The Mathematical Association of Victoria

Trial Examination 2022

MATHEMATICAL METHODS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 24 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A- Muliple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The range of the function $f:[0,10]\to R, f(x)=3(x-1)^2+5$ is

A. $[8,248]$
B. $[5,248]$
C. $[0,248]$
D. $[0,10]$
E. $[1,5]$

Question 2

The height of water in a dam is modelled by $h(t)=-3\cos\left(\frac{\pi}{6}(t-10)\right)+4$ where h is the height of the water in metres at time t hours after 9 am on a certain day. The average height of water from 9 am to 9 pm on that day in metres is

A. 4
B. 1
C. $\frac{1-\sqrt{3}}{6}$
D. $\frac{3\sqrt{3}}{\pi}+4$
E. $\frac{1}{2}$

Question 3

For the graph of $y=-\pi\sin(2\pi x)-\pi$, the amplitude is

A. 1
B. 2
C. 2π
D. $-\pi$
E. π

Question 4

If $f(x)=2^{x-1}$ then $f(x)\times f(y)$ equals

A. $f(xy)$

B. $f(x+y)$

C. $f(x)+f(y)$

D. $f(x+y-2)$

E. $f(x+y-1)$

Question 5

Consider the function $g:(3,7]\to R,\ g(x)=\dfrac{-2}{x-3}+10$.

The rule and domain, respectively, for the inverse function g^{-1} are

A. $g^{-1}(x)=\dfrac{3x-32}{x-10}$ $\quad x\in(3,7]$

B. $g^{-1}(x)=\dfrac{3x}{x-10}-\dfrac{32}{x-10}$ $\quad x\in(-\infty,9.5]$

C. $g^{-1}(x)=\dfrac{-2}{x-3}+10$ $\quad x\in(-\infty,9.5]$

D. $g^{-1}(x)=\dfrac{10x-32}{x-3}$ $\quad x\in[7,3)$

E. $g^{-1}(x)=\dfrac{3x-32}{x-10}$ $\quad x\in[7,3)$

Question 6

When the transformation described by $T:R^2\to R^2, T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ has been applied to a certain exponential function its image rule becomes $y=e^{2x}$. The original exponential rule is

A. $y=-e^{\frac{2x}{3}}$

B. $y=e^{\frac{2x}{3}}$

C. $y=\dfrac{1}{3}e^{-2x}$

D. $y=3e^{-\frac{x}{2}}$

E. $y=3e^{-2x}$

SECTION A - continued

Question 7

The set of simultaneous equations

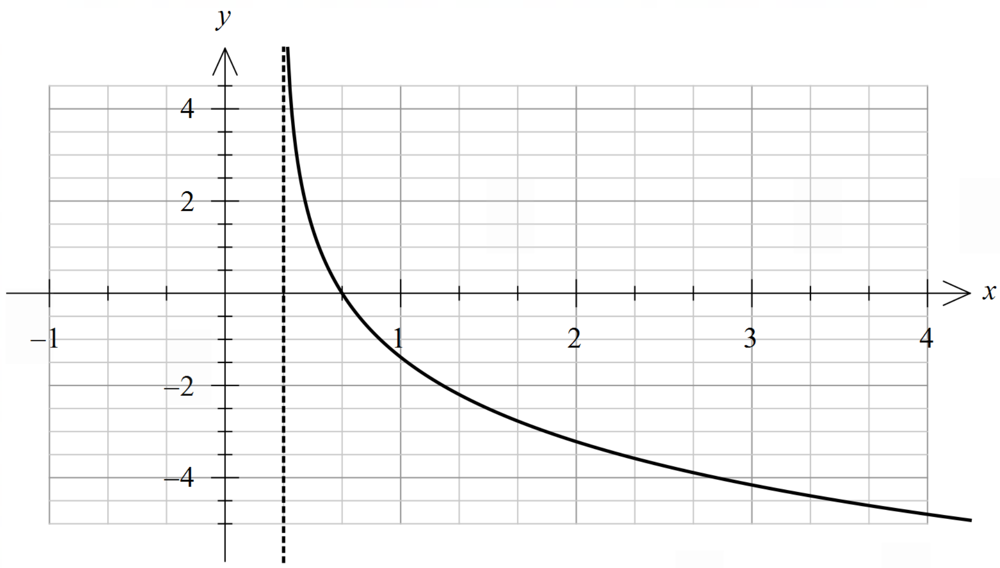
$$2x-2y=m$$
$$6x-6y=m$$

where $m \in R\setminus\{0\}$ is a real constant, will

A. represent parallel lines
B. represent perpendicular lines
C. intersect at the point (m,m)
D. have infinite solutions
E. have a unique solution

Question 8

Part of the graph of $y=a\log_e(bx+c)$ where *a, b* and *c* are real constants is shown below.



Given that the graph goes through the point $(3,-6\log_e(2))$ and the *x*-intercept of the graph is at $\left(\frac{2}{3},0\right)$, the rule for the function could be

A. $y=-2\log_e\left(x+\frac{1}{3}\right)$

B. $y=-2\log_e\left(x-\frac{1}{3}\right)$

C. $y=-\log_e\left(x-\frac{1}{3}\right)$

D. $y=-2\log_e(3x-1)$

E. $y=\log_e(3x-1)$

Question 9

A and B are two independent events. If $\Pr(A \mid B) = \frac{1}{2}$ and $\Pr(A' \cap B') = \frac{1}{5}$ then $\Pr(A \cap B')$ equals

A. $\frac{3}{10}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{3}{5}$

E. $\frac{1}{2}$

Question 10

Let $f(x)$ be the probability density function

$$f(x) = \begin{cases} \frac{x}{2} + k, & 0 \le x \le k \\ 0, & \text{otherwise} \end{cases} \quad \text{where } k \in R^+.$$

The median of $f(x)$ is

A. 0.4915

B. 0.5

C. $\frac{16\sqrt{5}}{75}$

D. $\frac{\sqrt{130} - 4\sqrt{5}}{5}$

E. 1.34

Question 11

A random sample of Australian secondary students was taken in 2020. The approximate 95% confidence interval for the proportion of students who missed more than 40 days of school was found to be $(0.3346, 0.4654)$. The number of students in the random sample is closest to

A. 144

B. 215

C. 216

D. 10 000

E. 19 347

Question 12

The probability distribution for the number of pets per household, X, in a particular suburb is shown in the table below, where $k \in R^+$.

x	0	1	2	3
$\Pr(X = x)$	0.2	$\frac{1+k^2}{10}$	$\frac{4-k}{10}$	$\frac{k}{20}$

The mean number of pets per household is

A. 0.2
B. 0.8
C. 1
D. 1.2
E. 2

Question 13

The graph of $f(x) = x^4 + bx^3 + x^2 - 2$, where b is a real contant, will have three stationary points when

A. $b \le -\frac{4\sqrt{2}}{3}$ or $b \ge \frac{4\sqrt{2}}{3}$
B. $-\frac{4\sqrt{2}}{3} \le b \le \frac{4\sqrt{2}}{3}$
C. $b < -\frac{4\sqrt{2}}{3}$ or $b > \frac{4\sqrt{2}}{3}$
D. $-\frac{4\sqrt{2}}{3} < b < \frac{4\sqrt{2}}{3}$
E. $b = -\frac{4\sqrt{2}}{3}$ or $b = \frac{4\sqrt{2}}{3}$

Question 14

The equation of the line which is perpendicular to the line with equation $y = 2x$ and is a tangent to the curve with equation $h(x) = \sqrt{1-3x}$ is

A. $y = -2x + \frac{25}{24}$
B. $y = -\frac{x}{2} - \frac{20}{3}$
C. $y = -3x + \frac{5}{4}$
D. $y = \frac{-x}{2} + \frac{5}{3}$
E. $y = -2x$

SECTION A - continued
TURN OVER

Question 15

The velocity of a particle, v ms^{-1}, at time t seconds, is given by the rule $v = \frac{-1}{(t+1)^2} + \frac{t}{8} + \frac{1}{2}$ for $t \geq 0$. The distance in metres, travelled in the first 2 seconds is closest to

A. 0.292
B. 0.569
C. 0.583
D. 0.736
E. 0.737

Question 16

The area bounded by the curves with equations $f(x) = e^{\sin(x)}$ and $g(x) = e^{\cos(x)}$ over the interval $[-\pi, 2\pi]$ can be found by evaluating

A. $2\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \left(g(x) - f(x)\right)dx$

B. $\int_{-\frac{3\pi}{4}}^{\frac{5\pi}{4}} \left(g(x) - f(x)\right)dx$

C. $\int_{\frac{\pi}{4}}^{-\frac{3\pi}{4}} \left(g(x) - f(x)\right)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(f(x) - g(x)\right)dx$

D. $\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \left(f(x) - g(x)\right)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(g(x) - f(x)\right)dx$

E. $\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \left(f(x) - g(x)\right)dx + \int_{\frac{5\pi}{4}}^{\frac{\pi}{4}} \left(g(x) - f(x)\right)dx$

Question 17

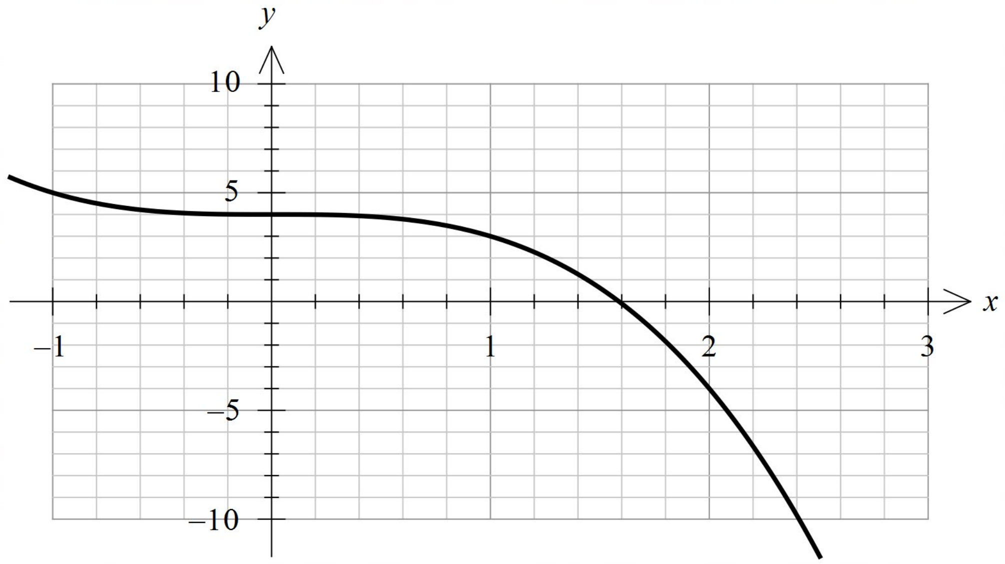
Consider $f(x) = -\log_e(12 - 4x)$ and $g(x) = -\frac{1}{\sqrt{2x-4}}$ over their maximal domains.

The range of $f(g(x))$ is

A. $(-\infty, -\log_e(12))$
B. $(-\infty, 3)$
C. $[2, \infty)$
D. $(-\infty, 0)$
E. R

Question 18

The graph of $y = f(x) = -x^3 + 4$ is shown below.



The definite integral $\int_0^2 \left(-x^3 + 4\right) dx$ is approximated using four rectangles of equal length and the right endpoint of each rectangle. Part of the x-axis always forms one of the sides of the rectangles.
The value of this approximation is

A. $\frac{7}{2}$
B. $\frac{23}{4}$
C. $\frac{7}{4}$
D. $\frac{23}{2}$
E. 4

Question 19

The size of the acute angle, in degrees correct to two decimal places, between the lines $2x - y = 6$ and $3x + 2y = 5$ is

A. 1.05
B. 7.13
C. 8.13
D. 41.19
E. 60.26

SECTION A - continued
TURN OVER

Question 20

Consider the graphs of $f(x)=2\sqrt{4x-a}$, where a is a real constant, and $g(x)=(x-1)(x+2)(x+3)$ over their maximal domains.
The maximum number of points of intersection will occur when

A. $-12<a<-8$ only
B. $-12\le a\le -8$
C. $-3\le a\le -2$
D. $a<-12$ or $a>-8$
E. $a>4$

END OF SECTION A

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TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

A function $h:[0,10] \to R, h(t) = 0.005t(t-10)^2(2t+5)$ models the height of liquid in a particular glass, measured in centimetres at time, t minutes.

a. Find the average height of liquid in the glass. 1 mark

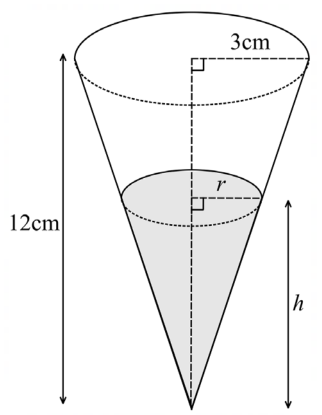
b. Find the maximum height of liquid in the glass.
Give your answer in cm correct to one decimal place. 1 mark

c. Find the average rate of change of the height of liquid in the glass for the time it is filling.
Give your answer correct to two decimal places in cm/min. 2 marks

d. Find the time when the height of liquid in the glass is increasing at its maximum rate.
Give your answer in minutes correct to two decimal places. 1 mark

SECTION B - Question 1 - continued

The glass is in the shape of a right circular cone with height 12 cm and radius 3 cm as shown below. r is the radius of the top surface of the liquid in centimetres and h is the height of the liquid in centimetres at time t minutes.



e. Show that the volume, V cm^3, of liquid in the glass is given by $V = \frac{\pi}{48}h^3$. 2 marks

f. Find the times when the glass is half full.
Give your answers in minutes correct to two decimal places. 2 marks

Let $h_1 : [a, 10+a] \to R, h_1(t) = 0.005(t-a)(t-a-10)^2(2(t-a)+5)$ model the height of liquid in a similar glass, in centimetres at time, t minutes. a is positive real constant.

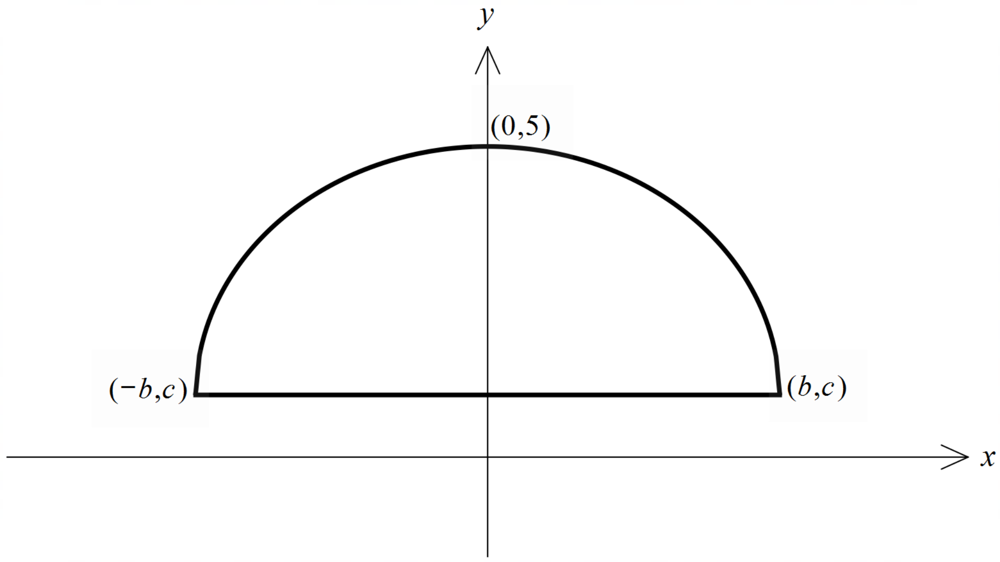
g. Find the values of a such that the average rate of change of $h_1(t)$ from $t = a$ to $t = 2 + a$ is equal to the gradient of the function $h_1(t)$ at the first time when the original glass was half full.
Give your answer correct to two decimal places. 2 marks

Question 2 (11 marks)
A semicircular arched window has the curved shape modelled by the function

$w:[-4,4] \to R, w(x) = \sqrt{r^2 - x^2} + c$ where $r > 0$ and $c \in R^+ \cup \{0\}$.

Let x be the horizontal distance in decimetres from the origin and let y be the vertical distance, in decimetres, from the x-axis.

The shape of the window is shown below, where the coordinates of the y-intercept are (0, 5) and the coordinates of the extremes of the window are $(-b, c)$ and (b, c).



a. State the radius of the semicircle. 1 mark

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b. Hence find the rule for w . 1 mark

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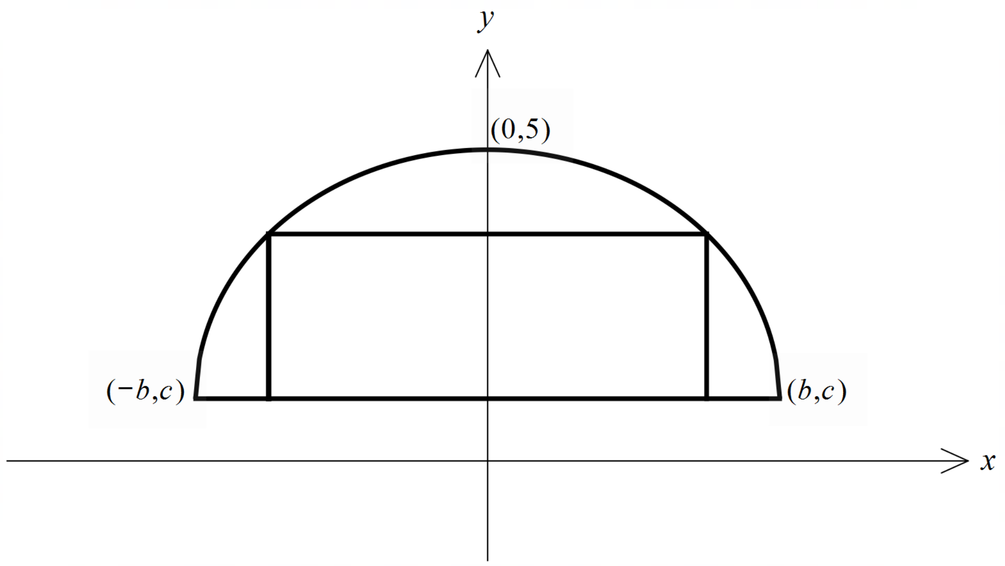
A team of specialised glaziers are brought in to insert a rectangular section of stained coloured glass that will sit inside the semicircular shape as shown below. The rectangular section touches the semicircle curve at the points (x_1, y_1) and $(-x_1, y_1)$. The glaziers will put the stained glass in the rectangle that has the maximum area.



c. Show that the area of the rectangle, A_r, in terms of x_1 is $A_r = 2x_1\sqrt{16 - x_1^2}$. 1 mark

d. Hence find the maximum possible area of the rectangle, in dm^2, that will fit inside the semicircular shape. Justify that this area is a maximum. 2 marks

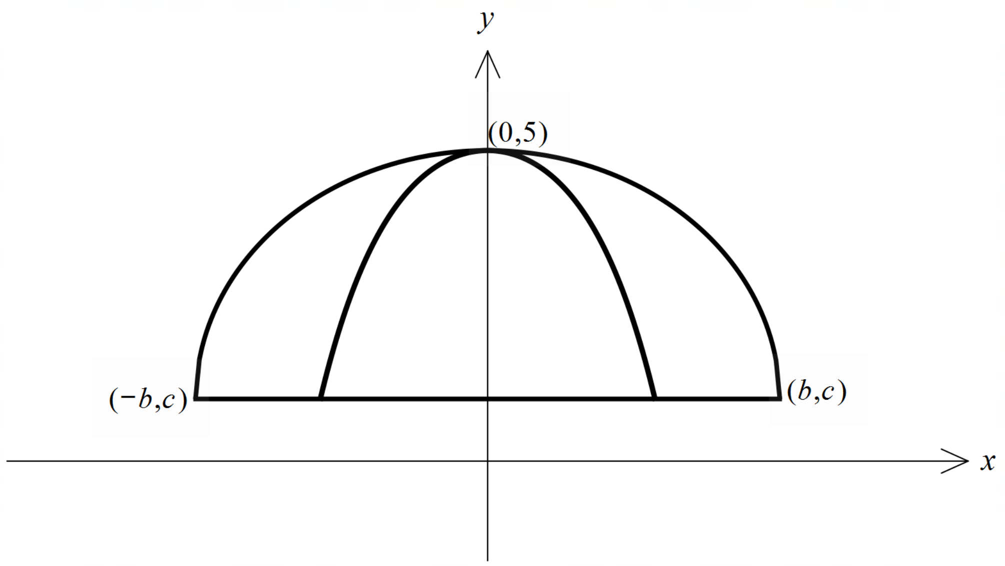
The glaziers will fill the rest of the semicircular window with plain glass.

e. Find the proportion of the semicircular window that is stained glass. 1 mark

The curved section of a different window is modelled by the rule

$h(x) = -\frac{1}{2}\left(e^{x} + e^{-x}\right) + d$ where $d \in R^{+} \cup \{0\}$.

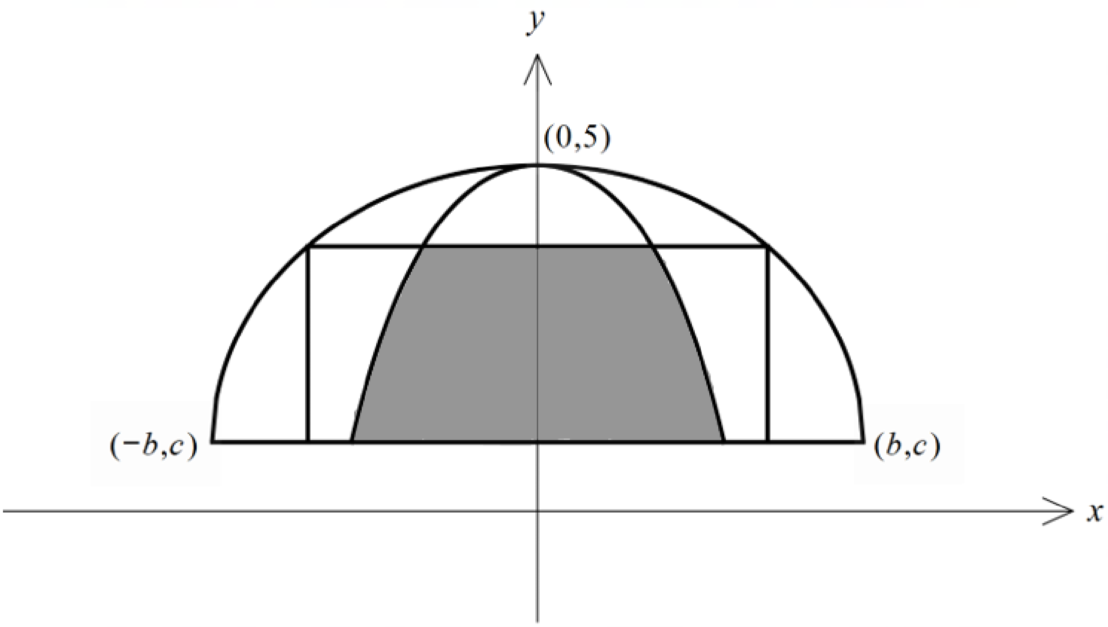
Let *x* be the horizontal distance in decimetres from the origin and let *y* be the vertical distance, in decimetres, from the *x*-axis. The *y*-intercept of *h* is $(0,5)$.



f. Find the value of *d*. 1 mark

g. State the coordinates of the point(s) where the window modelled by *h* intersects with the line $y = 1$. 1 mark

The glaziers decide to use a combination of both designs for the window as shown below. The rectangle is the one that has the maximum area within the semicircle.



Stained glass is to be put in the shaded area.

h. Find the area of the shaded region. Give your answer in dm^2 correct to one decimal place. 3 marks

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SECTION B - continued
TURN OVER

Question 3 (16 marks)
The lifespan of a domesticated cat, *Felis catus*, is normally distributed with a mean of 15 years and a standard deviation of 1 year.

a. **i.** Find the probably that a randomly selected cat has a lifespan of more than 17.5 years. Give your answer correct to four decimal places. 1 mark

ii. Given that a particular cat is at least 14 years old, what is the probability that it will live longer than 17.5 years? Give your answer correct to four decimal places. 2 marks

The masses of the same species of cat are normally distributed. The probability that a cat is less than 3.8 kg is 0.0478 and the probability a cat is more than 4.2 kg is 0.1487.

b. Find the mean and standard deviation of this distribution. Give your answers in kg correct to two decimal places. 2 marks

c. Twenty five of these cats are randomly selected. What is the probability that more than 20 of them will weigh between 3.8 and 4.2 kg? Give your answer correct to four decimal places. 2 marks

d. What is the least number of cats that would need to be sampled to ensure the probability that at least 21 of them will have weights between 3.8 and 4.2 kg is more than 0.95? 2 marks

The probability density function, $s(x)$ for the number of hours a cat sleeps per day is given by

$$s(x)=\begin{cases}\frac{x}{4}-3 & 12\le x\le 14\\ a\left(x-16\right)^2+b & 14<x\le 16\\ 0 & \text{elsewhere}\end{cases}$$ where a and b are real constants.

e. Find a and b. 2 marks

The graph of $s(x)$ is transformed to the graph of $p(x)$ where $p(x)$ is the probability density function for the number of hours a cat plays per day.

$$p(x)=\begin{cases}\frac{3x^2}{32}-\frac{3x}{2}+\frac{49}{8} & 8\le x\le 10\\ -\frac{x}{4}+3 & 10<x\le 12\\ 0 & \text{elsewhere}\end{cases}$$

f. i. Give a possible set of one or more transformations that has been applied to the graph of $s(x)$. 1 mark

SECTION B - Question 3 – continued
TURN OVER

ii. Find the variance of p. Give your answer correct to three decimal places. 2 marks

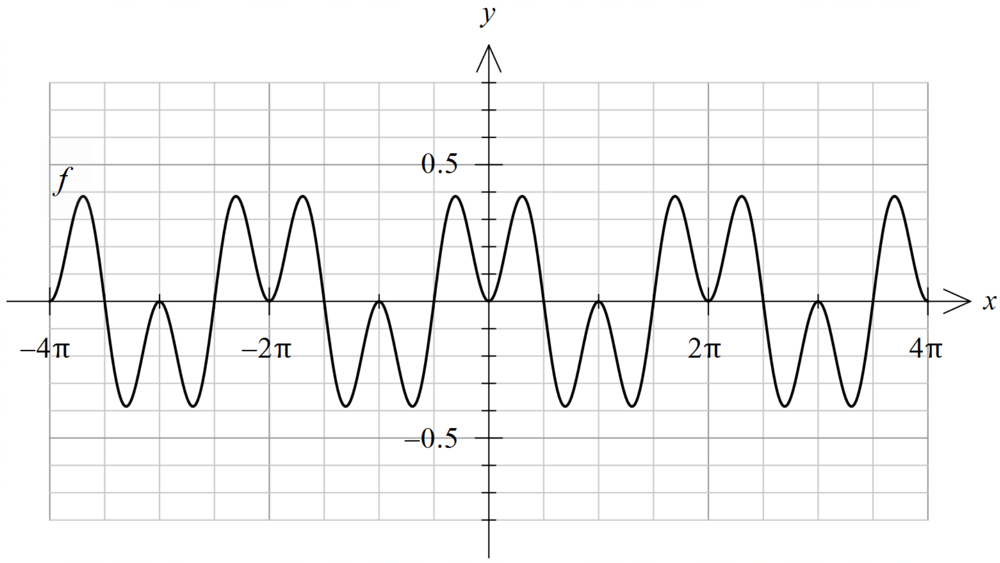
Kitty, a domesticated cat, has a small box of soft toys. There are 8 toy rats and k toy mice in the box. She randomly selects 3 toys, without replacement, from the box.

g. If the mean of the sample proportion of rats in the sample is $\frac{8}{17}$ find k. 2 marks

Question 4 (14 marks)

Let $f: R \to R, f(x) = \sin^2(x)\cos(x)$.

Part of the graph of *f* is shown below.



a. State the period of *f*. 1 mark

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b. State the range of *f*. 1 mark

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c. Find the general solution for the *x* intercepts of *f*. 1 mark

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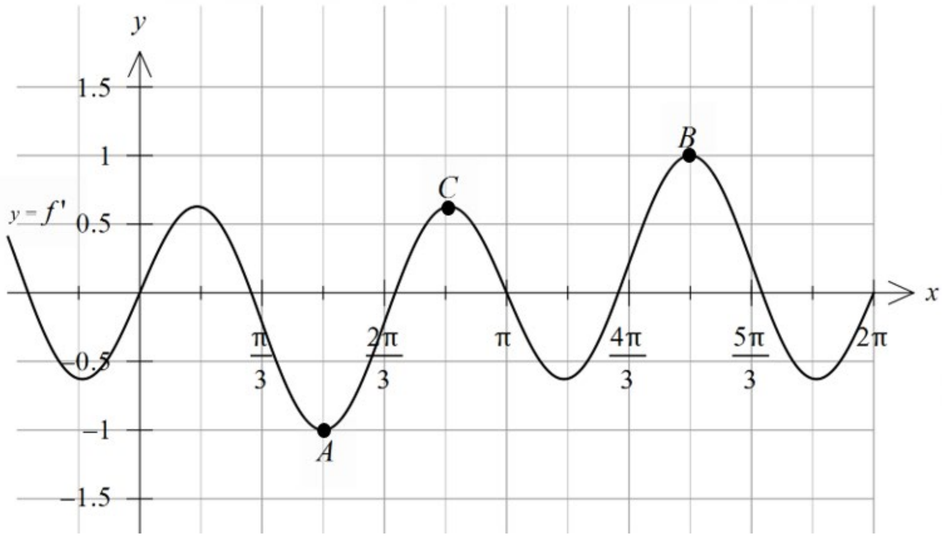
Now consider the graph of $h(x) = \log_e\left(\sin^2(x)\cos(x)\right)$.

d. Find the maximal domain of *h*. 2 marks

Part of the graph of the derivative of f is shown below. The points A, B and C are at the turning points of the graph of f'.



e. Find the equation of the line which passes through AB. 2 marks

f. **i.** Find the equation of the perpendicular bisector of the line segment AB. 1 mark

ii. Show that the perpendicular bisector does not pass through the point C. 2 marks

g. Find the area bounded by the graphs of f and f' over the interval $\left[0, 2\pi\right]$.
Give your answer correct to two decimal places. 1 mark

Now consider the family of curves $g(x) = f\left(kx\right)$, where $k \in R^+$.

h. Find the area bounded by the graphs of g and g' over one cycle, if the period of g is 4π.
Give your answer correct to two decimal places. 2 marks

i. Find the smallest integer value of k for which the area bounded by the graphs of g and g' over one cycle, is less than 3.1. 1 mark

Question 5 (8 marks)
Let $p: R \to R, p(x) = -(x-1)^3$.

a. Find the rule of p^{-1}. 1 mark

b. How many points of intersection do the graphs of p and p^{-1} have? 1 mark

Let $q: R \to R, q(x) = -a(x-1)^3$, where $a \in R^+$.

c. What are the possible number of solutions to $q(x) = q^{-1}(x)$? 1 mark

d. One of the values of a such that q is tangential to q^{-1} is $\frac{32}{27}$. Find the other value of a. 2 marks

e. Hence, find the values of a such that $q(x) = q^{-1}(x)$ has three solutions. 1 mark

Now consider the family of curves $t: R \to R, t(x) = -\left(x-1\right)\left(x^2 + bx + c\right)$.

f. Find the values of b in terms of c for which $t(x) = 0$ has two unique real solutions. 2 marks

END OF QUESTION AND ANSWER BOOK