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YEAR 12 *Trial Exam Paper*

2023

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1a.

Worked solution

$$\frac{dy}{dx}=\frac{(3x+5)(2)-2x(3)}{(3x+5)^2}$$
$$=\frac{6x+10-6x}{(3x+5)^2}$$
$$=\frac{10}{(3x+5)^2}$$

Mark allocation: 1 mark

- 1 answer mark for correctly applying the quotient rule to find the derivative:

  $$\frac{dy}{dx}=\frac{10}{(3x+5)^2}$$

Tips

- *You may find it helpful to begin by writing the quotient rule.*
- *There is no need to expand brackets in an answer unless this will lead to the expression being able to be simplified. However, you will not lose a mark for expanding brackets, provided this is done correctly.*

Question 1b.

Worked solution

Let $y = 3 - e^{1-2x}$. Use the chain rule, with $u = 1 - 2x$ and $y = 3 - e^{u}$.

$$f'(x)=\frac{dy}{dx}$$
$$=\frac{dy}{du}\frac{du}{dx}$$
$$=(-e^{u})(-2)=2e^{1-2x}$$

$$f'\left(\frac{1}{2}\right)= 2e^{1-2\left(\frac{1}{2}\right)}$$
$$= 2e^{0} = 2$$

Mark allocation: 2 marks

- 1 answer mark for applying the chain rule to find the derivative: $f'(x) = 2e^{1-2x}$
- 1 answer mark for correctly evaluating the derivative at $x = \frac{1}{2}$: $f'\left(\frac{1}{2}\right) = 2$

Tip

- *You may find it helpful to begin by writing the chain rule.*

Question 2

Worked solution

Solve the equations simultaneously.

Substitute $y = x - 7$ into the first equation:

$$ax + y = 3$$
$$ax + x - 7 = 3$$
$$ax + x = 3 + 7$$
$$x(a + 1) = 10$$
$$x = \frac{10}{a+1}$$

$\therefore x > 0$ if $a + 1 > 0$

$\therefore a > -1$

Now substitute $x = \dfrac{10}{a+1}$ into $y = x - 7$:

$$y = \frac{10}{a+1} - 7 = \frac{10 - 7(a+1)}{a+1}$$
$$= \frac{3-7a}{a+1}$$

From earlier working (above), we know that $a + 1 > 0$, so $y > 0$ if:

$$3 - 7a > 0$$
$$a < \frac{3}{7}$$

$$\therefore -1 < a < \frac{3}{7}$$

Mark allocation: 3 marks

- 1 method mark for expressing x in terms of a (or y in terms of a)
- 1 answer mark for obtaining $a > -1$
- 1 answer mark for obtaining $a < \dfrac{3}{7}$

Tip

- *In the fraction* $x = \dfrac{10}{a+1}$ *the numerator* (10) *is always positive, so x will only be positive if the denominator is also positive. The same idea is used to ensure that* $y = \dfrac{3-7a}{a+1}$ *is always positive.*

Question 3a.

Worked solution

$E(X) = 0 \times 0.6 + 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.1$
$\quad = 0.8$

Mark allocation: 1 mark

- 1 answer mark for finding the correct answer: $E(X) = 0.8$

Question 3b.

Worked solution

Let N be the number of walks on which one kangaroo is seen.

$$\Pr(N=2) = \binom{3}{2}(0.1)^2(0.9)$$
$$= 3(0.01)(0.9)$$
$$= 0.027 \quad \left(\text{or } \frac{27}{1000}\right)$$

An alternative method would be to list all the possibilities for seeing one kangaroo on two out of three walks.

If O = a walk on which one kangaroo was seen:

$\Pr(X = \text{one kangaroo on two walks}) = \Pr(OOO') + \Pr(OO'O) + \Pr(O'OO)$

$$= 0.1 \times 0.1 \times 0.9 + 0.1 \times 0.9 \times 0.1 + 0.9 \times 0.1 \times 0.1$$
$$= 3(0.1)^2(0.9)$$
$$= 3(0.01)(0.9)$$
$$= 0.027 \left(\text{or } \frac{27}{1000}\right)$$

Mark allocation: 2 marks

- 1 method mark for relevant calculations or tree diagram
- 1 answer mark for finding the correct answer: 0.027 or $\frac{27}{1000}$

Tip

- *You should be familiar with the formula for calculating binomial probabilities,* $\Pr(X = x) = \binom{n}{x}(p)^x(1-p)^{n-x}$. *It is on the formula sheet.*

Question 3c.

Worked solution

$$\begin{aligned}
&\Pr(\text{total} = 3 | \geq 1 \text{ on both walks})\\
&= \frac{\Pr(\text{total} = 3 \cap X \geq 1 \text{ on both walks})}{\Pr(X \geq 1 \text{ on both walks})}\\
&= \frac{\Pr(X=1)\times\Pr(X=2)+\Pr(X=2)\times\Pr(X=1)}{\left[\Pr(X\geq 1)\right]^2}\\
&= \frac{0.1\times 0.2+0.2\times 0.1}{0.4^2}\\
&= \frac{0.04}{0.16}\\
&= \frac{1}{4} \quad (\text{or } 0.25)
\end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for setting up a suitable conditional probability equation.
- 1 method mark for considering ways to obtain a total of 3, given at least one kangaroo on both walks.
- 1 answer mark for the correct answer: $\frac{1}{4}$ (or 0.25)

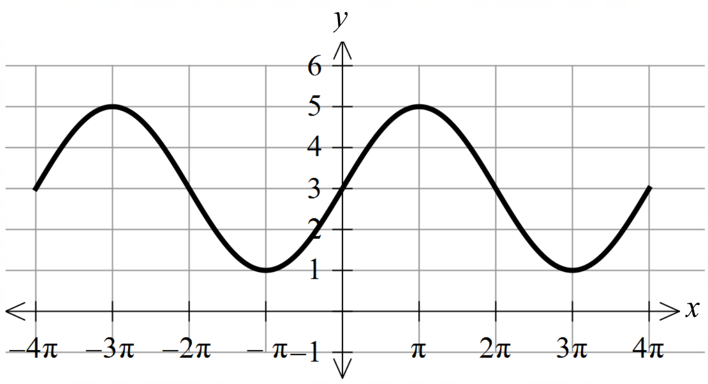
Tips

- *Remember that the conditional probability formula,* $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$, *is on the formula sheet.*
- *It can sometimes be tricky writing probability expressions using symbols, so setting up an initial probability expression using words may be helpful for understanding a problem.*
- *The only way to see a total of three kangaroos, when there is at least one kangaroo seen on each walk, is for there to be one kangaroo seen on one walk and two kangaroos on the other walk. Remember this can happen in either order:* 1, 2 *or* 2, 1.
- *For Mathematical Methods examination 1, you need to be familiar with multiplying and dividing decimals.*

Question 4a.i.

Worked solution

Consider the graph of f and a translation of c units to the right that will give a y-intercept at $(0, 3)$.



From the graph: $c = 2\pi$

Mark allocation: 1 mark

- 1 answer mark for finding $c = 2\pi$

Tip

- *Sketch a quick graph of* f*. We know that* $c > 0$ *so the graph of* f *is translated to the right. Look for the smallest distance to the right that will produce a y-intercept at* $(0, 3)$.

Question 4a.ii.

Worked solution

$c = 2k\pi$, where $k \in Z$

Mark allocation: 1 mark

- 1 answer mark for $c = 2k\pi$, where $k \in Z$

Tip

- *The graph will have a y-intercept at* $(0, 3)$ *if it is translated left or right by any multiple of* 2π*, so a general solution is required.*

Question 4b.

Worked solution

$$\text{Average value} = \frac{1}{\pi - 0}\int_0^{\pi} 3 + 2\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{\pi}\left[3x - 4\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(3\pi - 4\cos\left(\frac{\pi}{2}\right) - 0 + 4\cos(0)\right)$$

$$= \frac{1}{4}(3\pi + 4) \qquad \left(\text{or } 3 + \frac{4}{\pi}\right)$$

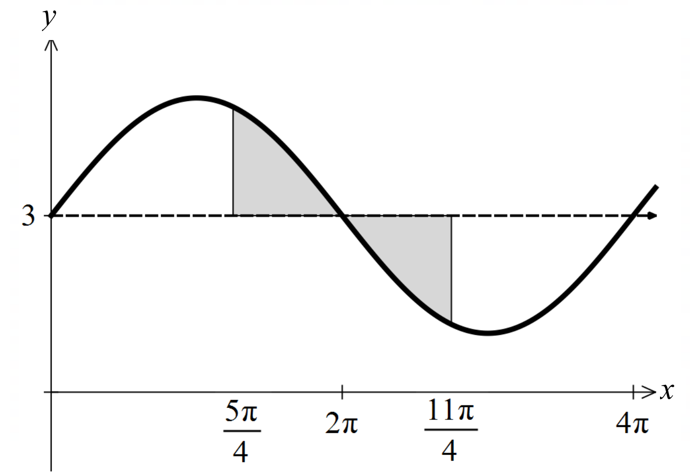
Mark allocation: 2 marks

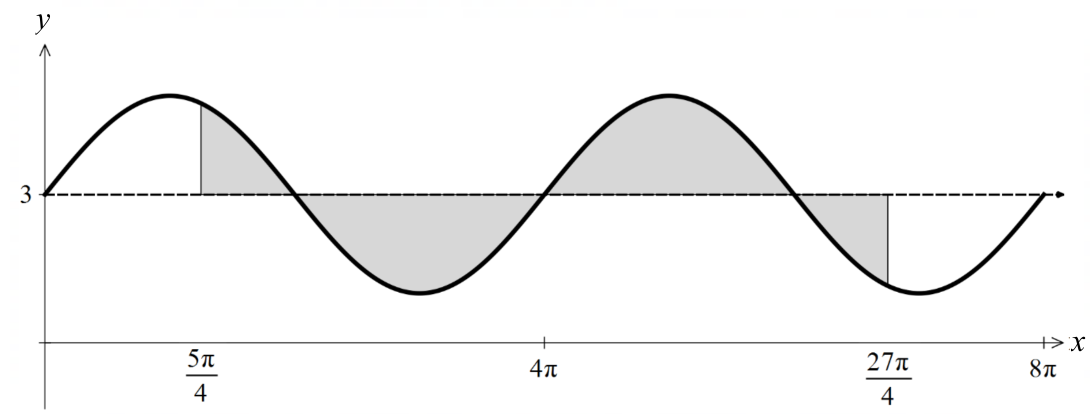
- 1 answer mark for setting up the average value equation
- 1 answer mark for finding the correct answer: $\frac{1}{\pi}(3\pi + 4)$ or $3 + \frac{4}{\pi}$

Question 4c.

Worked solution

Over any interval for which the area between the graphs of $y = f(x)$ and $y = 3$ has equal portions lying above the line $y = 3$ and lying below the line $y = 3$, the average value of f is 3.





Consideration of the symmetry properties of the graphs shows that the following two intervals have an average value of 3:

- the interval from $x = \frac{5\pi}{4}$ to $x = \frac{11\pi}{4}$
- the interval from $x = \frac{5\pi}{4}$ to $x = \frac{27\pi}{4}$.

$\therefore b = \frac{11\pi}{4}, \frac{27\pi}{4}$

Mark allocation: 2 marks

- 1 answer mark for finding $b = \frac{11\pi}{4}$
- 1 answer mark for finding $b = \frac{27\pi}{4}$

Question 5a.

Worked solution

$$A = \frac{1}{2}(1)[f(1) + f(2) + f(2) + f(3)]$$

$$= \frac{1}{2}(e^1 - 1 + 2e^2 - 2 + e^3 - 1)$$

$$= \frac{1}{2}(e + 2e^2 + e^3 - 4) \text{ square units}$$

Mark allocation: 2 marks

- 1 method mark for setting up the area equation
- 1 answer mark for finding the correct answer: $\frac{1}{2}(e + 2e^2 + e^3 - 4)$

Question 5b.

Worked solution

$e^x - 1 = m - e^x$

$2e^x = m + 1$

$e^x = \dfrac{m+1}{2}$

$x = \log_e\left(\dfrac{m+1}{2}\right)$

$\therefore k = \log_e\left(\dfrac{m+1}{2}\right)$

Mark allocation: 1 mark

- 1 answer mark for all steps shown clearly and the final answer written as $k = \ldots$

Tip

- *For a question that asks you to 'show that', clearly show all the steps used.*

Question 5c.

Worked solution

$$A = \int_0^k \left(m - e^x - (e^x - 1)\right) dx$$

$$= \int_0^k \left(m + 1 - 2e^x\right) dx$$

$$= \left[(m+1)x - 2e^x\right]_0^k$$

$$= (m+1)k - 2e^k + 2e^0$$

$$= (m+1)k + 2 - 2e^k$$

Substitute $k = \log_e\left(\frac{m+1}{2}\right)$, and $A = 6\log_e(3) - 4$

$$\left(m+1\right)\log_e\left(\frac{m+1}{2}\right) - 2e^{\log_e\left(\frac{m+1}{2}\right)} + 2 = 6\log_e(3) - 4$$

$$\left(m+1\right)\log_e\left(\frac{m+1}{2}\right) - 2\left(\frac{m+1}{2}\right) + 2 = 6\log_e(3) - 4$$

$$\left(m+1\right)\log_e\left(\frac{m+1}{2}\right) - m + 1 = 6\log_e(3) - 4$$

By equating parts:

$m + 1 = 6 \qquad \left(\text{or use } \frac{m+1}{2} = 3, \text{ or } -m + 1 = -4\right)$

$m = 5$

$\therefore\ k = \log_e\left(\frac{5+1}{2}\right)$

$= \log_e(3)$

Mark allocation: 3 marks

- 1 answer mark for setting up the integral expression for the area:
  $A = \int_0^k \left(m - e^x - (e^x - 1)\right) dx$
- 1 answer mark for finding the correct value of m: 5
- 1 consequential answer mark for the value of k (correct for the value of m found)

Question 6a.

Worked solution

$f(-1) = \log_e(1)$

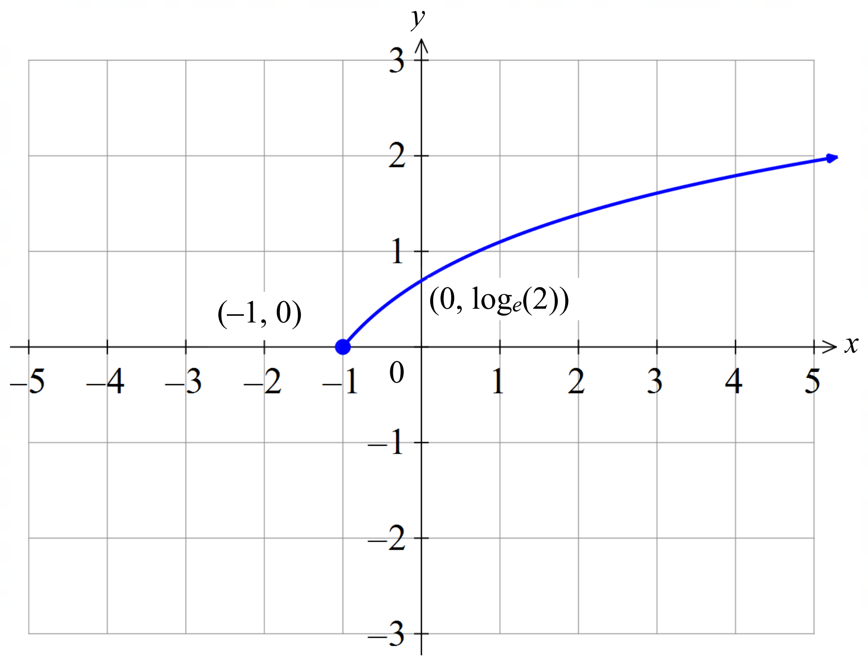
$= 0$

Mark allocation: 1 mark

- 1 mark for finding the correct answer: 0

Question 6b.

Worked solution



Mark allocation: 2 marks

- 1 answer mark for correct graph shape drawn over the domain $[-1, \infty)$. The graph should cross the y-axis between 0 and 1.
- 1 answer mark for axial intercepts labelled correctly.

Question 6c.i.

Worked solution

$f \circ g$ exists if ran $g \subseteq$ dom f

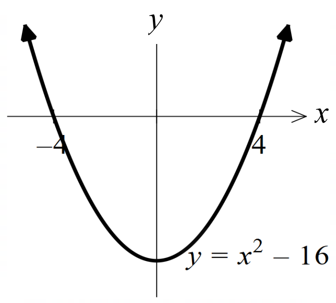
dom f : $[-1, \infty)$

$\therefore$ ran $g \subseteq [-1, \infty)$

$g(x) \geq -1$

$x^2 - 17 \geq -1$

$x^2 - 16 \geq 0$



First solve: $x^2 - 16 = 0$

$x = \pm 4$

Then consider where the graph of $y = x^2 - 16$ is above the x-axis:

$\therefore x \leq -4 \cup x \geq 4$

Since the domain is $[d, \infty)$:

$d = 4$

Mark allocation: 2 marks

- 1 method mark for identifying ran $g \subseteq [-1, \infty)$
- 1 answer mark for finding the correct answer: $d = 4$

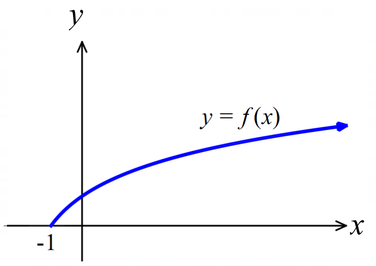
Tips

- *You need to be careful when solving inequalities. Using a graphical approach to these is helpful.*
- *For* $f \circ g$ *to exist, the maximum domain of g is* $(-\infty, -4] \cup [4, \infty)$. *Since the question states that g has domain* $[d, \infty)$, *we require* $d = 4$.

Question 6c.ii.

Worked solution

When $d = 4$, ran $g = [-1, \infty)$ with these numbers becoming the 'input' of f.
$\therefore$ Consider the graph of f when the domain is $[-1, \infty)$.



$\therefore$ ran $f \circ g = [0, \infty)$.

Mark allocation: 1 mark

- 1 answer mark for finding $[0, \infty)$

Tips

- ran $g = [-1, \infty)$, *which becomes the input for f, so we calculate the range of f when $x \in [-1, \infty)$. We know from* **part a** *that $f(-1) = 0$. This is useful here.*
- *Results from the earlier parts of questions are often useful for later parts of the same question.*

Question 6d.

Worked solution

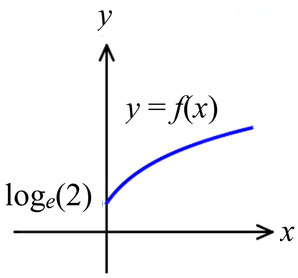
ran $h = [0, \infty)$

Work out ran f, when the input for f is $[0, \infty)$.

Consider the graph of f.

$f(0) = \log_e(2)$

ran $f \circ h = [\log_e(2), \infty)$



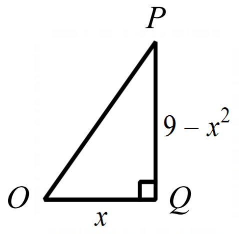
Mark allocation: 1 mark

- 1 answer mark for finding $[\log_e(2), \infty)$

Question 7a.

Worked solution

$$A = \frac{1}{2}x\left(9 - x^2\right)$$
$$= \frac{9}{2}x - \frac{1}{2}x^3$$



Mark allocation: 1 mark

- 1 answer mark for showing working that led to $A = \frac{9}{2}x - \frac{1}{2}x^3$

Tip

- *The width, OX, of the triangle is x units and the height, QP, of the triangle is f(x) units. So, height = f(x) =* $9 - x^2$.

Question 7b.

Worked solution

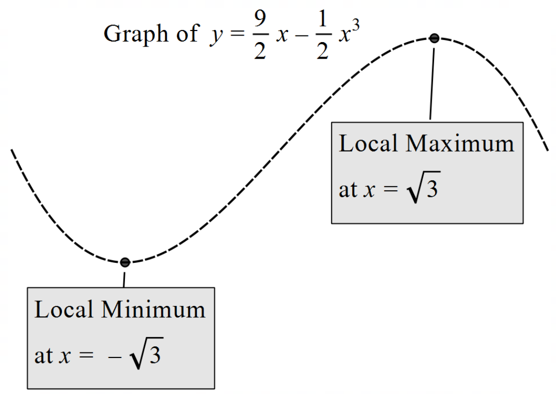
$$A = \frac{9}{2}x - \frac{1}{2}x^3$$

For turning points, solve $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = \frac{9}{2} - \frac{3}{2}x^2 = 0$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

Consideration of the shape of the graph of $y = \frac{9}{2}x - \frac{1}{2}x^3$ indicates the local maximum is at

$x = \sqrt{3}$, which is within the domain [1, 2].



For $x \in [1, 2]$, max area will occur when $x = \sqrt{3}$

$$\text{Max area} = \frac{9}{2}\sqrt{3} - \frac{1}{2}\left(\sqrt{3}\right)^3$$
$$= \frac{9}{2}\sqrt{3} - \frac{1}{2}\left(3\sqrt{3}\right)$$
$$= 3\sqrt{3} \text{ square units}$$

For $x \in [1, 2]$, the minimum area will occur at an endpoint, so compare values of $A(1)$ and $A(2)$:

$$A(1) = \frac{9}{2} - \frac{1}{2} = 4$$

$$A(2) = \frac{9(2)}{2} - \frac{2^3}{2} = 5$$

$\therefore$ Minimum area = 4 square units

Mark allocation: 3 marks

- 1 answer mark for finding the x value when the derivative equals 0: $x = \sqrt{3}$
- 1 answer mark for finding the correct maximum area: $3\sqrt{3}$
- 1 answer mark for finding the correct minimum area: 4

Tips

- *Maximum/minimum values may occur at a turning point or at an endpoint. Be aware that some questions will require consideration of the endpoints.*
- $A = \frac{9}{2}x - \frac{1}{2}x^3$ *is a negative cubic function. You should be familiar with the general shape of the graph of this type of function.*
- *Since* $\sqrt{1} < \sqrt{3} < \sqrt{4}$, *we know that* $1 < \sqrt{3} < 2$ *and, therefore,* $x = \sqrt{3}$ *is within the domain* [1, 2]

Question 8a.

Worked solution

$$f\left(-\frac{v}{6}\right) = \sqrt{6\left(-\frac{v}{6}\right) + v} - 3$$
$$= -3$$

Mark allocation: 1 mark

- 1 answer mark for finding the correct answer: –3

Question 8b.

Worked solution

Solve $f(x) = x$:

$$\sqrt{6x + v} - 3 = x$$
$$\sqrt{6x + v} = x + 3$$
$$6x + v = (x + 3)^2$$
$$6x + v = x^2 + 6x + 9$$
$$x^2 = v - 9$$
$$x = \pm\sqrt{v - 9}$$

Mark allocation: 2 marks

- 1 method mark for setting up the equation $f(x) = x$
- 1 answer mark for finding the correct answer: $x = \pm\sqrt{v - 9}$

Tips

- *Theoretically $f(x) = f^{-1}(x)$ could be solved to find any points of intersection; however this is very difficult to solve by hand. The point of intersection also lies on the line $y = x$ and, because it is much easier to solve $f(x) = x$, that is used instead.*
- *There is no reason to discard the negative square root in the answer. So the answer must contain both the positive and negative square roots.*

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Question 8c.

Worked solution

Point/s of intersection occur when $x = \pm\sqrt{v-9}$.

So, points of intersection are possible when $v \geq 9$, although $v = 9$ only gives one point of intersection.

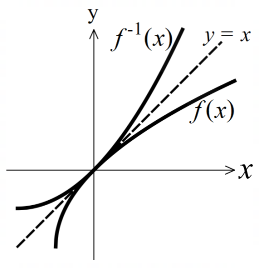
We should also consider whether there is an upper bound for v.

The following possibilities for points of intersection exist:

CASE 1

One point of intersection:

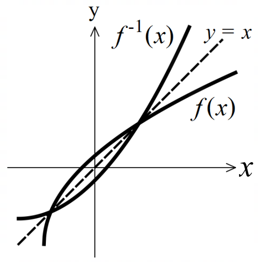
$v = 9$



CASE 2

Two points of intersection:

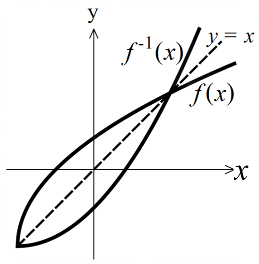
$9 < v < ?$



CASE 3

The last time there are two points of intersection:

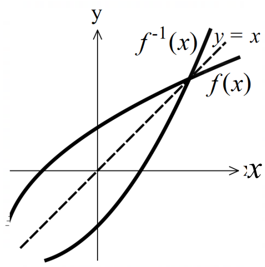
$v = ?$



CASE 4

One point of intersection:

$v > ?$



As the value of v increases from $v = 9$, the last time there are two points of intersection is when the functions intersect at their left-hand endpoints.

The critical value of v is when the endpoints coincide (the Case 3 graph above):

Endpoint of $f(x)$: $\left(-\frac{v}{6}, -3\right)$ Endpoint of $f^{-1}(x)$: $\left(-3, -\frac{v}{6}\right)$

These points coincide when $-\frac{v}{6} = -3$

$v = 18$

$\therefore$ There are two points of intersection if $9 < v \leq 18$.

Mark allocation: 2 marks

- 1 answer mark for finding $v > 9$
- 1 answer mark for finding $v \leq 18$

Tips

- *Different approaches are possible, but consideration of graphs is an efficient method.*
- *Once $v > 9$ is identified, consideration of whether there is an upper bound for v is required.*

END OF WORKED SOLUTIONS