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YEAR 12 *Trial Exam Paper*

2022

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes

Writing time: 2 hours

STUDENT NAME:

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 27 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions

- Write your **name** in the box provided above on this page, and on your answer sheet for multiple-choice questions.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

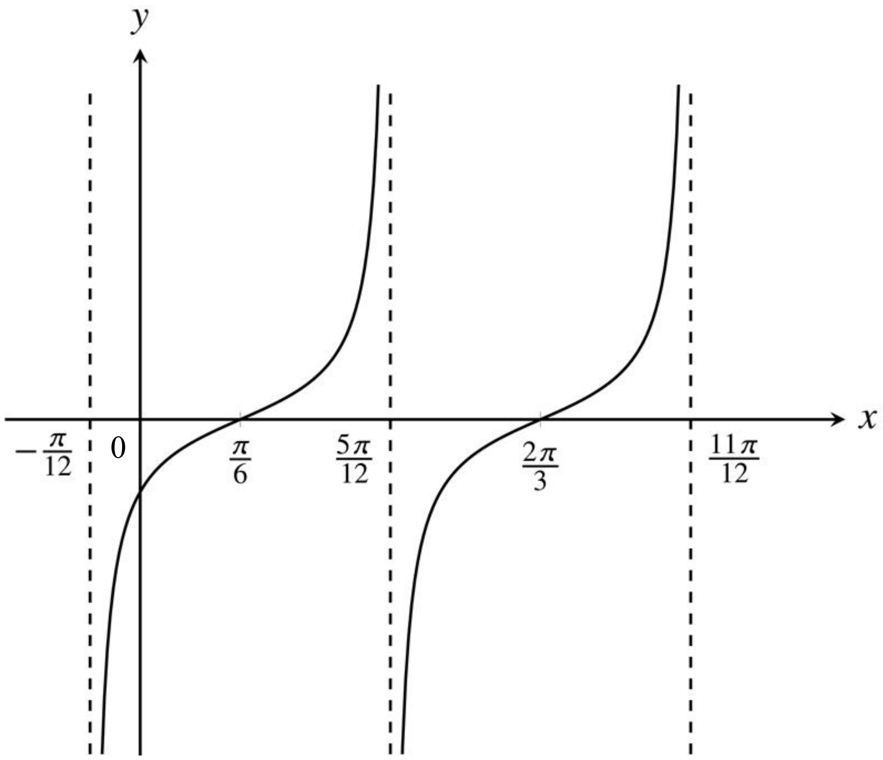
Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The graph below shows two periods of a circular function.



Which one of the following describes the circular function?

A. $y = \tan\left(\frac{x}{2} + \frac{\pi}{6}\right)$

B. $y = \tan\left(\frac{x}{2} - \frac{\pi}{6}\right)$

C. $y = \tan\left(2x + \frac{\pi}{6}\right)$

D. $y = \tan\left(2x - \frac{\pi}{6}\right)$

E. $y = \tan\left(2x - \frac{\pi}{3}\right)$

Question 2

If $\left(f(x)\right)^2 = f(2x)$, then the rule for f could be

A. $f(x) = \sqrt{\dfrac{x}{2}}$

B. $f(x) = 2\sqrt{x}$

C. $f(x) = \log_e\left(\dfrac{x}{2}\right)$

D. $f(x) = 2^x$

E. $f(x) = 2x^2$

Question 3

The expression $2 + 3\log_2(x) - \dfrac{1}{2}\log_2(y)$ where $x > 0$ and $y > 0$ is equivalent to

A. $\log_2\left(\dfrac{2x^3}{\sqrt{y}}\right)$

B. $\log_2\left(\dfrac{4x^3}{\sqrt{y}}\right)$

C. $\log_2\left(\dfrac{2+3x}{\sqrt{y}}\right)$

D. $\log_2\left(\dfrac{1+3x}{\sqrt{y}}\right)$

E. $\log_2\left(\dfrac{6x^3}{y}\right)$

Question 4

The function $f(x) = \left(x^2 - 1\right)e^x$ will have an inverse function for

A. $x \in R$

B. $x \in (-\infty, 1)$

C. $x \in \left[-1-\sqrt{2}, \infty\right)$

D. $x \in \left(-\infty, -1+\sqrt{2}\right)$

E. $x \in \left[-1-\sqrt{2}, -1+\sqrt{2}\right]$

Question 5

A box contains five red and four yellow balls. Three balls are drawn one at a time from the box, without replacement.

The probability that they are all red given that at least two are red is equal to

A. $\frac{10}{21}$

B. $\frac{25}{42}$

C. $\frac{1}{5}$

D. $\frac{1}{3}$

E. $\frac{1}{21}$

Question 6

If $\int_{-1}^{1} f(2x+1)dx = 3$ and $a\int_{-1}^{3} f(x)dx = 8$, then the value of a is equal to

A. $\frac{2}{3}$

B. $\frac{3}{4}$

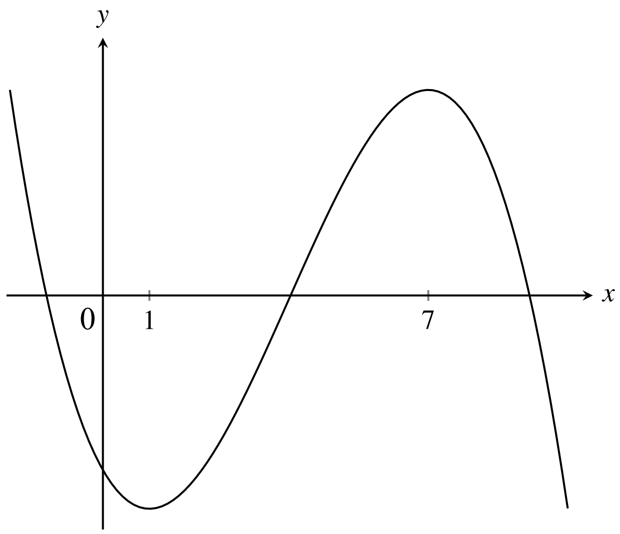
C. $\frac{4}{3}$

D. $\frac{3}{2}$

E. $\frac{1}{3}$

Question 7

The graph of $y = f(x)$ is shown below.



Given that $f(x)$ is a cubic polynomial with turning points at $x = 1$ and $x = 7$, the graph of the derivative function $f'(x)$ is strictly increasing for

A. $x \in (1, 7)$

B. $x \in [1, 7]$

C. $x \in (-\infty, 4)$

D. $x \in (-\infty, 4]$

E. $x \in [4, \infty)$

Question 8

For the continuous random variable X with probability density function

$$f(x) = \begin{cases} \log_e(x) & 1 \le x \le e \\ 0 & \text{otherwise} \end{cases}$$

the standard deviation σ of X is closest to

A. 0.4196

B. 0.1760

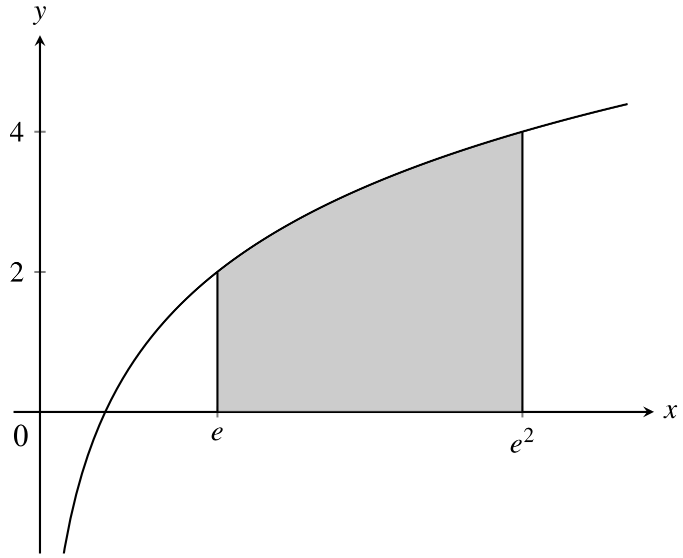
C. 2.4773

D. 1.5740

E. 1.0475

Question 9

The area bounded by the graph of $y = 2\log_e(x)$, the x-axis and the lines $x = e$ and $x = e^2$ is shown below.



Which one of the following gives the area of the shaded region?

A. $4e^2 - 4e - \int_2^4 e^{\frac{x}{2}}\,dx$

B. $2e^2 - 2e + \int_2^4 \left(e^2 - e^{\frac{x}{2}}\right)dx$

C. $4e^2 - 4e - \int_2^4 \left(e^2 - e^{\frac{x}{2}}\right)dx$

D. $2e^2 - 2e + \int_2^4 \left(e^2 - e^{2x}\right)dx$

E. $4e^2 - 4e + \int_2^4 \left(e^2 - e^{2x}\right)dx$

Question 10

The shortest distance from the origin, O, to the parabola $y = -(x-2)^2 + 1$ is closest to

A. 0.8346

B. 0.9082

C. 1.0451

D. 1.8813

E. 2.2361

Question 11

The functions $f(x) = \log_e(5-x)$ and $g(x) = \sqrt{4x+3}$ are defined over their maximal domains.

The maximal domain of the function $h = \dfrac{f}{g}$ is

A. $\left(-\infty, -\dfrac{3}{4}\right)$

B. $(-\infty, 5)$

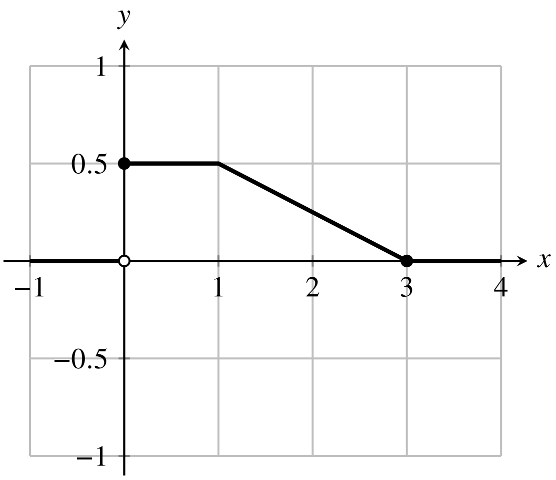
C. $\left(-\dfrac{3}{4}, \infty\right)$

D. $\left(-\dfrac{3}{4}, 5\right)$

E. $\left[-\dfrac{3}{4}, 5\right]$

Question 12

A continuous random variable X has the probability distribution shown by the graph below.



If $\Pr(X < a) = 0.75$, then a is equal to

A. 2

B. $\dfrac{2}{3}$

C. $3+\sqrt{2}$

D. $3-\sqrt{3}$

E. $3-\sqrt{2}$

Question 13

The average value of $f(x) = 3x^2 - ax$ over the interval $[-2, 4]$ is 8.

The value of a is

A. 4

B. 3

C. –2

D. –3

E. –4

Question 14

The sum of the solutions of the equation $\sin(2x) = \frac{1}{2}$ for $-\pi \le x \le \pi$ is

A. $-\frac{\pi}{12}$

B. $-\pi$

C. $\frac{\pi}{12}$

D. π

E. $\frac{3\pi}{2}$

Question 15

In a particular population it is known that 10% of people are left-handed. A random sample of 530 people is selected.

If $\Pr\left(\hat{P} > a\right) = 0.0671$, then a is closest to

A. 0.08

B. 0.12

C. 0.14

D. 0.16

E. 0.18

Question 16

Let $f : R \to R$ be a differentiable function.

For all $x > 0$ the derivative of $f\left(\log_e(2x)\right)$ is

A. $f'\left(\log_e(2x)\right)$

B. $\dfrac{f'(x)}{2x}$

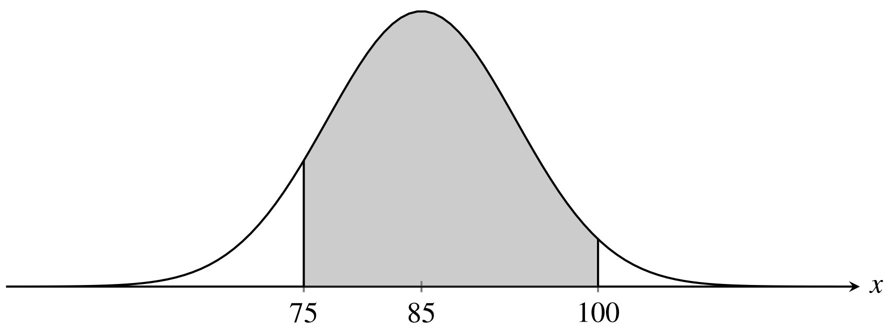
C. $\dfrac{2f'\left(\log_e(2x)\right)}{x}$

D. $\dfrac{f'\left(\log_e(2x)\right)}{x}$

E. $\dfrac{f'\left(\log_e(2x)\right)}{2x}$

Question 17

The diagram below shows the graph of a normal distribution curve with a mean of 85 and a standard deviation of 5.



Let Z be the standard normal distribution.

Which one of the following does **not** evaluate to the area shaded in the graph above?

A. $\Pr(-3 < Z < 2)$

B. $2\Pr(0 < Z < 3) + \Pr(2 < Z < 3)$

C. $2\Pr(0 < Z < 2) + \Pr(2 < Z < 3)$

D. $2\Pr(-3 < Z < 0) - \Pr(-3 < Z < -2)$

E. $\Pr(-2 < Z < 2) + \Pr(2 < Z < 3)$

Question 18

The probability distribution for a discrete random variable *X*, where $a \in R$, is shown in the table below.

x	1	2	3	4
$\Pr(X = x)$	a	$\frac{15a+2}{25}$	$\frac{25a^2+9}{25}$	$\frac{125a^3+4}{25}$

The expected value, $E(X)$, is equal to

A. $\frac{1}{5}$

B. $\frac{3}{5}$

C. $\frac{9}{5}$

D. $\frac{11}{5}$

E. $\frac{13}{5}$

Question 19

The graph of the cubic $y = 2x^3 + (p-1)x^2 + \frac{1}{3}(p+3)x + 2p$ has two stationary points if

A. $p \in R \setminus [-1,5]$

B. $p \in R \setminus (-1,5)$

C. $p \in (-1,5)$

D. $p \in [-1,5]$

E. $p \in R$

Question 20

Which one of the following transformations maps the graph of $y = \sin(x)$ to the graph of $y = 3\cos(2x) + 1$?

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 1 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3\pi}{4} \\ -1 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 1 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3\pi}{4} \\ 1 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3\pi}{4} \\ 1 \end{bmatrix}$

END OF SECTION A

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided. Write using blue or black pen.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (9 marks)

In a particular 24-hour period, the depth of water at the end of a pier is modelled by the function

$$h(t) = 5 - 2\cos\left(\frac{\pi}{6}(t-3)\right),\ \ 0 \le t \le 24$$

where h is measured in metres and t is measured in hours after midnight.

a. What are the maximum and minimum depths of water at the end of the pier?

1 mark

b. What is the depth of water at 6.00 am?

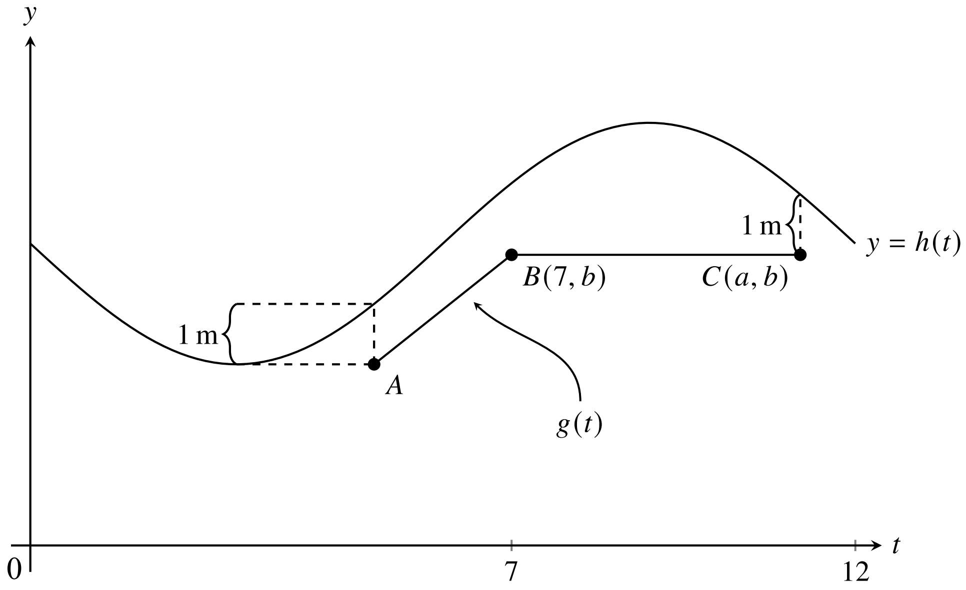
1 mark

c. At what times in the 24-hour period after midnight is the depth of water equal to four metres?

2 marks

The diagram below shows the depth of water $y = h(t)$. Point A occurs at the time when the water is one metre deeper than the time when it is at its most shallow. An empty ship berths at the end of the pier at this time.

The gradient of the line segment $y = g(t)$ is equal to the gradient of $h(t)$ at the point directly above point A.



d. Find the rule for $g(t)$.

2 marks

e. As the ship is loaded, it settles in the water and requires a depth of water equal to $g(t)$ metres. Loading stops at 7.00 am.

Find the distance in metres between the bottom of the ship and the sea floor when loading stops. Use the form $p - \frac{\sqrt{q}\pi}{q}$ where p and q are positive integers.

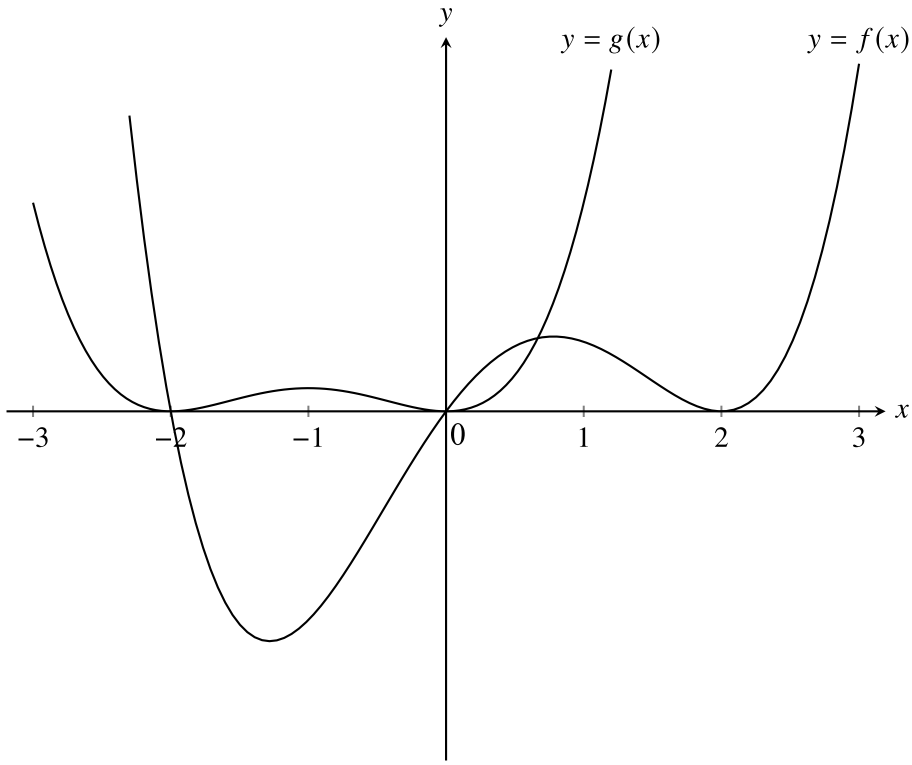
2 marks

f. Determine the coordinates of $C(a,b)$, correct to two decimal places.

1 mark

Question 2 (14 marks)

Part of the graphs of $f(x) = x(x+2)(x-2)^2$ and $g(x) = x^2(x+2)^2$ are shown below.



The graphs meet at $(-2,0)$, $(0,0)$ and at (a,b).

a. Find the values of a and b.

2 marks

b. Find the total area bounded by the graphs of f and g.

2 marks

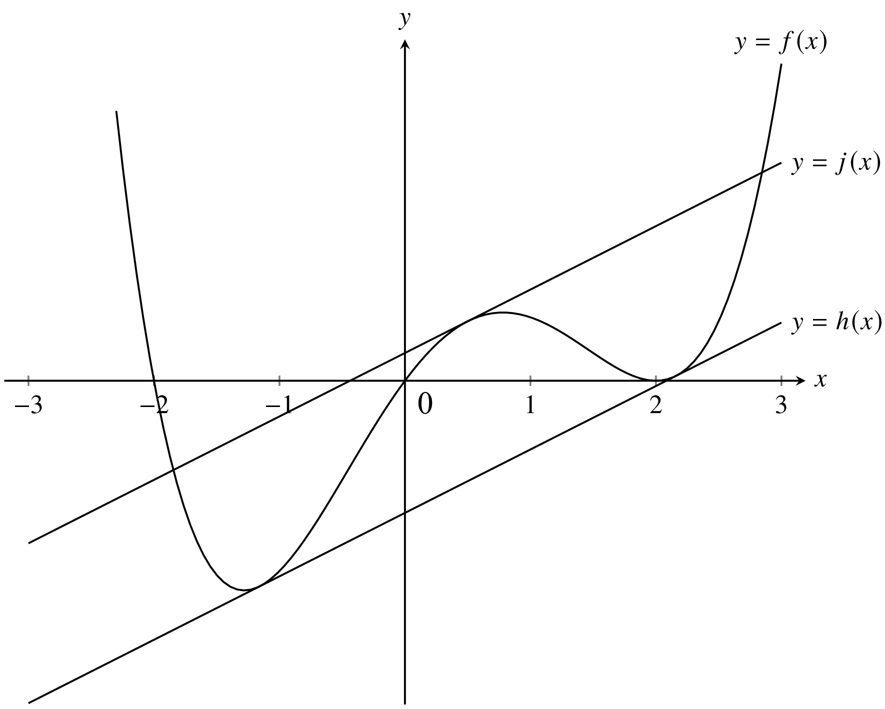
c. Find the equation of the tangent $h(x)$ to the graph of $f(x)$ at any point where $x = a$.

1 mark

d. Find the values of x in terms of a where the graphs of $f(x)$ and $h(x)$ meet and, hence, find the values of a for which there are only two points of intersection.

4 marks

The graphs of $y = f(x)$, the tangent $y = h(x)$ to $f(x)$ when $x = \frac{1}{2}\left(1+\sqrt{11}\right)$ and the tangent to the graph of $y = f(x)$, which is parallel to $h(x)$, are shown below.



e. Write down the equation of the tangent $h(x)$.

1 mark

f. Find the equation of the tangent $j(x)$ which is parallel to $h(x)$.

1 mark

g. Write down the equation of the line which is perpendicular to the graph of $y = j(x)$ and which passes through the point of intersection of $j(x)$ and $f(x)$.

1 mark

h. Hence, show that the distance between the tangents is $\frac{121\sqrt{10}}{160}$.

2 marks

Question 3 (14 marks)

There is a large hill outside the town of Hillsville. Athletes compete in an annual run from the centre of the town to the top of the hill.

The time taken for elite athletes to complete the run is a normally distributed random variable with a mean of 56 minutes and a standard deviation of 4 minutes.

a. Find the probability that a randomly selected elite athlete will complete the run in less than 53 minutes, correct to four decimal places.

1 mark

b. The probability that an elite athlete will complete the run in less than *m* minutes is 0.75.

Determine the value of *m*, correct to four decimal places.

1 mark

A runner is said to have completed the run in an excellent time if their run takes less than 54 minutes. The probability that an elite athlete completes the run in less than 54 minutes is 0.308538, correct to six decimal places.

c. In a random sample of 30 elite athletes, find the probability that, at most, ten complete the run in an excellent time. Give your answer correct to four decimal places.

2 marks

d. Let $\hat{P}$ be the random variable that represents the proportion of elite athletes who complete the run in an excellent time in random samples of 30 elite athletes.

Find $\Pr\left(\hat{P} > 0.15 | \hat{P} < 0.6\right)$. Give your answer correct to four decimal places.

2 marks

The time taken for amateur athletes to complete the run is modelled by the following probability density function:

$$f(x) = \begin{cases} \frac{1}{9000}(x-50)(80-x)\left(e^{\frac{1}{15}(x-50)} - 1\right) & 50 \le x \le 80 \\ 0 & \text{otherwise} \end{cases}$$

e. Determine the median time, in minutes, for an amateur runner to complete the run. Give your answer correct to two decimal places.

2 marks

f. Determine the standard deviation of the time taken for an amateur runner to complete the run. Give your answer in minutes correct to four decimal places.

2 marks

SECTION B – Question 3 – continued
TURN OVER

g. Find the probability that a randomly selected amateur runner completes the run in an excellent time (that is, in less than 54 minutes) given that they complete the run in less than 60 minutes. Give your answer correct to four decimal places.

2 marks

h. A sample of 50 amateur runners was taken and used to create the confidence interval $(0.0747, 0.2453)$ for the proportion of runners who finished the race in an excellent time.

What is the percentage level of this confidence interval? Give your answer correct to the nearest per cent.

2 marks

Question 4 (10 marks)

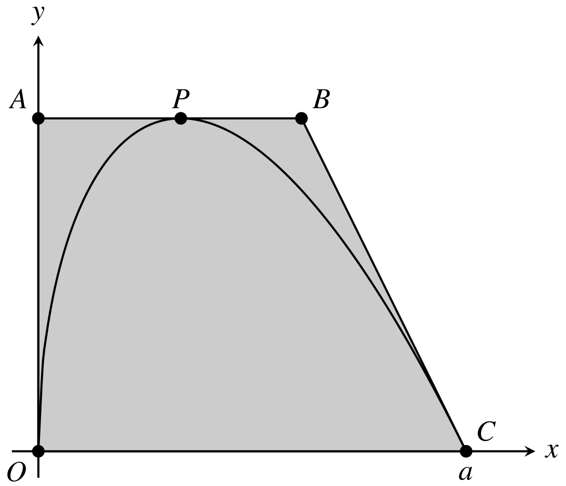
Let $f:[0,\infty)\to R$, $f(x)=\sqrt{x}(a-x)$, $a>0$.

Let P be the local maximum of the graph of f.

C is point $(a,0)$.

The gradient of the line joining B and C is equal to the gradient of the tangent to f at C.

Points A and B are joined by a horizontal line.



a. Find the coordinates of A in terms of a.

2 marks

b. Determine the coordinates of B in terms of a.

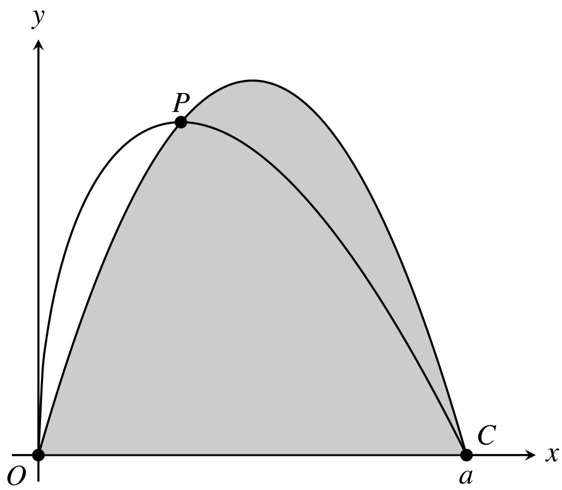
2 marks

c. Determine the area A_1 of the trapezium $OABC$.

Give your answer in the form $\frac{m\sqrt{n}-q}{p}a^{\frac{5}{2}}$ where m, n, p and q are positive integers.

2 marks

The diagram below shows the graph of f and the parabola which passes through the points O, P and C.



d. Determine the equation of the parabola in terms of a.

2 marks

e. Determine the area A_2 of the shaded region shown on the graph on page 22 in terms of a.

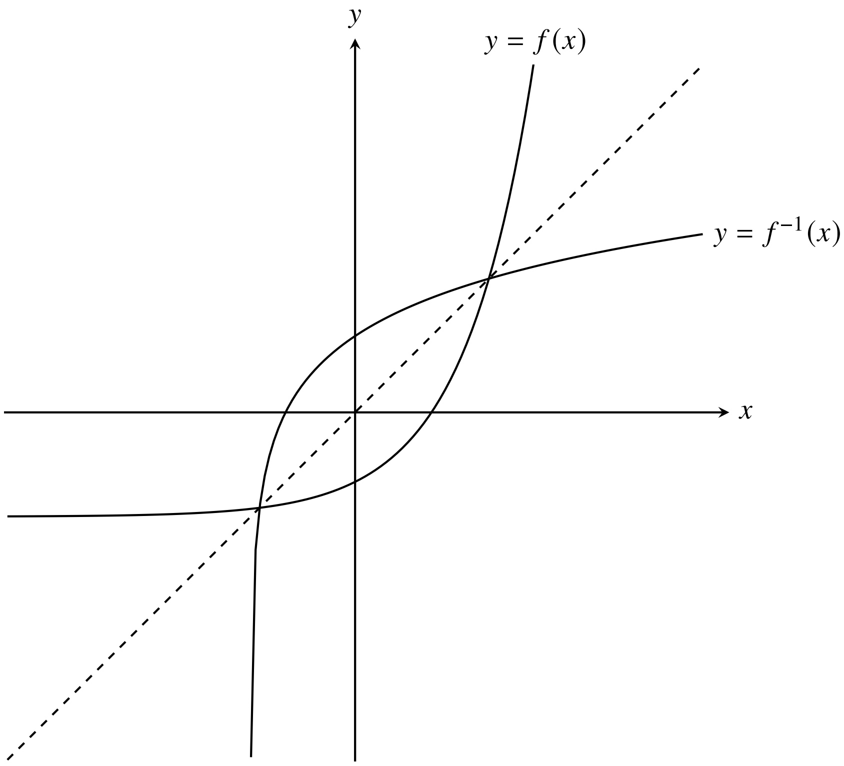
1 mark

f. Determine $\frac{A_1}{A_2}$. Give your answer in the form $\frac{p}{q}\left(r-\sqrt{s}\right)$ where p, q, r and s are positive integers.

1 mark

Question 5 (13 marks)

The graphs of $y = f(x) = e^{\frac{x}{2}} - 3$ and $y = f^{-1}(x)$ are shown below.



a. Find the rule for f^{-1}.

1 mark

b. Find the coordinates of the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$. Give your answer correct to three decimal places.

2 marks

Let $g: R \to R$, $g(x) = e^{\frac{x}{2}} - k$ where k is a real constant.

c. Suppose that the graphs of $y = g(x)$ and $y = g^{-1}(x)$ intersect only once.

i. Show that the value of k for which the graphs of g and g^{-1} meet only once is $k = 2 - \log_e(4)$.

1 mark

ii. Write down the coordinates of the point of intersection of the graphs of g and g^{-1} if they meet only once.

1 mark

Suppose that the graphs of $y = g(x)$ and $y = g^{-1}(x)$ do not intersect.

d. Write down the set of values of k for which the graphs of $y = g(x)$ and $y = g^{-1}(x)$ do not intersect.

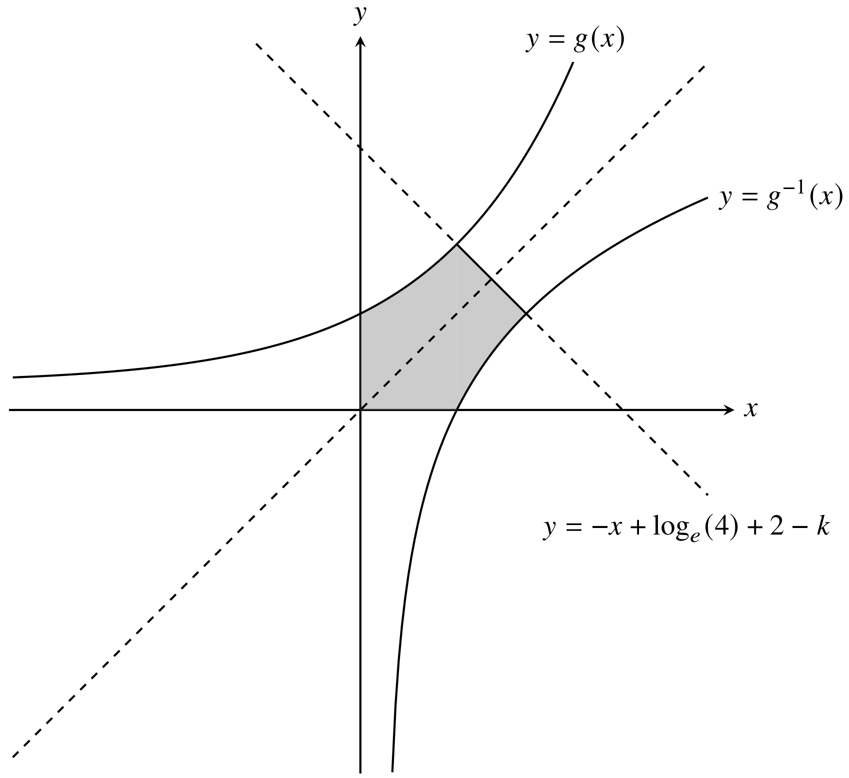
1 mark

e. Show that the equation of the line with a gradient of -1 that is perpendicular to the graph of $y = g(x)$ is $y = -x + \log_e(4) + 2 - k$.

1 mark

The diagram below shows the graphs of $y = g(x)$, $y = g^{-1}(x)$ and the line $y = -x + \log_e(4) + 2 - k$.

Assume that the graphs of $y = g(x)$ and $y = g^{-1}(x)$ do not cross more than once.



f. The shaded area in the diagram above is bounded by the line $y = -x + \log_e(4) + 2 - k$ and the graphs of $y = g(x)$ and $y = g^{-1}(x)$. It is denoted by $A(k)$.

Determine the area of $A(k)$. Give your answer in the form $ak^2 + bk + c$, where a, b and c are real constants.

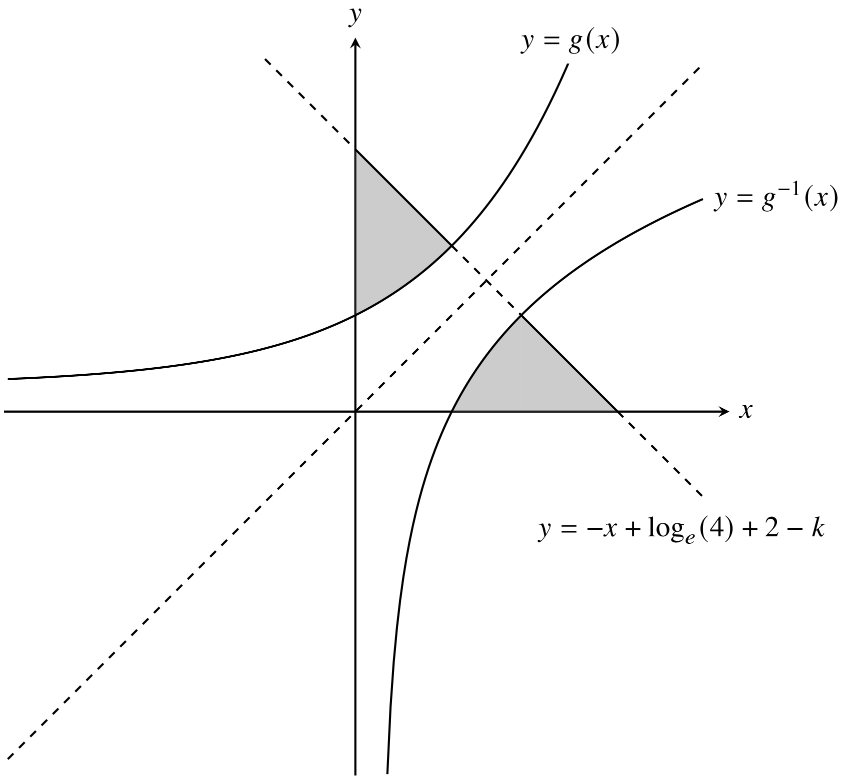
2 marks

g. Write down the minimum value of $A(k)$ and the value of k at which it occurs.

2 marks

h. Determine the total area, B, of the regions shaded below.

2 marks



END OF QUESTION AND ANSWER BOOK