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Student Name……………………………………

MATHEMATICAL METHODS UNITS 3 & 4

TRIAL EXAMINATION 2

2023

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section A and Section B.
Section A consists of 20 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 28 of this exam.
Section B consists of 5 extended-answer questions.
Section A begins on page 2 of this exam and is worth 20 marks.
Section B begins on page 11 of this exam and is worth 60 marks.
There is a total of 80 marks available.
All questions in Section A and Section B should be answered.
In Section B, where more than one mark is allocated to a question, appropriate working must be shown.
Where a numerical answer is required, an exact value must be given unless otherwise directed.
Diagrams in this exam are not to scale except where otherwise stated.
Students may bring one bound reference into the exam.
Students may bring into the exam one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory does not need to be cleared. For approved computer-based CAS, full functionality may be used.
A formula sheet can be found on pages 26 and 27 of this exam.

SECTION A – Multiple-choice questions

Question 1

The period of the function $f(x) = 2\tan\left(\frac{2\pi x}{3}\right)$ is

A. $\frac{3}{2}$

B. 3

C. $\frac{3\pi}{2}$

D. 6

E. 3π

Question 2

Let $f: R \to R$, $f(x) = \sin\left(\frac{x}{2}\right)$. The average rate of change of f between $x = \frac{\pi}{3}$ and $x = \pi$ is

A. $\frac{3}{4\pi}$

B. $\frac{1}{2} - \frac{\pi}{3}$

C. $1 - \frac{3}{4\pi}$

D. $-\frac{\pi}{3} - \frac{1}{2\pi} + 1$

E. $-\pi - \frac{1}{6\pi} + 1$

Question 3

Let $g: [-1, 2] \to R$, $g(x) = -x^2(x-2)$. The maximum value of g is

A. -1

B. 0

C. $\frac{32}{27}$

D. $\frac{4}{3}$

E. 3

Question 4

An experiment involving n trials has a probability of success in each trial of p.
Let X be the discrete random variable representing the number of successes in these trials, where X has a binomial distribution. Given that the mean and variance of X are 12 and 9.12 respectively, then

A. $n = 16$ and $p = 0.75$
B. $n = 20$ and $p = 0.6$
C. $n = 20$ and $p = 0.4$
D. $n = 50$ and $p = 0.24$
E. $n = 50$ and $p = 0.76$

Question 5

For $x > 0$, the graph of $y = 2\log_e(x) - \log_e(x+1) + 1$ is identical to the graph of

A. $y = \log_e\left(\frac{2x}{x+1}\right)$
B. $y = \log_e\left(\frac{ex^2}{x+1}\right)$
C. $y = \log_e\left(\frac{x^2}{e(x+1)}\right)$
D. $y = \log_e(2x(x+1))$
E. $y = \log_e(x^2(x+1))$

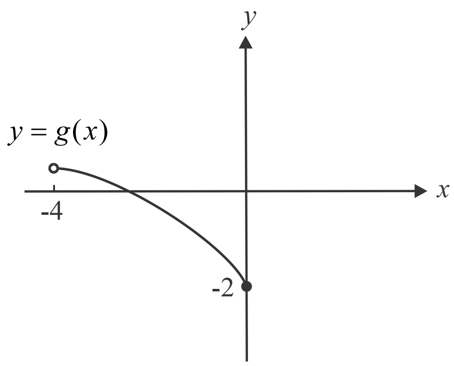
Question 6

The simultaneous linear equations $ax + 4y = a + 6$ and $x + ay = -2$ will have infinitely many solutions when

A. $a = -2$ only
B. $a = 2$ only
C. $a = -2$ or $a = 2$
D. $a \in R\backslash[-2, 2]$
E. $a \in R$

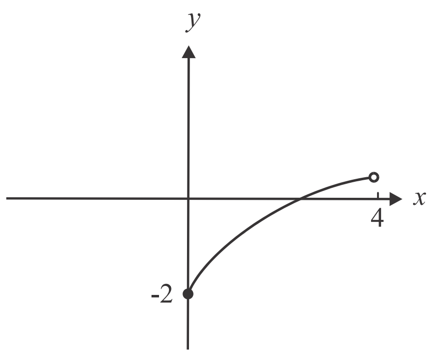
Question 7

The graph of $y = g(x)$ is shown below.



The graph of the inverse function $g^{-1}(x)$ could be

A.



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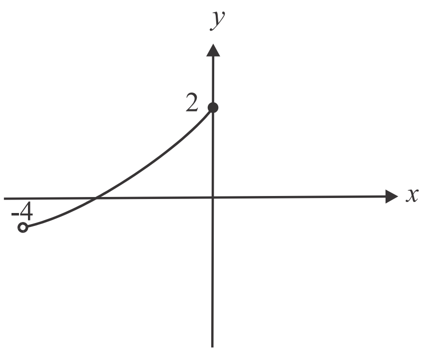
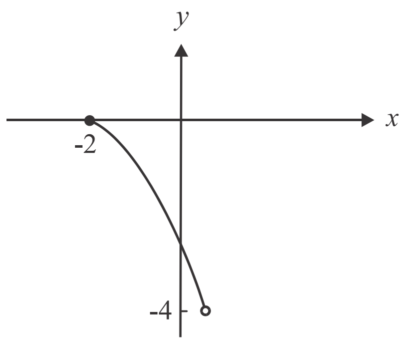
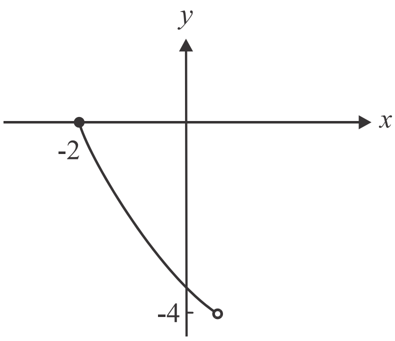


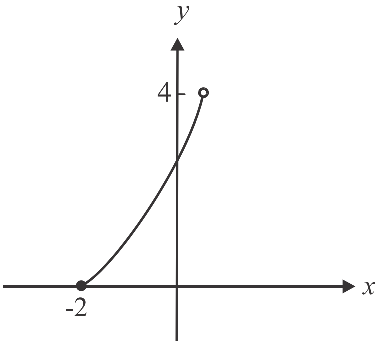
C.



D.



E.



Question 8

Which one of the following functions is **not** continuous?

A. $f(x)=\begin{cases} x+1 & x<0 \\ x^2+1 & x\ge 0 \end{cases}$

B. $f(x)=\begin{cases} x & x<0 \\ x(x-1) & x\ge 0 \end{cases}$

C. $f(x)=\begin{cases} e^x & x\le 0 \\ 1-x & x>0 \end{cases}$

D. $f(x)=\begin{cases} x^2 & x<0 \\ x+1 & x\ge 0 \end{cases}$

E. $f(x)=\begin{cases} 0 & x\le 0 \\ \sqrt{x} & x>0 \end{cases}$

Question 9

The system of linear equations

$$2x-z=1$$
$$y+z=3$$

has a general solution given by

A. $x=k,\quad y=2-2k,\quad z=2k+1,$ for all $k\in R$

B. $x=\frac{k-4}{2},\ y=k,\ z=3-k,$ for all $k\in R$

C. $x=\frac{4-k}{2},\ y=k,\ z=3-k,$ for all $k\in R$

D. $x=\frac{k+1}{2},\ y=k+3,\ z=k,$ for all $k\in R$

E. $x=\frac{k-1}{2},\ y=3-k,\ z=k,$ for all $k\in R$

Question 10

The probability distribution for the discrete random variable X is shown in the table below.

x	-1	0	1	3
$\Pr(X=x)$	$2a$	$4a$	$3a$	a

The variance of X is

A. 0.76
B. 1.16
C. 1.24
D. 1.54
E. 1.56

Question 11

Let $h(x)=\log_e \sqrt{(x-k)(x+k)}$, where k is a positive, real constant. The maximal domain of h is

A. $x \in R$
B. $x \in (-k, k)$
C. $x \in R \setminus (-k, k)$
D. $x \in [-k, k]$
E. $x \in R \setminus [-k, k]$

Question 12

The heights of ten year old boys in a particular country are normally distributed with a mean of 137 cm.
Given that 4% of these ten year old boys have a height of more than 145 cm, then the standard deviation of the distribution is closest to

A. 2.0 cm
B. 3.2 cm
C. 3.9 cm
D. 4.6 cm
E. 5.4 cm

Question 13

The function $g: R \to R,\ g(x)=ax^3+bx^2-cx+1,$ where a, b and c are positive constants, has no stationary points when

A. $a<-\frac{b^2}{6c}$
B. $a<-\frac{b^2}{3c}$
C. $a>-\frac{b^2}{4c}$
D. $a>\frac{b^2}{6c}$
E. $a>\frac{b^2}{3c}$

Question 14

Consider the algorithm below which uses Newton's method to estimate the solution to an equation of the form $f(x)=0$, with a tolerance of 10^{-4}.

```
Inputs:    f(x), a function of x
           x0, an initial estimate for the solution of f(x)=0
Define:    newton_method(f(x),x0)
           df(x) ← the derivative of f(x)
           xcurrent ← x0
           i ← 0
           While i < 100  Do
                xnext ←xcurrent −f(xcurrent) ÷ df(xcurrent)
                If −10^-4 < xnext − xcurrent < 10^-4   Then
                     Return  xnext, f(xcurrent)
                Else
                     xcurrent ← xnext
                     Return  f(xcurrent)
                     i ← i + 1
           EndWhile
```

The algorithm is implemented as follows.

```
newton_method(5 − x^3, 1)
```

The output after the second iteration of the while loop would be closest to

A. -10.398

B. -7.704

C. -1.452

D. 1.862

E. 2.333

Question 15

A bucket contains x green tennis balls and y yellow tennis balls. Two balls are randomly selected from the bucket without replacement. The probability that these two balls are the same colour is

A. $\dfrac{1}{2xy}$

B. $\dfrac{x^2+y^2}{x+y}$

C. $\dfrac{x^2+y^2}{(x+y^2)}$

D. $\dfrac{x(x-1)+y(y-1)}{(x+y)^2}$

E. $\dfrac{x(x-1)+y(y-1)}{(x+y)(x+y-1)}$

Question 16

For random samples of four retirees, $\hat{P}$ is the random variable representing the proportion of retirees who would like to find part-time work.
If $\Pr(\hat{P}=1)=\dfrac{1}{81}$, then $\Pr(\hat{P}>0.5)$ is closest to

A. 0.1111
B. 0.2723
C. 0.4074
D. 0.8852
E. 0.9871

Question 17

Let $f:(-\infty,2)\to R,\ f(x)=\dfrac{1}{x-2}$.
A tangent to the graph of *f* at $x=a$, has a negative vertical axis intercept.

The possible values of *a* are

A. $a\in(-\infty,1)$

B. $a\in(-\infty,2)$

C. $a\in\left(-\dfrac{1}{2},1\right)$

D. $a\in\left(-\dfrac{1}{2},2\right)$

E. $a\in[0,2)$

Question 18

The graph of h is obtained from the graph of $y = f(x)$, where $f(x) = x(e^x - 1)$, by a dilation of a factor of 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation of 1 unit in the negative x direction and a translation of 2 units in the positive y direction.

The rule of h is

A. $h(x) = 2 - 2e^x(x+1)$

B. $h(x) = 2 - 2(x+1)(e^x - 1)$

C. $h(x) = 2 - 2(x-1)(e^{x-1} - 1)$

D. $h(x) = 2 - 2(x+1)(e^{x+1} - 1)$

E. $h(x) = 2 - (x+1)(e^{\frac{x+1}{2}} - 1)$

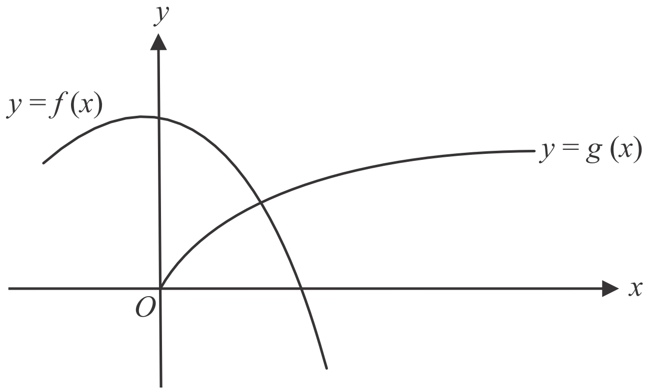
Question 19

A quartic function has the rule $y = g(x)$.
The graph of the gradient function g' crosses the x-axis at $(-4,0), (-3,0)$ and $(1,0)$, and has a maximum turning point at $x = a$ and a minimum turning point at $x = b$ where $b > a$.
The graph of $y = g(x)$ has a minimum turning point when

A. $x = -4$ only
B. $x = -3$ only
C. $x = 1$ only
D. $x = -4$ and $x = 1$ only
E. $x = -4$ and $x = -3$ only

Question 20

Let $f: R \to R,\ \ f(x) = 2 - x^2$ and $g: [0, \infty) \to R,\ \ g(x) = \sqrt{x}$.
Part of the graphs of *f* and of *g* are shown below.



Let *A* represent the area enclosed by the graph of *f*, the graph of *g* and the *x*-axis.

Let *B* represent the area enclosed by the graph of *f*, the graph of *g* and the *y*-axis.

The ratio of *A* to *B* is

A. $\dfrac{3\sqrt{2}}{4} - 1 : 1$

B. $\dfrac{4\sqrt{2}}{3} - 1 : 1$

C. $\dfrac{2 \times 2^{\frac{3}{4}}}{3} : 1$

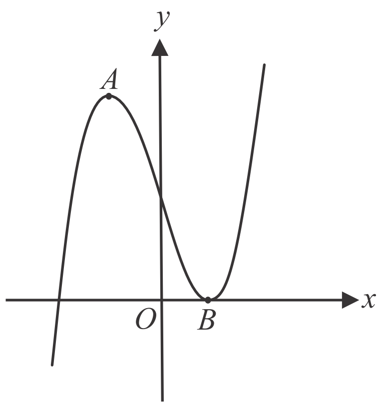
D. $\dfrac{2 \times 2^{\frac{3}{4}}}{3} + 1 : 1$

E. $\dfrac{4\sqrt{2}}{3} - \dfrac{2 \times 2^{\frac{3}{4}}}{3} : 1$

SECTION B

Question 1 (14 marks)

Consider the cubic $f : R \to R,\ f(x) = x^3 - 3x + 2$. Part of the graph of *f* is shown below.



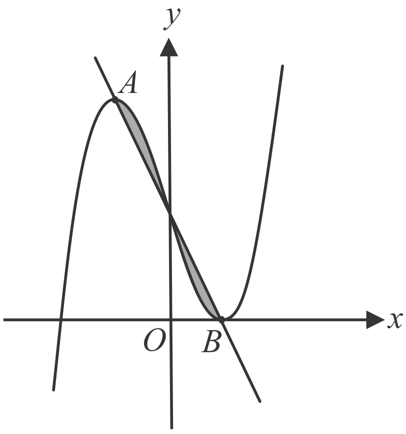
The function *f* has turning points at points *A* and *B*.

a. Find the coordinates of *A* and *B*. 2 marks

b. Find the distance *AB*. 1 mark

c. Find the equation of the straight line through *A* and *B*. 2 marks

d. The diagram below shows part of the graph of $y = f(x)$ and the straight line through A and B.



The region enclosed by the graph of $y = f(x)$ and the straight line through A and B is shaded.
Find the area of this shaded region. 3 marks

Let $g: R \to R,\ g(x) = x^3 + (k-4)x + 2k,\ k \in R$.

e. Find the value of k for which $f(x) = g(x)$ for all x. 1 mark

f. Find all solutions to $g'(x) = 0$ in terms of k. 1 mark

g. Find the value of k for which g has only one stationary point. 1 mark

h. Find the values of k for which g has three x-intercepts. 3 marks

Question 2 (9 marks)

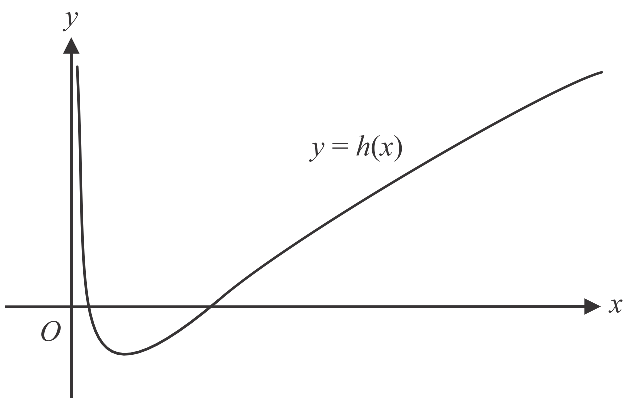
Consider the functions $f(x)=x^2-1$ and $g(x)=\log_e(x)$ where both functions have a maximal domain.
Let h be the function $h:D\to R,\ h(x)=(f\circ g)(x)$.

a. **i.** Explain why h exists. 1 mark

ii. Find D. 1 mark

iii. Find the range of h. 1 mark

The graph of $h: D \to R, h(x) = [\log_e(x)]^2 - 1$ is shown below.



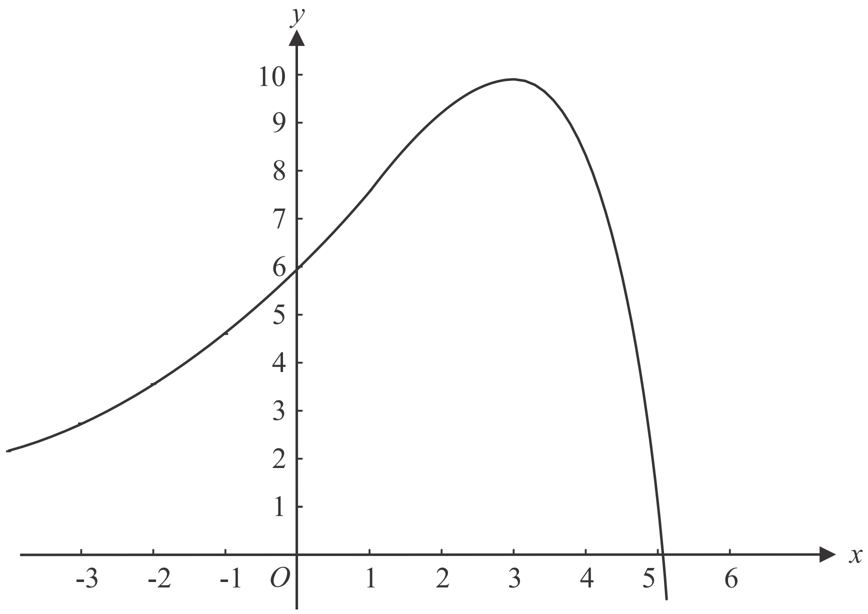
b. Find the area enclosed by the graph of h and the x-axis. 3 marks

c. Find the acute angle that the tangent to h at $x = e$ makes with the positive direction of the horizontal axis. Give your answer correct to the nearest degree. 1 mark

d. A tangent to the graph of *h* has a gradient of $\log_e(2)$ and an *x*-intercept at the point $(a, 0)$ where $a > e$.
Find the *y*-intercept of this tangent. 2 marks

Question 3 (12 marks)

Consider the function $f: R \to R,\ f(x) = (5-x)e^{\frac{x}{2}} + 1$.
Part of the graph of $y = f(x)$ is shown below.



a. Find $f'(x)$. 1 mark

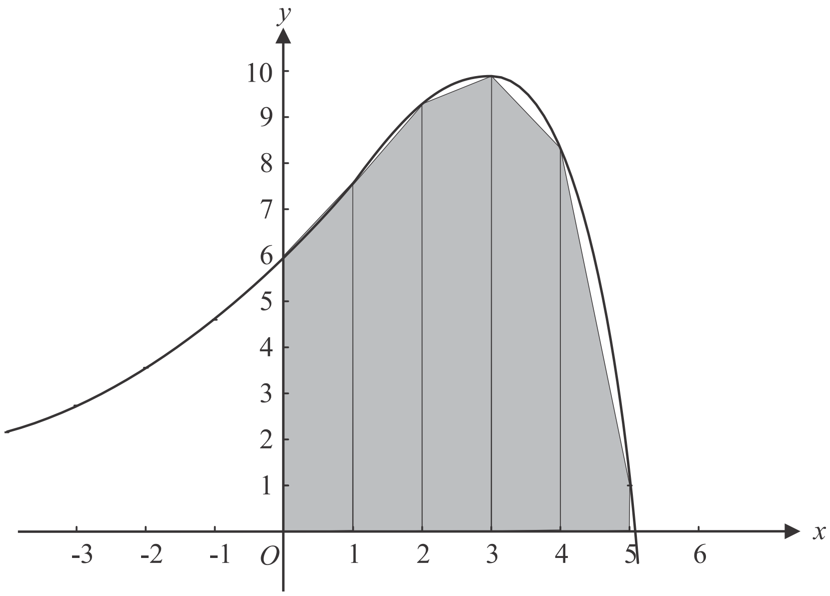
b. Find the values of x for which the **gradient function** of f is strictly increasing. 1 mark

Let $g: (-\infty, a] \to R,\ g(x) = f(x)$.

c. **i.** Find the value of a such that g has a maximal domain, given that the inverse function g^{-1} exists. 1 mark

ii. Find the domain of g^{-1}. 1 mark

Five trapeziums of equal width are used to approximate the area between the function $f(x)=(5-x)e^{\frac{x}{2}}+1$ and the x-axis from $x=0$ to $x=5$. The top corners of each trapezium lie on the graph of $y=f(x)$ as shown in the graph below.



d. Find the total area of the five trapeziums. Give your answer in square units correct to one decimal place. 2 marks

e. One of the five trapezium, over the interval $x \in [p, q]$, overestimates the actual area between the graph of $y=f(x)$ and the x-axis for that interval.

i. Write down the values of p and q. 1 mark

ii. Explain why this trapezium overestimates the actual area. Include a reference to the gradient function $f'(x)$ in your explanation. 1 mark

Newton's method is used to find an approximate *x*-intercept of *f*, with an initial estimate of $x_0 = 5$.

f. Show that $x_1 = 5 + e^{-\frac{5}{2}}$. 2 marks

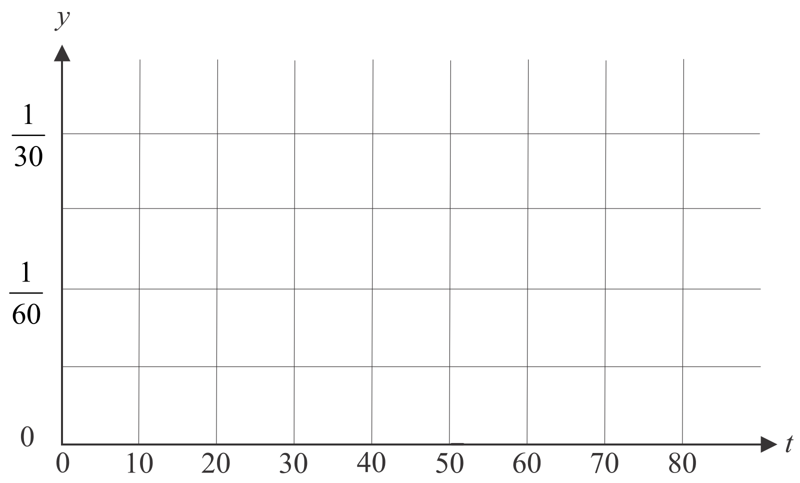
g. Find the minimum value of *n*, where $n = 0, 1, 2, 3...$ for which x_n equals the actual value of the *x*-intercept of the graph of *f* correct to four decimal places. 2 marks

Question 4 (15 marks)

The store manager of a large store in a shopping centre works overtime on some of her shifts.
The continuous random variable *T* models the overtime, *t*, in minutes, that she works each shift, and has a probability density function *f* where

$$f(t)=\begin{cases}\dfrac{1}{30} & 0\le t<10\\ \dfrac{1}{1200}(50-t) & 10\le t\le 50\\ 0 & \text{elsewhere}\end{cases}$$

a. **i.** Sketch the graph of *f* on the set of axes below. 3 marks



ii. Find $\Pr(5\le T\le 30)$. 2 marks

iii. Find the value of *k* such that 90% of the overtime shifts that the store manager has to do, will be less than *k* minutes. Express your answer correct to one decimal place. 2 marks

iv. The mean overtime worked by the store manager each shift is $\frac{155}{9}$ minutes.

Find the standard deviation of the overtime worked by the store manager each shift in minutes, correct to one decimal place. 1 mark

b. Store data shows that the probability that a shopper entering the store will make a purchase is 0.72. Assume that the probability of one shopper making a purchase is independent of another shopper making a purchase.
Find the probability, correct to three decimal places, that of four randomly chosen shoppers who enter the store

i. all will make a purchase. 1 mark

ii. more than one will make a purchase. 1 mark

iii. exactly three will make a purchase given that more than one will make a purchase. 2 marks

c. Shoppers can pay for a bag to put their purchases in. The store manager wants to determine how many bags might be needed by estimating the population proportion, *p*, of shoppers who pay for a bag.
In a random sample of 100 shoppers, 48 paid for a bag.
Let $\hat{P}$ be the random variable representing the proportion of shoppers who paid for a bag, in random samples of 100 shoppers.

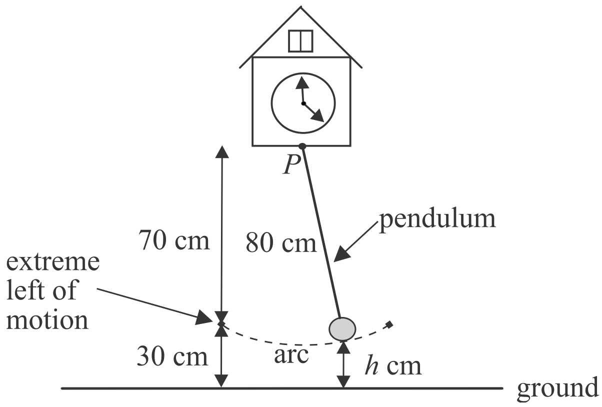
i. Find the standard deviation of $\hat{P}$, correct to four decimal places. 1 mark

ii. Find an approximate 95% confidence interval for *p* using the sample proportion of shoppers who paid for a bag, correct to four decimal places. 1 mark

iii. By increasing the number of shoppers in the sample, the store manager can reduce the width of the confidence interval and therefore get a better estimate of the number of bags needed.
If $\hat{p} = 0.48$, how many shoppers would be required in samples in order to reduce the width of the confidence interval found in part **ii**. by 50%? 1 mark

Question 5 (10 marks)

The pendulum on a clock is made up of a straight rod with a circular weight attached to one end.
The other end of the rod is fixed to the clock at point *P* which is 100 cm vertically above the ground.
The pendulum (including the rod and weight) has a length of 80 cm and swings back and forth so that the weight follows the path of an arc.
When the weight is at the extreme left or right of its motion, it is 30 cm vertically above the ground as shown in the diagram below.



The height of the weight above the ground, *h*, can be modelled by the function

$$h(t) = 25 + a\cos(\pi t), \quad t \geq 0$$

where *t* is the time, in seconds, after the pendulum starts its motion, *a* is a positive constant and *h* is measured in centimetres.

a. Explain why $a = 5$. 1 mark

The movement of the pendulum, from one side to the other, regardless of direction, is referred to as a 'swing'.

b. How many swings does the pendulum make each minute? 1 mark

c. Find the proportion of time during each swing when the weight is no more than 27.5 cm above the ground. Give your answer to the nearest whole percent. 2 marks

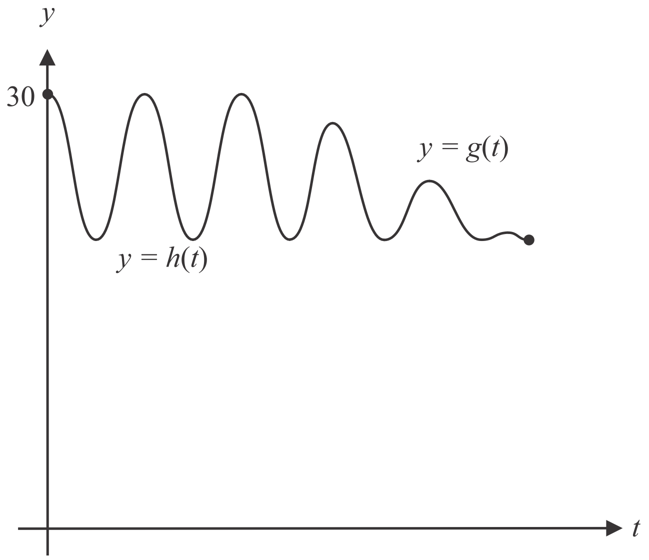
d. Find the average height of the weight above the ground during the first half second of motion. Give your answer, in cm, correct to two decimal places. 2 marks

After five seconds, the mechanism which maintains the pendulum's swing malfunctions causing a gradual decrease in the extent of each successive swing.
The height of the weight above the ground after the malfunction, g, can be modelled by the function

$$g(t) = k + (10 - t)(\cos(\pi t) + 1), \quad 5 < t \leq 10$$

where t is the time, in seconds, after the pendulum first started its motion, g is measured in centimetres and k is a constant.

The graph of h, for $0 \leq t \leq 5$, and of g for $5 < t \leq 10$, have a smooth and continuous join as shown in the diagram below.



e. Show that $k = 20$. 1 mark

f. Find the difference in the height reached by the weight between one side and the other, of the next swing that occurs after the malfunction. Give your answer, in cm, correct to two decimal places. 3 marks

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$		$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int \frac{1}{x}dx=\log_e(x)+c,\ \ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$		$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$		$\int \cos\ (ax)\,dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+2f(x_{n-1})+f(x_n)\right]$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\operatorname{var}(X) = \sigma^2 = \mathrm{E}((X - \mu)^2) = \mathrm{E}(X^2) - \mu^2$
binomial coefficient	$\dbinom{n}{x} = \dfrac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\,(x - \mu)^2\, p(x)$
binomial	$\Pr(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$	$\sigma^2 = np(1-p)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\operatorname{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

MATHEMATICAL METHODS

TRIAL EXAMINATION 2

MULTIPLE - CHOICE ANSWER SHEET

STUDENT NAME:..

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ▬ C D E

The answer selected is B. Only one answer should be selected.

1. A B C D E
2. A B C D E
3. A B C D E
4. A B C D E
5. A B C D E
6. A B C D E
7. A B C D E
8. A B C D E
9. A B C D E
10. A B C D E
11. A B C D E
12. A B C D E
13. A B C D E
14. A B C D E
15. A B C D E
16. A B C D E
17. A B C D E
18. A B C D E
19. A B C D E
20. A B C D E