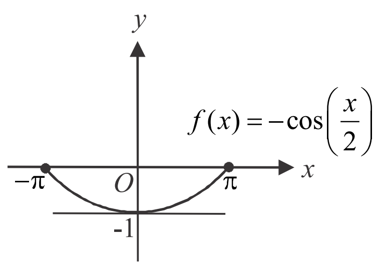
The graph of f is shown.
The point $(0,-1)$ lies on this graph.



A couple of tangents are drawn to the graph as shown.
The y-intercept of these two tangents is less than -1.

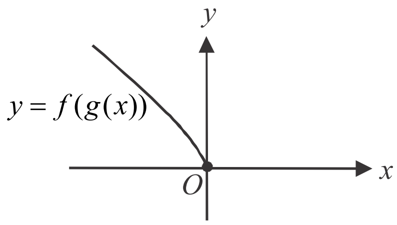


The horizontal tangent shown, has a y-intercept of -1.
This is the maximum possible value of the y-intercept i.e. of c.
The answer is B.



Question 18

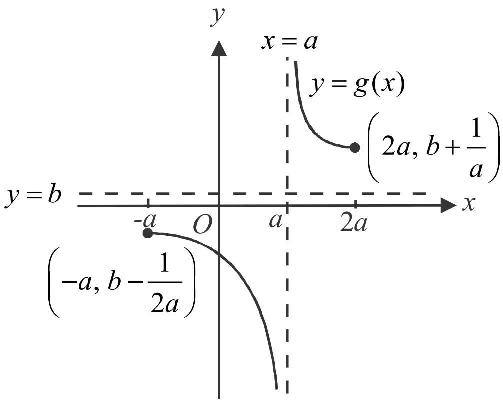
$f(g(x)) = \sqrt{x^2 - 2x}$
Note that $f(g(x))$ is defined because
$r_g = [0, \infty)$ and $d_f = [0, \infty)$ i.e. $r_g \subseteq d_f$
So $d_{f(g(x))} = d_g$
$= (-\infty, 0]$
Do a quick sketch of $f(g(x))$ to find the range.



$r_{f(g(x))} = [0, \infty)$
Option B offers this combination.
The answer is B.

Question 19

A possible sketch is shown.



$g(-a) = \frac{1}{-2a} + b$

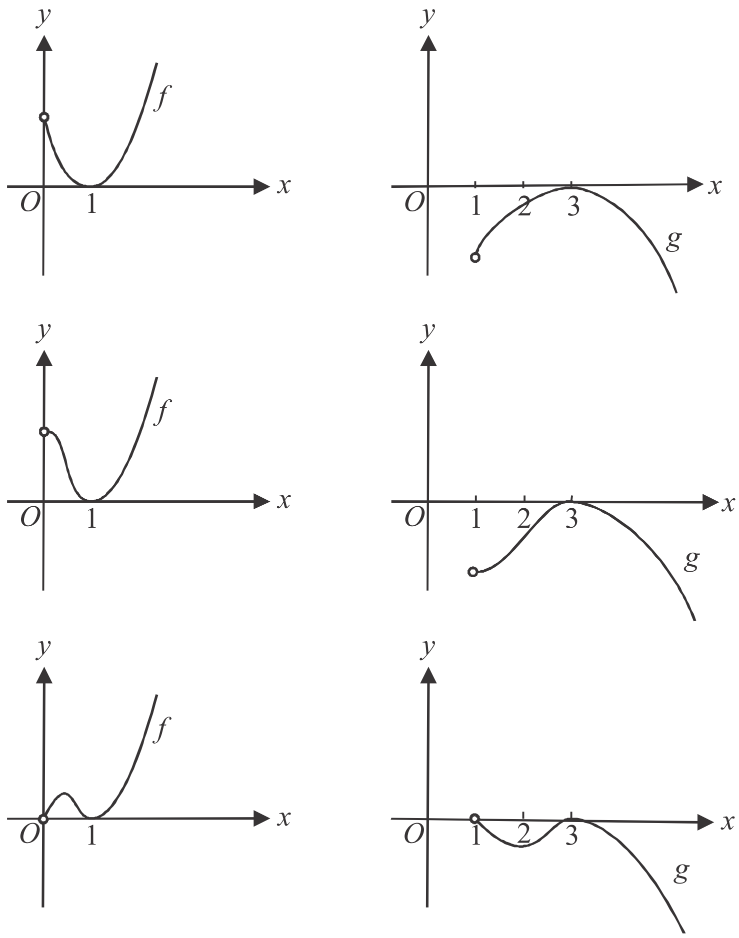
So the left endpoint of the left hand branch of the graph of *g* occurs at $\left(-a, b - \frac{1}{2a}\right)$.

$g(2a) = \frac{1}{a} + b$

So the right endpoint of the right hand branch of the graph of *g* occurs at $\left(2a, b + \frac{1}{a}\right)$.

$r_g = \left(-\infty, b - \frac{1}{2a}\right] \cup \left[b + \frac{1}{a}, \infty\right)$
The answer is E.

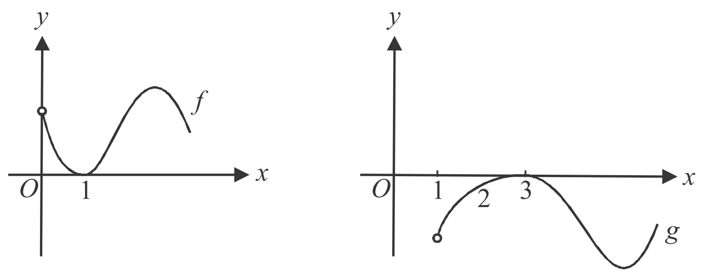
Question 20



The first three possible graphs of *f* shown above (on the left hand side) are all when *f* is a positive cubic i.e. the minimum turning point lies to the right of the maximum turning point. The transformed graphs for each of these graphs of *f* are shown to their right i.e. the graph of *g*.
The transformation from *f* to *g* involves

- a dilation by a factor of 2 from the *y*-axis
- a reflection in the *x*-axis
- a translation 1 unit to the right.

One of these graphs of *g* has a minimum turning point which occurs at a point where $1 < x < 3$.
These values are not offered in the answers.
The other possible graph of *f* is shown below.



This graph shows the graph of a negative cubic i.e. the minimum turning point lies to the left of the maximum turning point.
The transformed graph, *g*, has a minimum turning point occurring at a point where $x > 3$.
The answer is E.

SECTION B

Question 1 (10 marks)

a. Find the x-coordinate of the maximum turning point.
Solve $f'(x)=0$ for x using CAS.

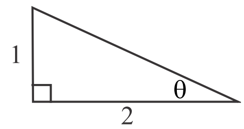
$$x=\frac{3}{e}$$ **(1 mark)**

The graph of f is strictly decreasing for $x\in\left[\frac{3}{e},\infty\right)$. **(1 mark)**

b. $f'\left(\frac{3}{\sqrt{e}}\right)=-\frac{1}{2}$ **(1 mark)**

c.

$$\tan(\theta)=\frac{1}{2}$$



$$\theta=\tan^{-1}\left(\frac{1}{2}\right)$$
$$=0.4636...$$
$$=26.5650...^\circ$$

So the tangent makes an angle of $180^\circ-26.5650...^\circ=153.4349...^\circ$ or 153° (to the nearest degree) with the positive direction of the x-axis. **(1 mark)**

d. The tangent at $x=\frac{3}{\sqrt{e}}$ has a slope of $-\frac{1}{2}$.
So the tangent at $x=a$ has a slope of 2 (ie the negative reciprocal).
Solve $f'(x)=2$ for x. **(1 mark)**

$$x=\frac{3}{e^3}$$

So $a=\frac{3}{e^3}$. **(1 mark)**

e. P is the point $(p, f(p))$ i.e. $\left(p, -p\log_e\left(\frac{p}{3}\right)\right)$.

Equation of tangent at P is $y=p-(\log_e(p)-\log_e(3)+1)x$ using CAS. **(1 mark)**
The y-intercept of this tangent is p. **(1 mark)**

f. From part **e.**, the y intercept of the tangent at $x=p$ is p.

So the y-intercept of the tangent at $x=\frac{3}{\sqrt{e}}$ is $\frac{3}{\sqrt{e}}$.

The tangent perpendicular to this tangent occurs at $x=a$ i.e. at $x=\frac{3}{e^3}$ from part **d.**

So the y-intercept of this perpendicular tangent is $\frac{3}{e^3}$. **(1 mark)**

The distance between the $y-$intercepts is $\frac{3}{\sqrt{e}}-\frac{3}{e^3}=\frac{3e^{\frac{5}{2}}-3}{e^3}$

$$=\frac{3(e^{\frac{5}{2}}-1)}{e^3}$$

(1 mark)

Question 2 (11 marks)

a. The right endpoint of both the function n and the function s occurs at $P(1.5, 0.5)$.
So $a = 1.5$ **(1 mark)**

b. Define n and s on your CAS.
$n(0) = 1.1$ and $s(0) = 0$
So the length of the dam wall is 1.1 km. **(1 mark)**

c. $\text{area} = \int_0^{1.5} (n(x) - s(x))dx$

$= 1.3591...$

$= 1.36 \text{ km}^2$ (to 2 decimal places)

(1 mark)

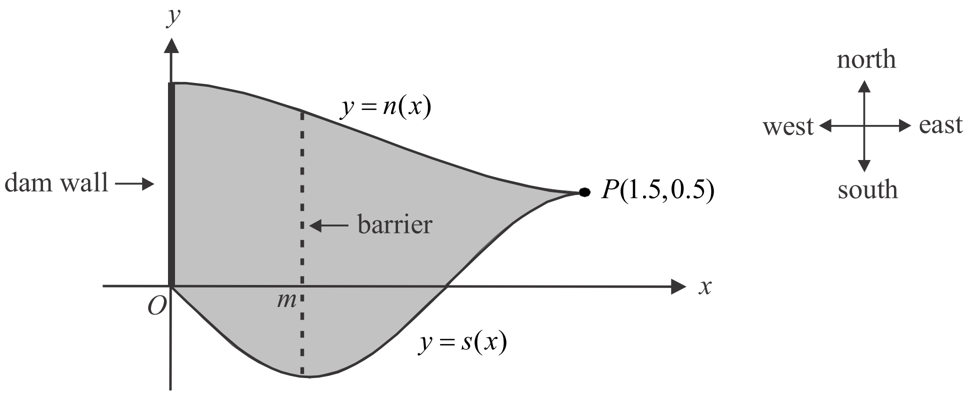
d. $\text{distance } OP = \sqrt{(1.5-0)^2 + (0.5-0)^2}$

$= 1.5811...$

$= 1.58 \text{ km}$ (to 2 decimal places) **(1 mark)**

e. The graph of $y = s(x)$ is symmetrical about the line $x = 0.5$ between $x = 0$ and $x = 1$.
So a line running in an east-west direction that passes through $(0.2, s(0.2))$ would also pass through the point $(0.8, s(0.2))$. So the length of barrier required is 0.6 km.
(1 mark)

f. We want the shaded area east of the barrier to equal the shaded area to the west.



Solve $\int_m^{1.5} (n(x) - s(x))dx = \int_0^m (n(x) - s(x))dx$ for m. **(1 mark)**

$m = 0.4975...$

(1 mark)

length of barrier required $= n(0.4975...) - s(0.4975...)$

$= 1.4513...$

$= 1.45 \text{ km}$ (to 2 decimal places) **(1 mark)**

g. Let $D = n(x) - s(x)$

Solve $\frac{dD}{dx} = 0$, for x **(1 mark)**

$x = 0.4033...$ **(1 mark)**

Make sure that you have limited the domain of D to $x \in [0, 1.5]$ if you hadn't initially put the domains of n and s into your CAS.

maximum length $= n(0.4033...) - s(0.4033...)$

$= 1.4762...$

$= 1.48 \text{ km}$ (to 2 decimal places) **(1 mark)**

Question 3 (10 marks)

a. The graph of $y=\sqrt{x+1}-1$ needs to be translated 1 unit to the right and 1 unit up to become the graph of $y=\sqrt{x}$.
So $c=1$ and $d=1$.

(1 mark)

b. The endpoint of the graph of f is $(-1,-1)$.
The endpoint of the image graph is $(-2,-1)$.
The graph of f has been dilated by a factor of 2 from the y-axis.
So $a=2$ and $b=1$.

(1 mark)

c. $h(x)=\sqrt{x+n}-n$
Since $\sqrt{x+n}\geq 0$, then $\sqrt{x+n}-n>0$ for $n<0$.
So $h(x)>0$ for $n<0$.

(1 mark)

d. $h(x)=\sqrt{x+n}-n$
Let $y=\sqrt{x+n}-n$
Swap x and y for inverse.
$x=\sqrt{y+n}-n$
Solve for y by hand or CAS.
$(x+n)^2=y+n$
$y=(x+n)^2-n$
So $h^{-1}(x)=(x+n)^2-n$ **(1 mark)**
$d_h=[-n,\infty)$
$r_h=[-n,\infty)$
So $d_{h^{-1}}=r_h$
$=[-n,\infty)$

(1 mark)

e. Method 1
Solve $h(x)=h^{-1}(x)$ for x
$x=-n$ or $x=1-n$

Method 2
Solve $h(x)=x$ for x
$x=-n$ or $x=1-n$

Since h and h^{-1} intersect on the line $y=x$, the points of intersection are
$(-n,-n)$ and $(1-n,1-n)$
(1 mark) **(1 mark)**

f. Area enclosed by graphs of h and h^{-1}

$$=\int_{-n}^{1-n}(h(x)-h^{-1}(x))\,dx \quad \text{(note that } h(x)>h^{-1}(x) \text{ for } x\in(-n,1-n))$$

$$=\frac{1}{3}$$

So area enclosed is a constant i.e. $\frac{1}{3}$. **(1 mark)**

g. Solve $\frac{1}{q-0}\int_0^q x^2dx=\frac{1}{1-0}\int_0^1\sqrt{x}\,dx$ for q. **(1 mark)**

$q=\pm\sqrt{2}$, but $q>0$
So $q=\sqrt{2}$. **(1 mark)**

Question 4 (17 marks)

a. **i.** $0.2+k+0.1+0.4=1$ so $k=0.3$

(1 mark)

ii. $\mathrm{E}(X)=10\times 0.2+20\times 0.3+40\times 0.1+100\times 0.4$
$=\$52$

(1 mark)

iii.
$$\begin{aligned}\mathrm{sd}(X)&=\sqrt{\mathrm{Var}(X)}\\&=\sqrt{\mathrm{E}(X^2)-[\mathrm{E}(X)]^2}\end{aligned}$$
$$\begin{aligned}\text{Now } \mathrm{E}(X^2)&=10^2\times 0.2+20^2\times 0.3+40^2\times 0.1+100^2\times 0.4\\&=4300\end{aligned}$$
$$\begin{aligned}\text{So } \mathrm{sd}(X)&=\sqrt{4300-52^2}\\&=39.9499...\\&=40 \text{ (to the nearest dollar)}\end{aligned}$$

(1 mark)

iv. Let the random variable R represent the distribution of the amount of the next four fishing licences purchased.
$R\sim \mathrm{Bi}(4,0.5)$ ie $p=0.1+0.4$
$=0.5$

Method 1 using CAS

$\Pr(R=3)=0.25$ (using CAS binomPdf(4,0.5,3))

Method 2 by hand

$$\begin{aligned}\Pr(R=3)&={}^4C_3(0.5)^3(0.5)^1\\&=0.25\end{aligned}$$

(1 mark)

b. **i.** Let the random variable Y represent the distribution of the weight, in kg, of Golden Perch fish that have been inspected.
$Y\sim \mathrm{N}(5.2,0.4^2)$

$$\begin{aligned}\Pr(Y>5.5)&=0.22662... \quad \text{(using CAS normCdf}(5.5,\infty,5.2,0.4))\\&=0.2266 \text{ (to 4 decimal places)}\end{aligned}$$

(1 mark)

ii. $(0.22662...)^5=0.0005978...$
$=0.0006$ (to 4 decimal places)

(1 mark)

iii. $\Pr(Y<m)=0.2$ (using CAS invNorm(0.2,5.2,0.4))
$m=4.863351...$
Maximum weight is 4.8634 kg (to 4 decimal places).

(1 mark)

c. **i.** Define $f(x)$ on your CAS.

$$\text{E}(X)=\int_{26}^{60} x\times f(x)\,dx$$ **(1 mark)**

$=40.0424...$

$=40.04\text{ cm}$ (to two decimal places)

(1 mark)

ii. $$\int_{30}^{60} f(x)\,dx=0.90141...$$

$=0.90$ (to two decimal places)

(1 mark)

iii. $\Pr(X>50\mid X>30)$

$$=\frac{\Pr(X>50\cap X>30)}{\Pr(X>30)}$$

$$=\frac{\Pr(X>50)}{0.90141...}$$ (from part **c.ii.**) **(1 mark)**

$=0.14323...$

$=0.1432$ (to four decimal places)

(1 mark)

d. **i.** Let W represent the distribution of the number of anglers questioned who didn't have a licence.

$W\sim \text{Bi}(25, 0.08)$

$\Pr(W\geq 2)=0.60527...$ (using CAS binomCdf(25, 0.08, 2, 25)

$=0.6053$ (to four decimal places)

(1 mark)

ii. $$\Pr\left(\hat{P}<0.1\right)=\Pr\left(\frac{W}{n}<0.1\right)$$ **(1 mark)**

$=\Pr(W<2.5)$

$=\Pr(W\leq 2)$ (since W is an integer)

$=0.67683...$ (using CAS binomCdf(25, 0.08, 0, 2)

$=0.6768$ (to four decimal places)

(1 mark)

e. **i.** approximate confidence interval $$=\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

(from the formula sheet)

Now $$\frac{0.1092+0.1504}{2}=0.1298.$$

Sample proportion is 0.1298.

(1 mark)

ii. The 95% confidence interval for this other state, (0.1092, 0.1504) does not contain the proportion of the original state which was 0.08.

(1 mark)

Question 5 (12 marks)

a. Define f on your CAS.
Solve $f(x)=0$ for x.
$x=a$ or $x=0$
x-intercepts are $(a,0)$ and (0, 0)

(1 mark)

b. Solve $f'(x)=0$ for x.
$x=\frac{2a}{3}$ or $x=0$ **(1 mark)**

Since the graph of f is a negative cubic, the local maximum occurs to the right of the local minimum, and since a is a positive constant, then $\frac{2a}{3}>0$, and so the local maximum occurs at $x=\frac{2a}{3}$.

$$f\left(\frac{2a}{3}\right)=\frac{4a^3}{27}$$

Local maximum is $\left(\frac{2a}{3},\frac{4a^3}{27}\right)$.

(1 mark)

c. **i.** The graphs intersect when

$$-x^3+ax^2=ax$$
$$-x^3+ax^2-ax=0$$
$$-x(x^2-ax+a)=0$$

There is a solution when $x=0$.
Consider the quadratic equation.
$x^2-ax+a=0$

$$\Delta=(-a)^2-4\times1\times a \text{ (discriminant)}$$
$$=a^2-4a$$ **(1 mark)**

If there is just one point of intersection i.e. at $x=0$, then we want the quadratic equation to have no solutions i.e. $\Delta<0$.
Solve $a^2-4a<0$ for a

$$0<a<4$$

So $0<a<4$ for one point of intersection.

(1 mark)

ii. <u>Method 1</u>
From part **i.**, there is one point of intersection at $x=0$.
For one more point of intersection, which would give a total of exactly two points of intersection, we require $\Delta=0$.

$$a^2-4a=0$$
$$a(a-4)=0$$
$$a=0 \text{ or } a=4$$ **(1 mark)**

but a is a positive constant so reject $a=0$.
For exactly two points of intersection $a=4$.

(1 mark)

Method 2

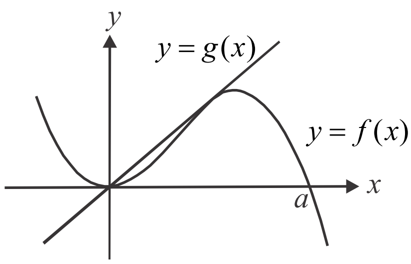
If $a = 4$, $-x(x^2 - 4x + 4) = 0$, so one solution is $x = 0$. **(1 mark)**

For the quadratic factor, $\Delta = (-4)^2 - 4 \times 1 \times 4$ (the discriminant)

$= 0$

so there is one more solution which gives a total of two solutions. **(1 mark)**

d. Both graphs pass through the origin. The graph of *g* will be a tangent to the graph of *f* when it touches the graph of *f* at one other point.
In total, the graphs will intersect exactly twice.
So from part **c. ii**, $a = 4$.



$f(x) = -x^3 + 4x^2$ (when $a = 4$)

$g(x) = 4x$ (when $a = 4$)

solve these two equations simultaneously for *x*,
$x = 0$ or $x = 2$

$f(2) = 8$ (when $a = 4$)

Point of tangency is $(2, 8)$.

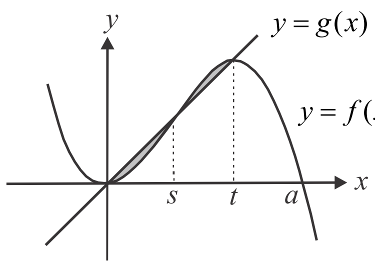
(1 mark)

e. **i.** Solve $f(x) = g(x)$ for *x* using CAS.
Given that $a > 4$ and $r < s < t$, then

$r = 0$, $s = \dfrac{a - \sqrt{a(a-4)}}{2}$ and $t = \dfrac{a + \sqrt{a(a-4)}}{2}$

(1 mark) – one correct value
(1 mark) – all 3 correct values

ii. Draw a graph.



Solve $\int_0^s (g(x) - f(x))\,dx = \int_s^t (f(x) - g(x))\,dx$ for a. **(1 mark)**

$$a = 0 \text{ or } a = \frac{9}{2}$$

$$\text{But } a > 4 \text{ so } a = \frac{9}{2}.$$

(1 mark)