2021 Mathematical Methods Trial Exam 1 Solutions

Q1a $(x-1)^{\frac{2}{3}}=(x-2)^{\frac{2}{3}}$, $(x-1)^2=(x-2)^2$, $(x-1)^2-(x-2)^2=0$,

$(x-1+x-2)=0$, $2x=3$, $x=\frac{3}{2}$

Q1b $e^{4x-4}-2e^{2x-2}-3=0$, $\left(e^{2x-2}\right)^2-2\left(e^{2x-2}\right)-3=0$,

$\left(e^{2x-2}-3\right)\left(e^{2x-2}+1\right)=0$, $e^{2x-2}-3=0$, $2x-2=\log_e 3$,

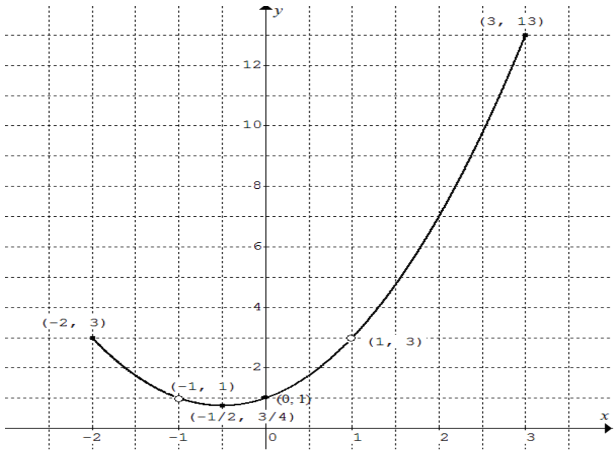
$x=\frac{1}{2}(\log_e 3+2)$

Q2a $f(x)=\frac{(x-1)\left(x^3+2x^2+2x+1\right)}{x^2-1}=\frac{(x-1)\left(x^3+2x^2+2x+1\right)}{(x-1)(x+1)}$

$=\frac{x^3+2x^2+2x+1}{x+1}=x^2+x+1$

Q2b $f'(x)=2x+1$

Q2c



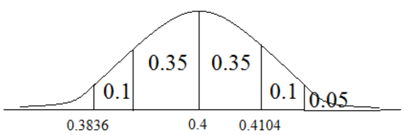
Q3 $\Pr(X=1)=\binom{n}{1}p(1-p)^{n-1}=(1-p)^{n-2}$,

$np(1-p)^{n-1}-(1-p)^{n-2}=0$, $(1-p)^{n-2}(np(1-p)-1)=0$

$\therefore\ np^2-np+1=0$, $\therefore\ p=\frac{n\pm\sqrt{n^2-4n}}{2n}=\frac{1\pm\sqrt{1-\frac{4}{n}}}{2}$,

$\therefore\ 0\le 1-\frac{4}{n}<1$, $\therefore\ n\ge 4$

Q4a

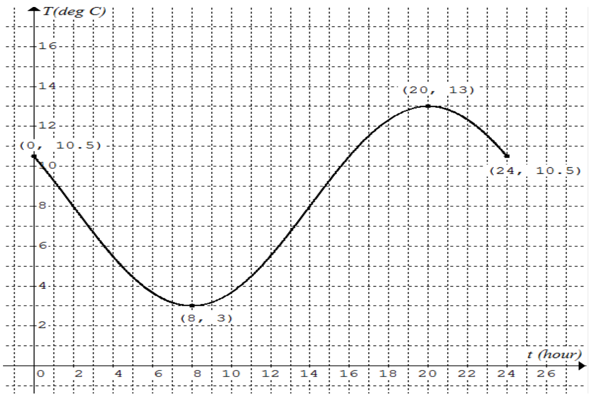


$\Pr\left(0.3836<\hat{P}<0.4104 \mid \hat{P}>0.3836\right)=\frac{0.8}{0.95}=\frac{16}{19}$

Q4b $p+z\sqrt{\frac{p(1-p)}{n}}=0.4+1.04\sqrt{\frac{0.4\times 0.6}{n}}\approx 0.4104$,

$\therefore\ n\approx 2400$

Q5a



Q5b Average temperature in $[0, 24]=8°$ C

Q5c Average rate of change $=\frac{13-3}{20-8}=\frac{5}{6}$ °C per hour

Q5d $T<\alpha$ for 4 hours, i.e. $3<T<\alpha$ for 2 hours, $\therefore$ α is the temperature at $t=8+2=10$

$\therefore\ T(10)=8-5\sin\left(\pi\left(\frac{10}{12}-\frac{1}{6}\right)\right)=8-5\sin\frac{2\pi}{3}=8-\frac{5\sqrt{3}}{2}$

Q5e

$T(t)=8-5\sin\left(\pi\left(\frac{t-1}{12}-\frac{1}{6}\right)\right)=8-5\sin\left(\pi\left(\frac{t}{12}-\frac{1}{4}\right)\right)$, $\therefore\ b=-\frac{1}{4}$

Q6a For $0\le h\le 2$, $V=Ah=3\sqrt{3}\,h$; and for $2<h\le 4$,

$V=3\sqrt{3}\times 2+2\sqrt{3}(h-2)=2\sqrt{3}\,h+2\sqrt{3}=2\sqrt{3}(h+1)$

$\therefore\ V(h)=\begin{cases}3\sqrt{3}\,h, & 0\le h\le 2\\ 2\sqrt{3}(h+1), & 2<h\le 4\end{cases}$

Q6b For $h>2$, $V=2\sqrt{3}(h+1)$, $\frac{dV}{dt}=2\sqrt{3}\frac{d}{dt}(h+1)=2\sqrt{3}\frac{dh}{dt}$,

$\therefore\ \frac{dh}{dt}=\frac{1}{2\sqrt{3}}\frac{dV}{dt}=\frac{1}{2\sqrt{3}}\times\frac{1}{500}=\frac{\sqrt{3}}{3000}$

Q7a $f(e)=g(e)$, $\therefore$ $1=\sqrt{ae+b}$; $f'(x)=\frac{1}{x}$, $g'(x)=\frac{a}{2\sqrt{ax+b}}$,

$f'(e)=g'(e)$, $\therefore$ $\frac{1}{e}=\frac{a}{2\sqrt{ae+b}}$ $\therefore$ $a=\frac{2}{e}$, $1=\sqrt{2+b}$, $b=-1$

Q7b $g'(x)=\frac{1}{e\sqrt{\frac{2}{e}x-1}}$, $\frac{2}{e}x-1>0$, $x>\frac{e}{2}$

Since $(x-e)^2>0$ for $x\ne e$, $\therefore$ $x^2-2ex+e^2>0$, $x^2>2ex-e^2$,

$\therefore$ $x>\sqrt{2ex-e^2}$ for $x>\frac{e}{2}$

$g'(x)-f'(x)=\frac{1}{e\sqrt{\frac{2}{e}x-1}}-\frac{1}{x}=\frac{1}{\sqrt{2ex-e^2}}-\frac{1}{x}>0$ for $x>\frac{e}{2}$ and $x\ne e$

$\therefore$ $g'(x)>f'(x)$ for $x\in\left(\frac{e}{2},e\right)\cup(e,\infty)$

Q8a $f(-5)=-f(5)=-f(5-7)=-f(-2)=-2$ or
$f(-5)=f(-5+7)=f(2)=-f(-2)=-2$

Q8b $f'(-5)=f'(2)=f'(-2)=\frac{1}{3}$

Q8c
$\int_{-5}^{9} f(x)dx=\int_{-7}^{7} f(x)dx=\int_{-7}^{0} f(x)dx+\int_{0}^{7} f(x)dx=\int_{-7}^{0} f(x)dx-\int_{-7}^{0} f(x)dx=0$

Q9a $R\cap R^{+}=R^{+}$

Q9b $b=f(a)$, $(a,b)\in f$, .: $(b,a)\in f^{-1}$ i.e. $(b,a)\in g$

$f'(x)=e^{x}+\frac{1}{x}$ for $x>0$, .: $f'(a)=e^{a}+\frac{1}{a}=\frac{ae^{a}+1}{a}$

.: $g'(b)=\frac{1}{f'(a)}=\frac{a}{ae^{a}+1}$

Please inform mathline@itute.com re conceptual and/or mathematical errors

