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YEAR 12 *Trial Exam Paper*

2023

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

SECTION A – Multiple-choice questions

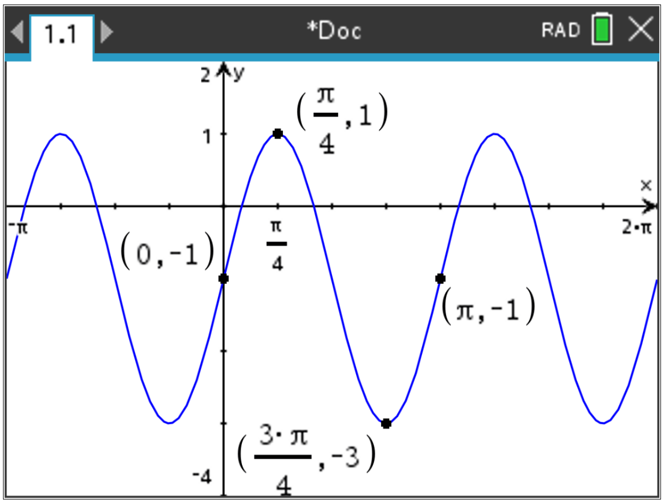
Question	Answer
1	*C*
2	*B*
3	*D*
4	*A*
5	*E*
6	*D*
7	*D*
8	*B*
9	*A*
10	*C*
11	*B*
12	*D*
13	*B*
14	*E*
15	*E*
16	*D*
17	*E*
18	*D*
19	*B*
20	*B*

Question 1

Answer: C

Explanatory notes

The period and range can be observed from a sketch or worked out by hand.



Distractors:

A: Period of $\sin(x)$

B: Period found by incorrectly dividing π by 2 instead of 2π, range of $\sin(x)-2$

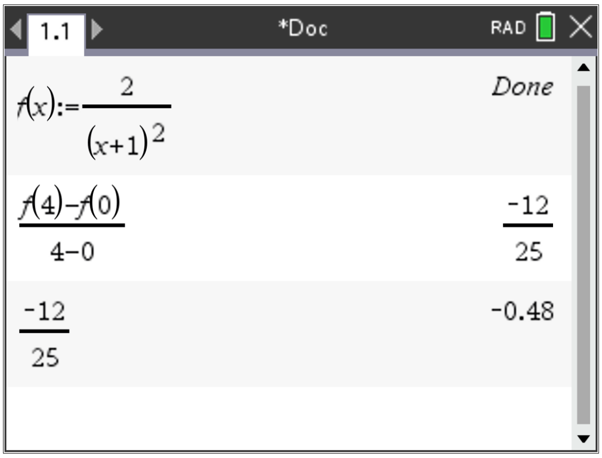
D: Period of $\sin(x)$, range of $\sin(2x)-1$

E: Range of $2\sin(x)$

Question 2

Answer: B

Explanatory notes



Distractors:

A: Average value of f from 0 to 4

C: Instantaneous rate of change at $x=4$

D: Instantaneous rate of change at $x=0$

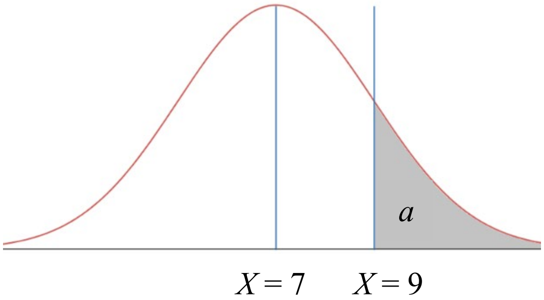
E: Integral of f from 0 to 4

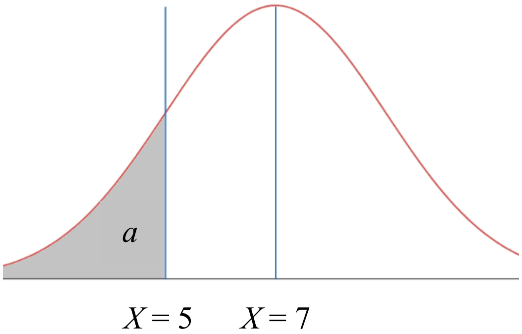
Question 3

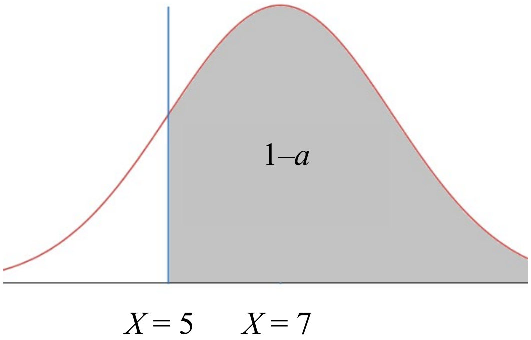
Answer: D

Explanatory notes

Draw a bell curve and mark the mean as 7 and then mark 9 and 5 (since 5 is the same distance from the mean as 9 is). Then the area before 5 is a and the area after 5 is $1-a$.







Distractors:

A: Equals 0.5

B: Equals $1-a$

C: Equals $\frac{1-a}{2}$

E: Equals $1-2a$

Tip

- *Recall that for continuous random variable X,* $\Pr(X > x) = \Pr(X \geq x)$ *since the probability that a continuous random variable will take on an exact value is 0.*

Question 4

Answer: A

Explanatory notes

Y can be prime only if it is the product of 1 and another number.

Y	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Distractors:

B: If 1 is mistakenly taken as a prime

C: Probability of a prime sum (i.e. if *Y* were the sum of the numbers)

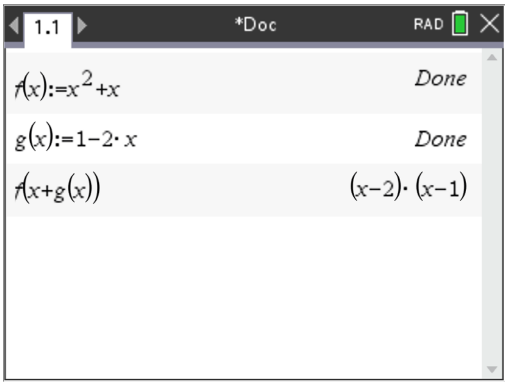
D: Proportion of the numbers 1 to 6 that are prime

E: Probability that *Y* is a square number

Question 5

Answer: E

Explanatory notes



Distractors:

A: Incorrect expansion of $(1-x)^2$

B: Incorrect expansion of $(1-x)(2-x)$

C: Looks similar to $(1-x)(2-x)$, which is a correct answer

D: $g(x+f(x))$

Note: The correct answer, E, is not in the form that naturally arises when completing this problem tech-free.

Question 6

Answer: D

Explanatory notes

If the graph of $y = f(x-2)$ were translated left 2 units, then it would still bound an area of 5 with the x-axis between $x = 1$ and $x = 4$. Hence, $\int_1^4 f(x)\,dx = 5$. Then we have

$$\begin{aligned}\int_1^4 \left(3f(x) - 2x\right)dx &= 3\int_1^4 f(x)\,dx - \int_1^4 2x\,dx \\ &= 3\times 5 - \left[x^2\right]_1^4 \\ &= 15 - [16-1] \\ &= 0\end{aligned}$$

Distractors:

A: The 3 is applied to $f(x)$ and to $2x$

B: The $-2x$ is evaluated as if it were a -2

C: The 3 is ignored

E: The $-2x$ is evaluated as if it were a $-x$

Question 7

Answer: D

Explanatory notes

Think about this purely algebraically: the equation to solve to find the x values of the intersections would be a quartic. Quartics can have up to 4 real solutions.

Question 8

Answer: B

Explanatory notes

This system can be solved on the CAS. Note that the CAS will choose itself which variable to let the parameter be equal to, so the form specified in the multiple-choice responses can be forced by including a fourth equation, $x = k$, in the solve command.

Distractors:

A: Would be correct if the second equation was $+z$

C: Would be correct if the second equation was $+z$ and $+5$

D: Would be correct if the second equation equalled $+5$

E: Correct relationship between x and y, but z is wrong

Question 9

Answer: A

Explanatory notes

Apply the dilation: $2\sqrt{x-2}$

Apply the translation: $2\sqrt{x+1}$

Apply the reflection: $2\sqrt{-x+1}$

Distractors:

B: Dilation and reflection steps applied to the wrong variables

C: Reflected in the x-axis, not the y-axis

D: Translated 3 units right, not left

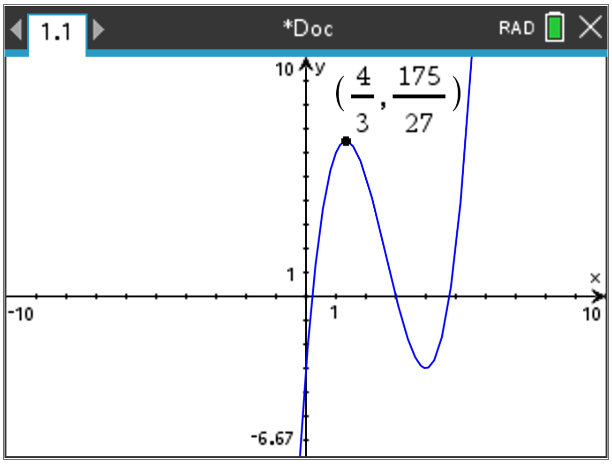
E: Dilation and reflection steps applied to the wrong variables, translation applied to $\frac{x}{2}$ rather than to x

Question 10

Answer: C

Explanatory notes

A sketch of $y = f(x)$ reveals that the maximum value of x, such that the graph to the left of this point is one-to-one, is $x = \frac{4}{3}$.



Distractors

A: The third x-intercept

B: The first x-intercept

D: The second x-intercept

E: The x value of second turning point

Question 11

Answer: B

Explanatory notes

The margin of error for a confidence interval is $z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, which can be seen as part of the confidence interval formula on the formula sheet. Increasing the value of n will mean a larger denominator and therefore a smaller margin of error.

Distractors:

A: The population size has no effect on a confidence interval (N is not part of the formula)

C: The population proportion has no effect on a confidence interval (p is not part of the formula)

D: $\hat{p}$ could make the interval wider or narrower, depending on whether it moves closer to 0.5 or farther away. Therefore, it is not guaranteed that the interval will narrow (e.g. if $\hat{p} < 0.5$, then the interval would widen if it were increased to a number closer to 0.5)

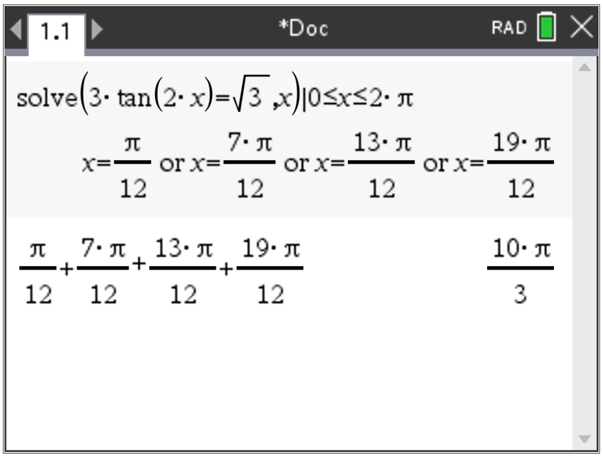
E: $\mathrm{E}\left(\hat{P}\right) = p$ so this is incorrect for the same reason that C is incorrect. Astute students may be able to realise this and therefore eliminate both C and E immediately since they cannot both be true in a multiple-choice question.

Question 12

Answer: D

Explanatory notes

This can be solved quickly with a CAS.



Distractors:

A: The first solution, not the sum of the solutions

B: The sum of the solutions over $[0,\pi]$

C: The sum of the solutions to $3\tan(x) = \sqrt{3}$ over $[0,2\pi]$

E: The sum of the solutions to $\tan(2x) = \sqrt{3}$ over $[0,2\pi]$

Question 13

Answer: B

Explanatory notes

Since the area is `w × sum` and `w` is $\frac{x_n - x_0}{n}$, we require `sum` to be $f(x_0) + f(x_1) + \ldots + f(x_{n-1})$. Thus, in the while loop

- `sum` must be continually adding on the next function value
  - `sum ← sum + f(x)`
- `x` must be increasing to the next left-endpoint
  - Since they all have width `w`, we have `x ← x + w`
- `i` must continually iterate until it equals `n-1`
  - `i ← i + 1`

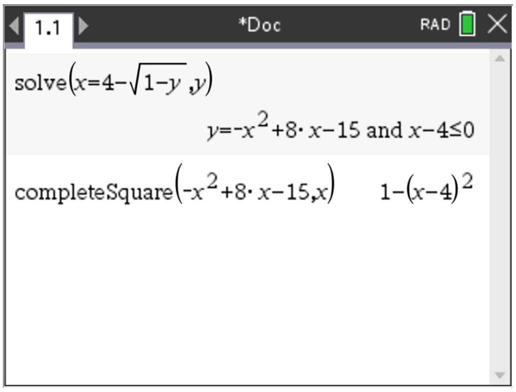
Tip

- *Pseudocode is new in the Study Design this year, so make sure you have lots of examples of it in your bound reference.*

Question 14

Answer: E

Explanatory notes



An alternative method would be sketching the graph of $y = h(x)$ and then sketching each option on the same set of axes to see which looks like a mirror reflection in the line $y = x$.

Distractors:

A: Correct rule, incorrect domain (same domain as h)

B: Correct domain, incorrect rule

C: Correct rule, incorrect domain (domain of h^{-1} if h were the positive root, rather than the negative root)

D: Incorrect rule, incorrect domain

Tip

- *It is important to recognise that $(x-4)^2$ and $(4-x)^2$ are the same. If this is confusing, then you can demonstrate this by expanding each expression.*

Question 15

Answer: E

Explanatory notes

Let X be the number of the customers who order an entrée in a random sample of five customers. Then $X \sim \text{Bi}(5, 0.75)$

$$\begin{aligned}\Pr\left(\hat{P} \geq 0.75\right) &= \Pr(X \geq 0.75 \times 5) \\ &= \Pr(X \geq 3.75) \\ &= \Pr(X \geq 4) \\ &= 0.6328\end{aligned}$$

Distractors:

A: $\Pr\left(\hat{P} = 0.75\right)$

B: $\Pr\left(\hat{P} \geq 0.75\right)$ if $n = 4$

C: $\Pr\left(\hat{P} > 0.75\right)$ if $n = 4$

D: $\Pr\left(\hat{P} = 1\right)$

Question 16

Answer: D

Explanatory notes

The graph of $y = f(x)$ must be a graph which has positive gradients before $x = 0$, has a stationary point of inflection at $x = 0$ (since the gradient before and after $x = 0$ must be positive), and has a gradient which tends to 0 as x tends to ∞. D is the only graph that satisfies all of these criteria.

Distractors:

A: The graph of $y = f''(x)$ (i.e. incorrectly differentiating instead of anti-differentiating)

B: A plausible looking derivative graph apart from asymptotic behaviour

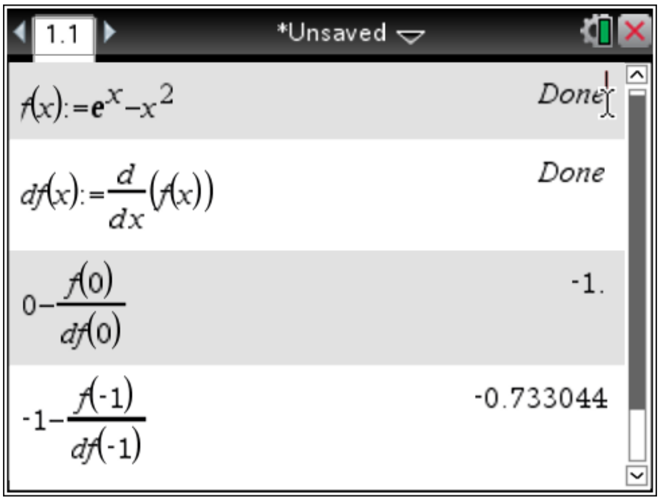
C: A plausible looking anti-derivative graph apart from asymptotic behaviour

E: The graph of $y = -f'(x)$

Question 17

Answer: E

Explanatory notes



Distractors:

A: The value of x_0

B: The value of x_1

C: The value of x_3

D: The actual x-intercept, correct to three decimal places

Tip

- *Newton's method is on the formula sheet. Defining on your CAS both f and its derivative will make implementing the algorithm more efficient.*

Question 18

Answer: D

Explanatory notes

The product rule must be applied with each part of the expression being differentiated with the chain rule, as well. The derivative of $f(2x)$ with respect to x is $f'(2x)\times\frac{d}{dx}(2x)$ and the derivative of $g(x^2)$ is $g'(x^2)\times\frac{d}{dx}(x^2)$.

Distractors:

A: The product of the two derivatives, done separately

B: The product rule applied correctly, but chain rule applied incorrectly

C: The 2 and $2x$ swapped

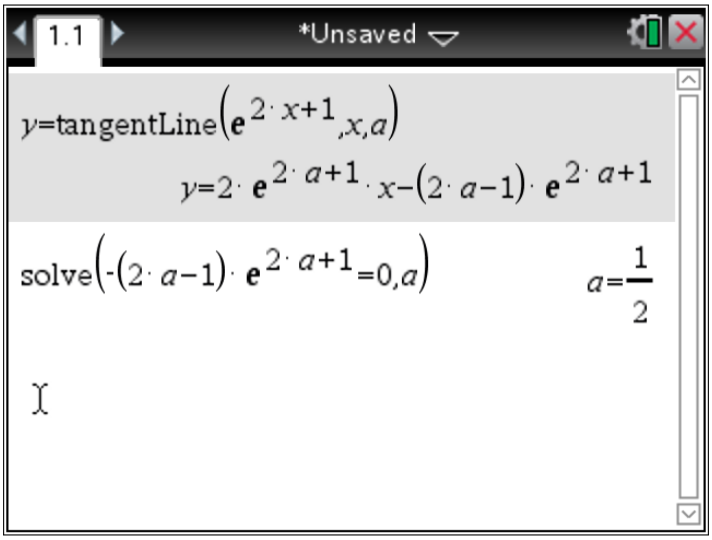
E: The derivative of $\frac{f(2x)}{g(x^2)}$

Question 19

Answer: B

Explanatory notes

The tangent line at $x = a$ can be found in $y = mx + c$ form and then c can be made to equal 0 (or the point $(0,0)$ could be substituted into the equation of the tangent).



Distractors:

A: The y -intercept

C: The y value of correct point

D: The gradient of the tangent line

E: The a value if the equation were $y = e^x$

Tip

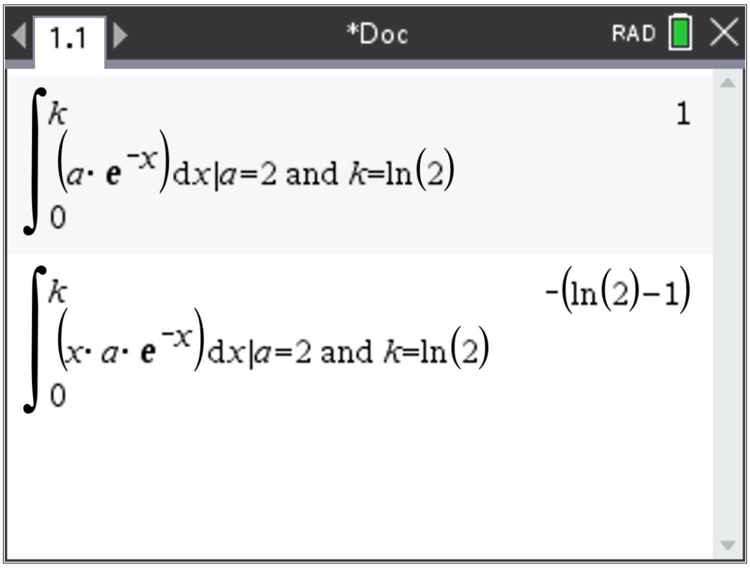
- *Use your CAS to find the equation of a tangent line. This will help you avoid doing a lot of unnecessary algebra by hand.*

Question 20

Answer: B

Explanatory notes

The values of a and k can be found by solving a pair of simultaneous equations, but trial and error is perhaps the easier method here – substitute in each pair to check.



Tip

- *Recall that a probability density function must integrate to 1 over its domain.*

Distractors:

A: Both values incorrect

C: The values of k and a, respectively

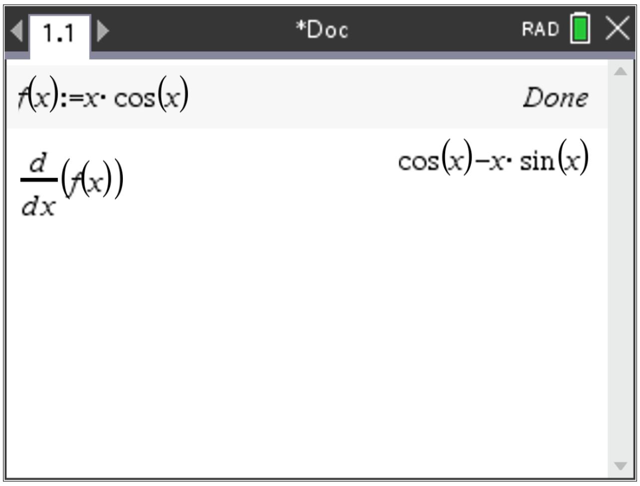
D: Correct value of a, incorrect value of k

E: Correct value of k, incorrect value of a

Question 1a.

Worked solution

$\cos(x) - x\sin(x)$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1b.

Worked solution

$$\begin{aligned} f'(2n\pi) &= \cos(2n\pi) - 2n\pi\sin(2n\pi) \\ &= 1 - 2n\pi(0) \\ &= 1 \end{aligned}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1c.

Worked solution

$\cos(x) \le 1$ for all x.

Multiplying both sides by x will not change the sign when $x \ge 0$.

Therefore, $x\cos(x) \le x$ for $x \ge 0$.

Mark allocation: 1 mark

- 1 mark for a correct and clearly communicated explanation

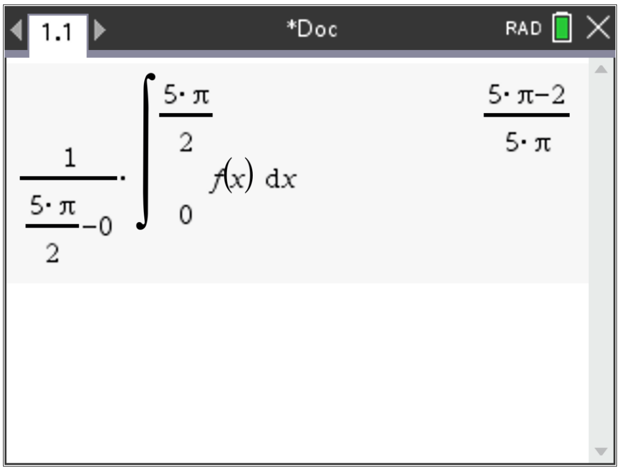
  Note: An algebraic argument is required. The mark should not be awarded for simply sketching $y = x\cos(x)$ and $y = x$ or for showing that the inequality is true for certain values of x.

Question 1d.

Worked solution

$$\frac{1}{\frac{5\pi}{2}-0}\int_0^{\frac{5\pi}{2}} f(x)\,dx = \frac{5\pi-2}{5\pi}$$

$$= 1-\frac{2}{5\pi}$$



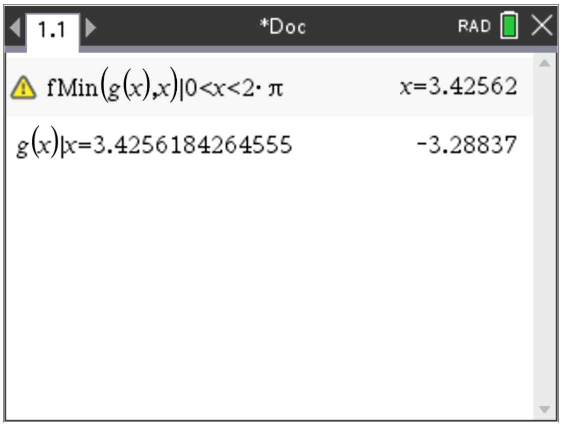
Mark allocation: 2 marks

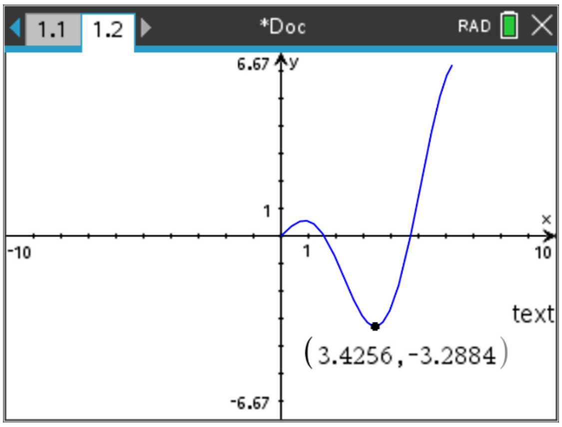
- 1 mark for applying the average value formula
- 1 mark for the correct answer expressed in the correct form (as specified)

Question 1e.

Worked solution

$(3.426, -3.288)$





Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1f.

Worked solution

The average rate of change between two points is the gradient of the chord connecting those two points. By inspection this is 1, since the points intersect with $y = x$.

The average rate of change formula could also be applied: $\frac{g(2\pi) - g(0)}{2\pi - 0} = \frac{2\pi}{2\pi} = 1$

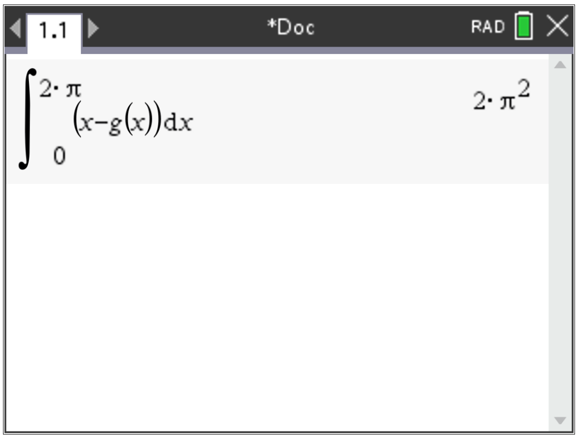
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1g.

Worked solution

$\int_0^{2\pi} (x - g(x))\,dx = 2\pi^2$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 1h.

Worked solution

First, we need a function to represent the distance between $y = g(x)$ and $(\pi, 0)$:

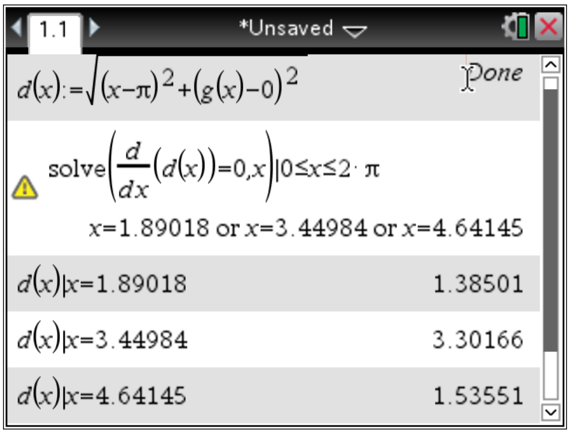
$$d(x) = \sqrt{(x-\pi)^2 + (g(x)-0)^2}$$

Finding the minimum of this function involves checking stationary points and endpoints. By inspection (i.e. simply looking at the graph of $y = g(x)$), we can tell that neither endpoint of $y = g(x)$ will be the closest point on the graph to $(\pi, 0)$. Therefore, we only need to investigate the stationary points of d.

Solving $d'(x) = 0$ for x reveals that there are three stationary points. These points are at $x = 1.89018$, $x = 3.44984$ and $x = 4.64145$.

We need to check the distance at each of these points.

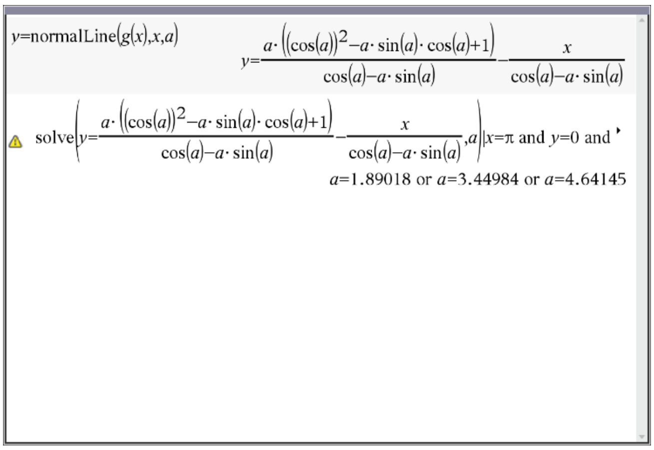
Checking the distance at each of these three points reveals that the minimum distance is 1.385, which occurs at $x = 1.89018$.



Mark allocation: 3 marks

- 1 mark for writing an expression for the distance between the graph of $y = g(x)$ and the point $(\pi, 0)$
- 1 mark for finding the x value of the minimum: $x = 1.890$
- 1 mark for the correct answer for the minimum distance: 1.385

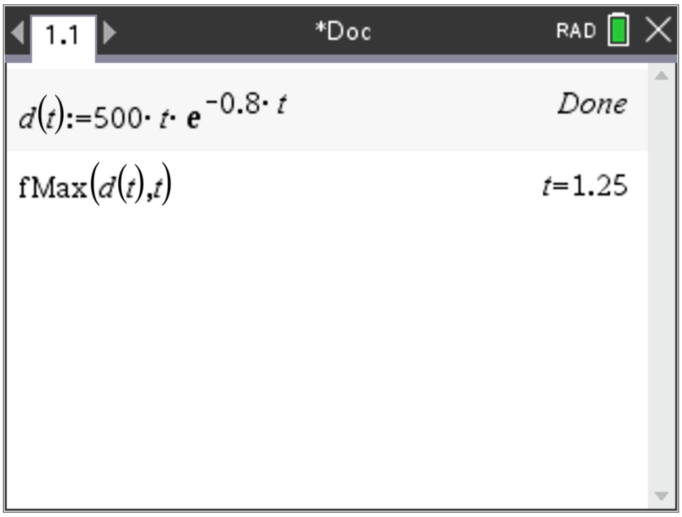
Note: Alternatively, perpendicular lines can be used to find the three non-endpoints to investigate, $x = 1.89018$, $x = 3.44984$ and $x = 4.64145$, by solving for the points on $y = g(x)$ whose normal lines pass through $(\pi, 0)$.



Question 2a.

Worked solution

75 minutes



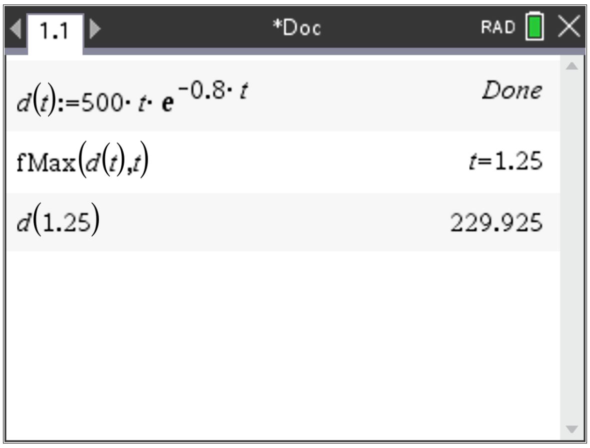
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 2b.

Worked solution

229.9 mg



Mark allocation: 1 mark

- 1 mark for the correct answer

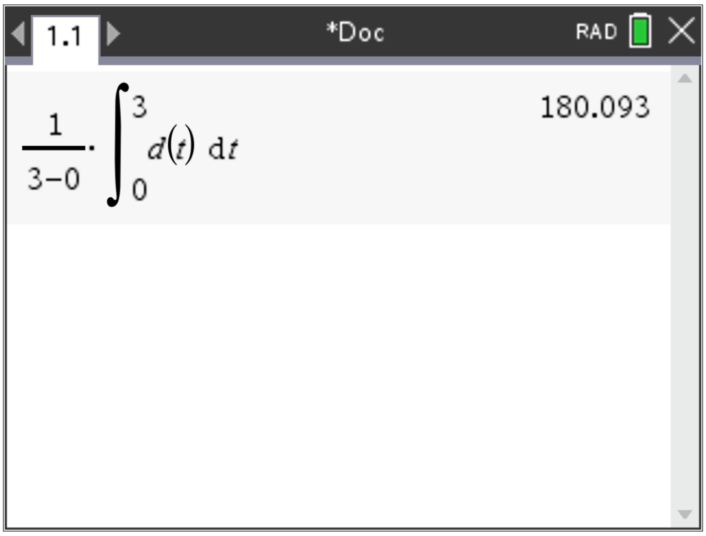
Question 2c.

Worked solution

Average value from $t=0$ to $t=3$ is

$$\frac{1}{3-0}\int_0^3 d(t)\,dt \approx 180$$

Therefore, 180 mg.



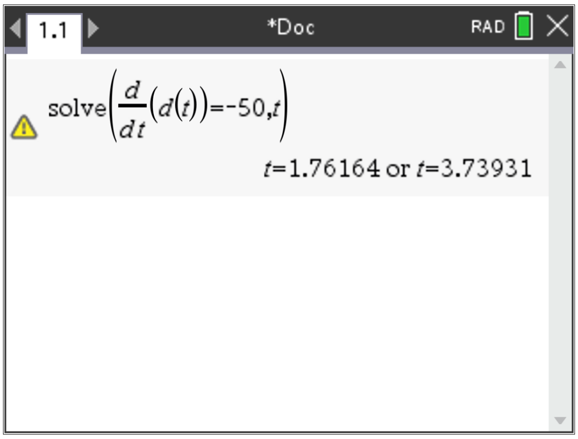
Mark allocation: 2 marks

- 1 mark for applying the average value formula
- 1 mark for the correct answer

Question 2d.

Worked solution

Solving $D'(t)=-50$ gives $t=1.762$ and $t=3.739$.



Mark allocation: 2 marks

- 1 mark for setting $d'(t)$ to equal -50
- 1 mark for the correct answers

Question 2e.

Worked solution

$D(4) \approx 81.52$

Mark allocation: 1 mark

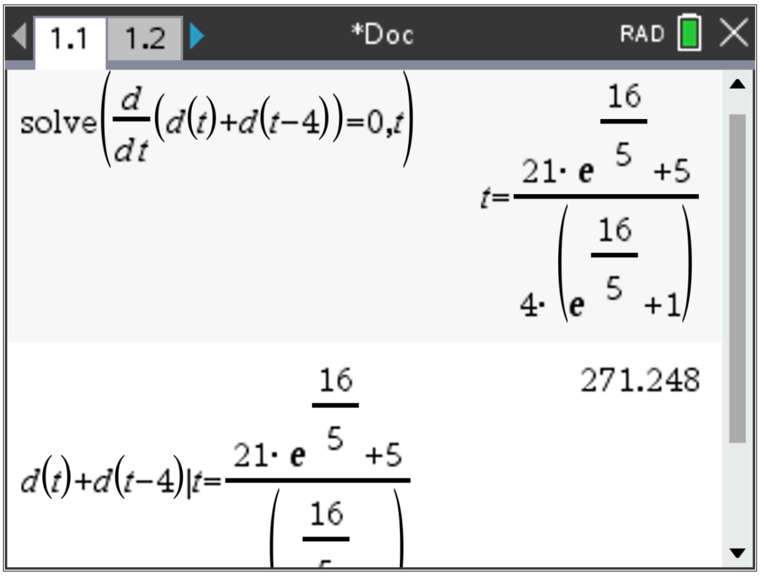
- 1 mark for the correct answer

Question 2f.

Worked solution

Total amount of the drug in the patient's system after the second dose is administered is $d(t)+d(t-4)$.

The maximum value of $d(t)+d(t-4)$ is 271 mg.



Mark allocation: 1 mark

- 1 mark for the correct answer

  Note: Since this question only awards 1 mark, no working out is required and therefore the maximum value can be found by simply sketching or using a maximum-finder command on the CAS.

Question 2g.

Worked solution

The total amount of the drug in the patient's system after the second dose is administered is given by $d(t)+d(t-a)$, where a is the value of t for which the second dose is administered.

The goal is to find the value of a such that the maximum of $d(t)+d(t-a)$ is 330.

Setting the derivative of $d(t)+d(t-a)$ to be 0 gives $t=\dfrac{(4a+5)e^{\frac{4a}{5}}+5}{4\left(e^{\frac{4a}{5}}+1\right)}$.

Therefore, the maximum of $d(t)+d(t-a)$ occurs at this value of t. Substituting this value of t into $D(t)+D(t-a)$ gives the maximum value of $d(t)+d(t-a)$ in terms of a. Solving for when this is equal to 330 gives $a=2.52383$, which is 2 hours and 31 minutes.

$\frac{d}{dt}(d(t)+d(t-a))$	$\left(-400\cdot\left(e^{\frac{4\cdot a}{5}}+1\right)\cdot t+(400\cdot a+500)\cdot e^{\frac{4\cdot a}{5}}+500\right)\cdot e^{\frac{-4\cdot t}{5}}$
$\text{solve}\left(\frac{d}{dt}(d(t)+d(t-a))=0,t\right)$	$t=\frac{(4\cdot a+5)\cdot e^{\frac{4\cdot a}{5}}+5}{4\cdot\left(e^{\frac{4\cdot a}{5}}+1\right)}$
⚠ $\text{solve}(d(t)+d(t-a)=330,a)\mid t=\frac{(4\cdot a+5)\cdot e^{\frac{4\cdot a}{5}}+5}{4\cdot\left(e^{\frac{4\cdot a}{5}}+1\right)}$	$a=-2.52383$ or $a=2.52383$
$60\cdot 0.5238283908503$	31.4297

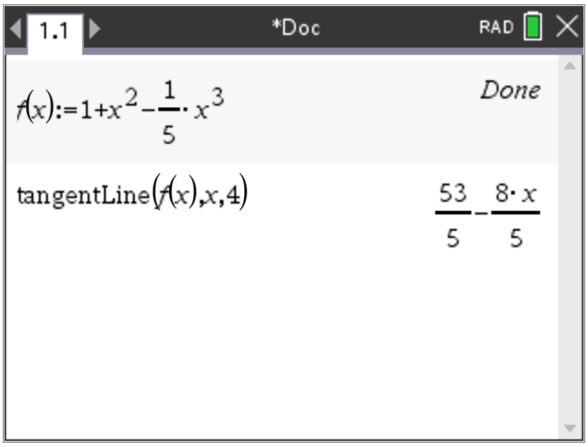
Mark allocation: 3 marks

- 1 mark for setting the derivative of $d(t)+d(t-a)$ to 0
- 1 mark for finding the correct value of t (i.e. $a=2.52383$)
- 1 mark for the correct answer (given as 2 hours, 31 mins)

Question 3a.

Worked solution

$$y=-\frac{8}{5}x+\frac{53}{5}$$



Mark allocation: 1 mark

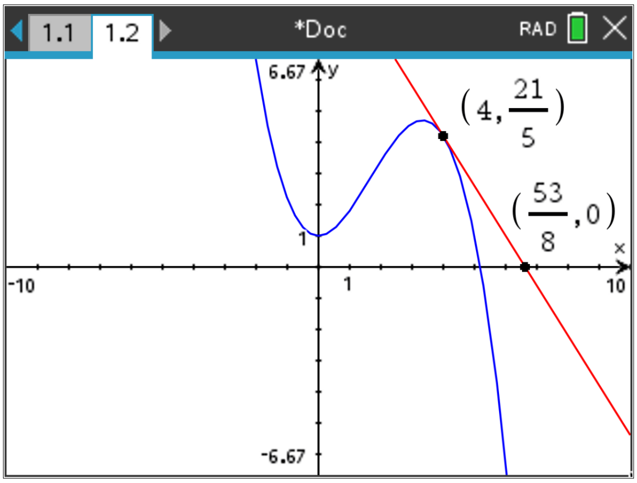
- 1 mark for the correct answer

Tip

- *Use your CAS to find the equation of a tangent line. This will help you avoid doing a lot of unnecessary algebra by hand.*

Question 3b.

Worked solution



Mark allocation: 2 marks

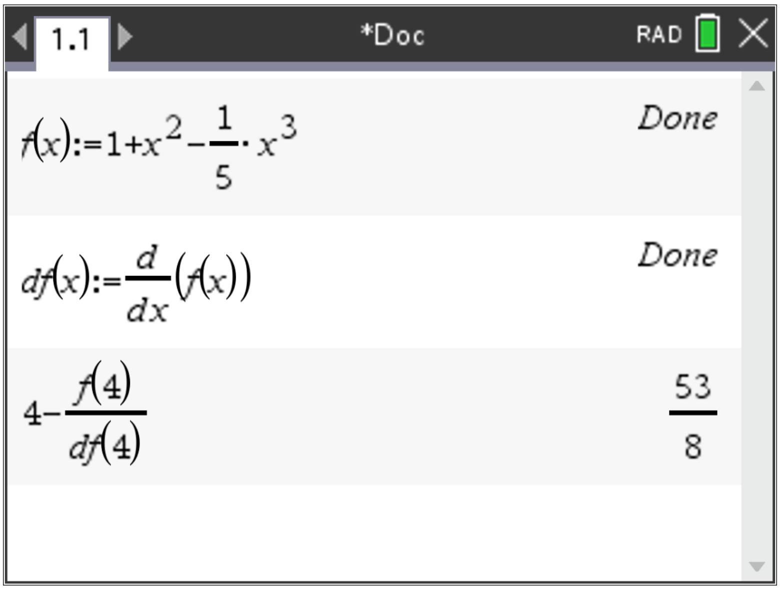
- 1 mark for sketching the correct line
- 1 mark for labelling the coordinates of the point of tangency and the x-intercept

Note: $\frac{53}{8}$ is a terminating decimal so 6.625 is also acceptable

Question 3c.

Worked solution

$x_1 = \frac{53}{8}$ or $x_1 = 6.625$



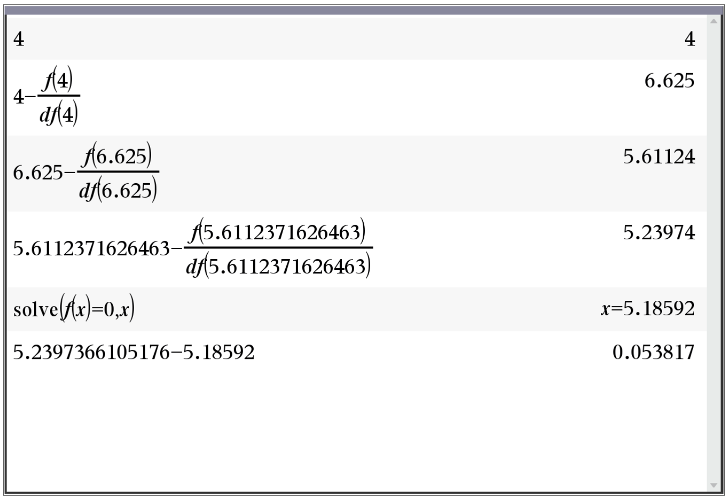
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3d.

Worked solution

0.0538

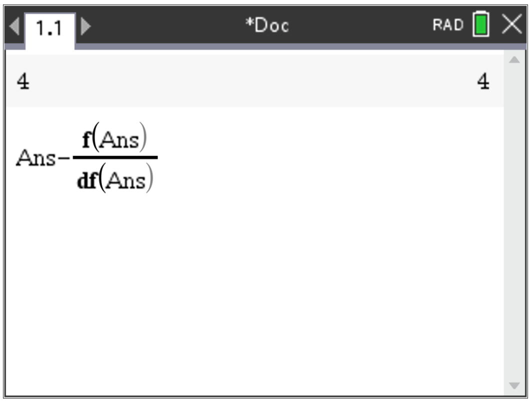


Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

- *To implement Newton's method efficiently on the CAS you can type in the value of x_0 and press enter. The, typing in* $\text{Ans}-\frac{\boldsymbol{f}(\text{Ans})}{\boldsymbol{df}(\text{Ans})}$ *(where* ***df*** *has already been defined as the derivative of f) will allow you to find each successive estimate for x by simply pressing enter repeatedly.*



Question 3e.

Worked solution

$x = 0$ and $x = \dfrac{10}{3}$ are the x values of the two stationary points. The tangents to these points are horizontal lines and will therefore never cross the x-axis. This means no further values of x_n can be found.

Mark allocation: 1 mark

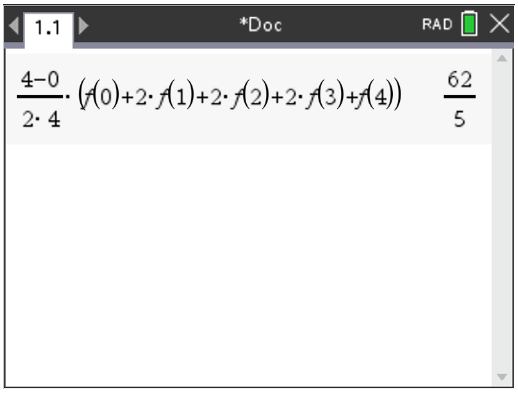
- 1 mark for a clear and valid explanation

  Note: Another example of a clear explanation would be that the derivative at $x = 0$ and $x = \dfrac{10}{3}$ is 0, and therefore the algorithm would try to divide by zero.

Question 3f.

Worked solution

$\dfrac{62}{5}$ or 12.4



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

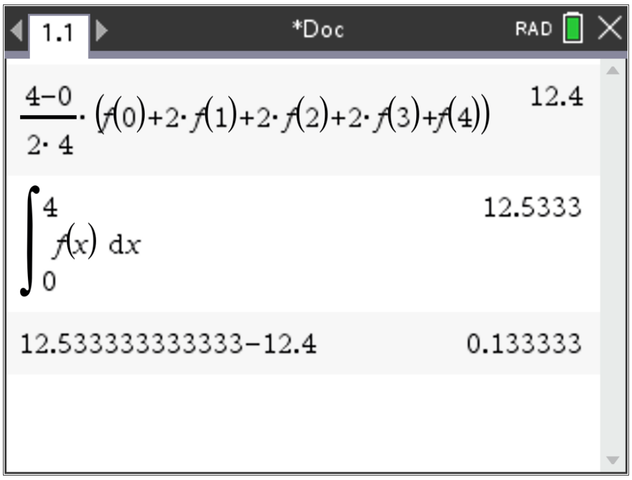
- *The trapezium rule approximation is new to the Study Design this year. Fortunately, the formula is simple to use and is provided on the formula sheet.*

Question 3g.

Worked solution

The actual area is $\int_0^4 f(x)\,dx = 12.5\dot{3}$.

Therefore, the difference between the actual area and the approximation, correct to two decimal places, is 0.13.



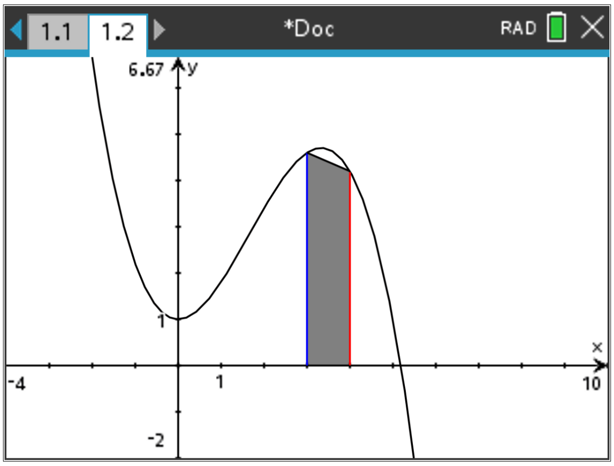
Mark allocation: 2 marks

- 1 mark for finding the value of the actual area as $12.5\dot{3}$ or $\frac{188}{15}$
- 1 mark for the correct answer

Note: The question starts with 'Hence', which implies a conditional mark will be awarded. If the response to **part f.** was incorrect but the difference between the actual area and that response was found correctly, then this answer mark is awarded.

Question 3h.

Worked solution



Mark allocation: 1 mark

- 1 mark for sketching the boundary lines of the correct trapezium

Question 3i.

Worked solution

5 and 6

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3j.

Worked solution

When `i=0`, `mid=5.5`, which is the midpoint of the initial lower and upper endpoints, `a` and `b`.

Then `b` becomes 5.5 and `a` remains 5.

Now `i` becomes 1, which is less than 3, so the while loop runs again.

`mid` becomes 5.25, and then `b` becomes 5.25 and `a` remains 5.

Now, `i` becomes 2, which is less than 3, so the while loop runs again.

`mid` becomes 5.125, and then `b` becomes 5.125 and `a` remains 5.

Finally, `i` becomes 3 and so the while loop does not run again. Therefore, the algorithm ends and the current value of `mid` is returned.

The value returned is 5.125.

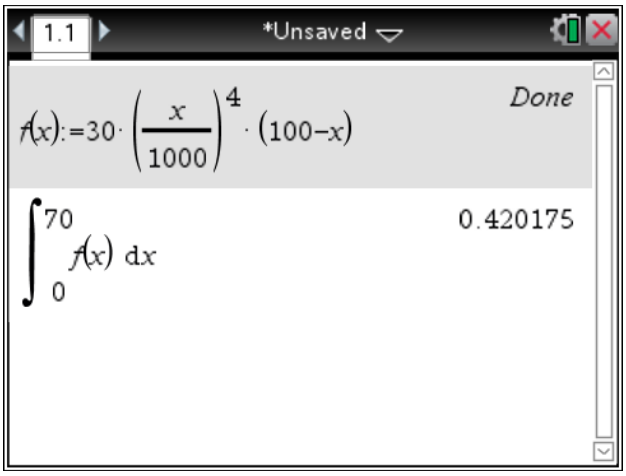
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4a.

Worked solution

$\Pr(X < 70) = 0.4202$



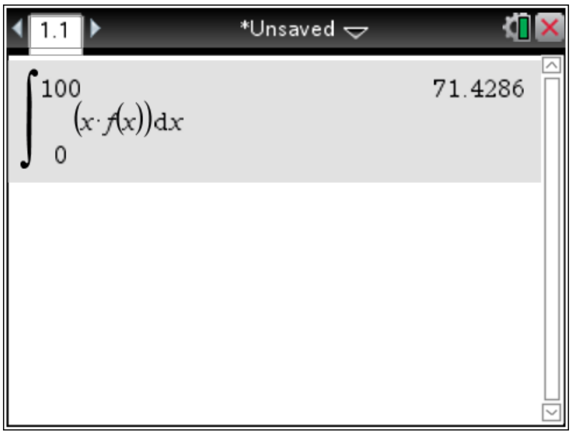
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4b.

Worked solution

$$\mathrm{E}(X)=\int_0^{100} x\,f(x)dx$$
$$=71.4$$



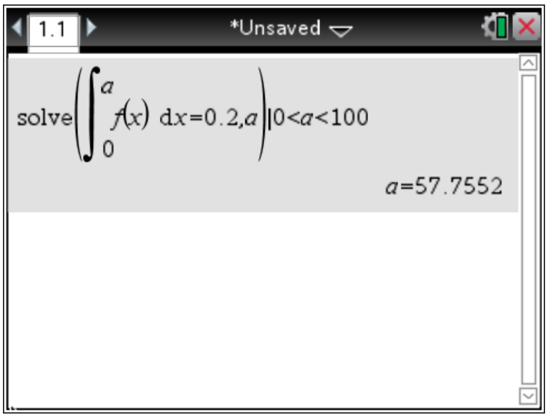
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4c.

Worked solution

Solving $\int_0^{a} f(x)dx=0.2$ for a gives $a=57.76$.



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4d.

Worked solution

First, we need to find the probability that a randomly selected globe burns out in less than 50 days.

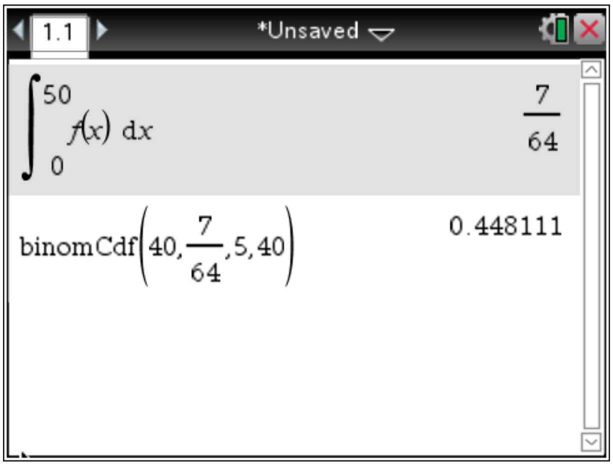
$$\Pr(X < 50) = \frac{7}{64}$$

Then, let D be the number of defective globes in a random sample of 40.

$$D \sim \text{Bi}\left(40, \frac{7}{64}\right)$$

So, the probability that at least five globes are defective is

$$\Pr(D \geq 5) = 0.4481$$



Mark allocation: 2 marks

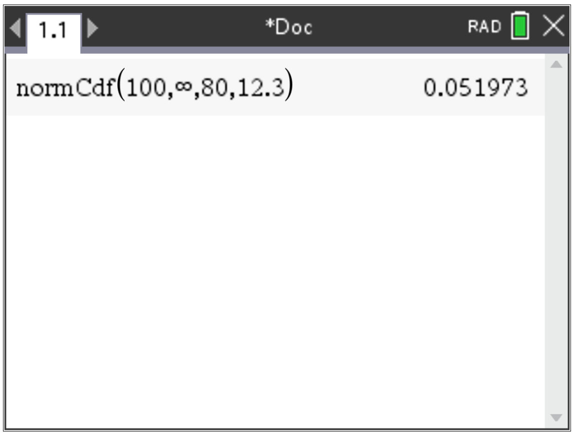
- 1 mark for finding $\Pr(X < 50) = \frac{7}{64}$
- 1 mark for the correct answer

Question 4e.

Worked solution

$Y \sim \mathrm{N}(80, 12.3^2)$

$\Pr(Y \geq 100) = 0.0520$



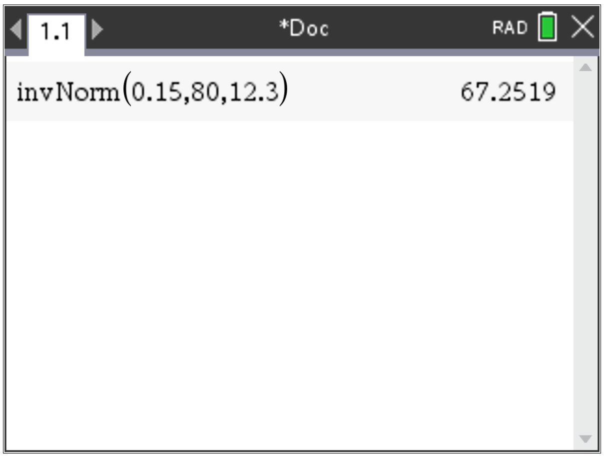
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4f.

Worked solution

Solving $\Pr(Y \leq k) = 0.15$ for k gives $k = 67.25$.



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4g.

Worked solution

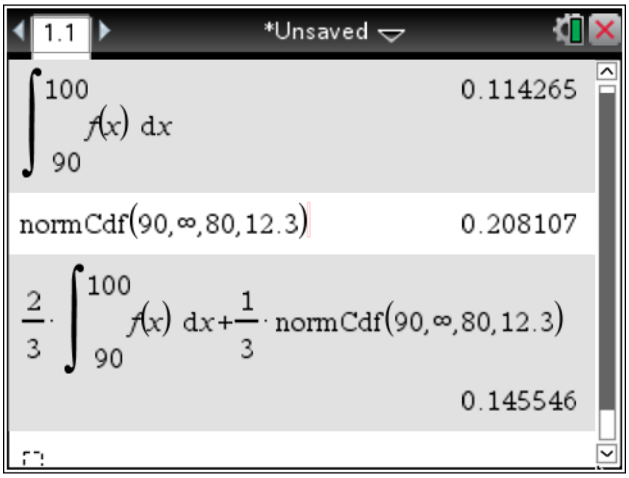
There are two ways the globe could last for more than 90 days.

The probability that an old globe lasts more than 90 days is $\Pr(X > 90) = 0.1143$.

The probability that a new globe lasts more than 90 days is $\Pr(Y > 90) = 0.2081$.

Therefore, the probability that a randomly selected one of the 600 globes lasts more than 90 days is

$$\frac{2}{3}\Pr(X > 90) + \frac{1}{3}\Pr(Y > 90) = 0.1455$$



Mark allocation: 2 marks

- 1 mark for a correct method shown clearly
  - Examples include: probability table, tree diagram, explicitly writing out $\frac{2}{3}\Pr(X > 90) + \frac{1}{3}\Pr(Y > 90)$
- 1 mark for the correct answer

Note: A correct final answer implies that students found the two relevant probabilities (0.1143 and 0.2081). Therefore, students writing simply $\Pr(X > 90)$ and $\Pr(Y > 90)$ as part of their response would be sufficient to achieve the first mark.

Tip

- *It may be helpful to draw a tree diagram to visualise this problem. The first branches would have options 'old globe' and 'new globe', and the next set of branches would have options 'more than 90 days' and 'less than 90 days'.*

Question 4h.

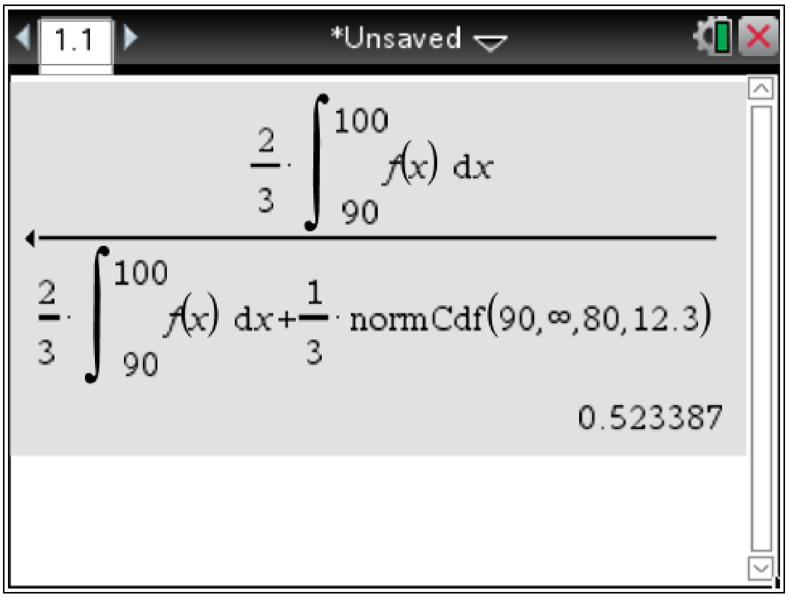
Worked solution

This is a conditional probability problem. The probability that a randomly selected globe from one of the 600 lasts more than 90 days was found in part **g.**

$$\Pr(\text{old} \mid >90) = \frac{\Pr(\text{old} \cap >90)}{\Pr(>90)}$$

$$= \frac{\frac{2}{3}\Pr(X > 90)}{\frac{2}{3}\Pr(X > 90) + \frac{1}{3}\Pr(Y > 90)}$$

$$= 0.52$$



Mark allocation: 2 marks

- 1 mark for clearly communicating that the conditional probability to be found is $\Pr(\text{old} \mid >90)$
- 1 mark for the correct answer

Question 4i.

Worked solution

The margin of error (ME) is half the width of the confidence interval (i.e. the difference between the point estimate and each boundary of the confidence interval), so

$$\text{ME} = 0.4963 - 0.4$$
$$= 0.0963$$

Therefore, the lower bound is

$$0.4 - 0.0963 = 0.3037$$

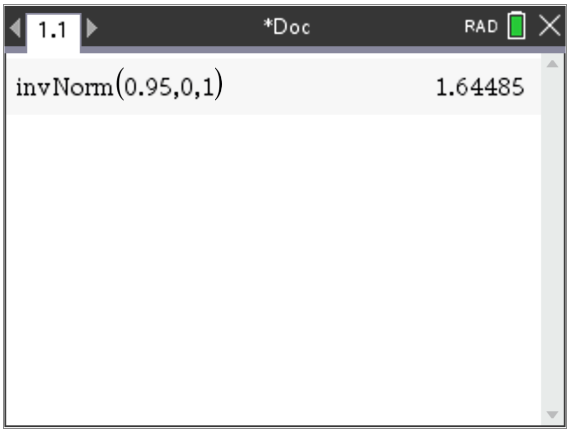
Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4j.

Worked solution

Solving $\Pr(-a < Z < a) = 0.9$ for a gives 1.645.



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4k.

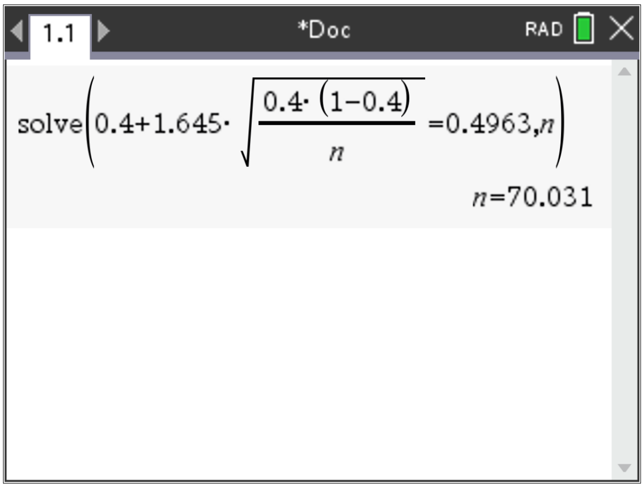
Worked solution

The upper bound of the confidence interval is given as 0.4963.

Therefore, $\hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = 0.4963.$

Since $\hat{p} = 0.4$ and $z = 1.645$, we have $0.4 + 1.645\sqrt{\dfrac{0.4(1-0.4)}{n}} = 0.4963.$

Solving for n gives $n = 70$



Mark allocation: 2 marks

- 1 mark for using an appropriate formula to solve for *n*, that is, lower bound, upper bound, margin of error
- 1 mark for the correct answer

Question 5a.

Worked solution

$f(1)=1$

$\therefore 1=a(k-1)$

$a=\dfrac{1}{k-1}$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5b.

Worked solution

$$\begin{aligned} f(x) &= ax(k-x^2) \\ &= akx-ax^3 \\ \therefore f'(x) &= ak-3ax^2 \end{aligned}$$

The stationary points occur when $f'(x)=0$.

$$\begin{aligned} ak-3ax^2 &= 0 \\ 3ax^2 &= ak \\ x^2 &= \frac{k}{3} \\ x &= \pm\frac{\sqrt{k}}{\sqrt{3}} \\ &= \pm\frac{\sqrt{3k}}{3} \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for setting $f'(x)=0$
- 1 mark for correct working to achieve $x=\pm\dfrac{\sqrt{3k}}{3}$

Tip

- *Since this is a 'show that' question, you must show your working out.*

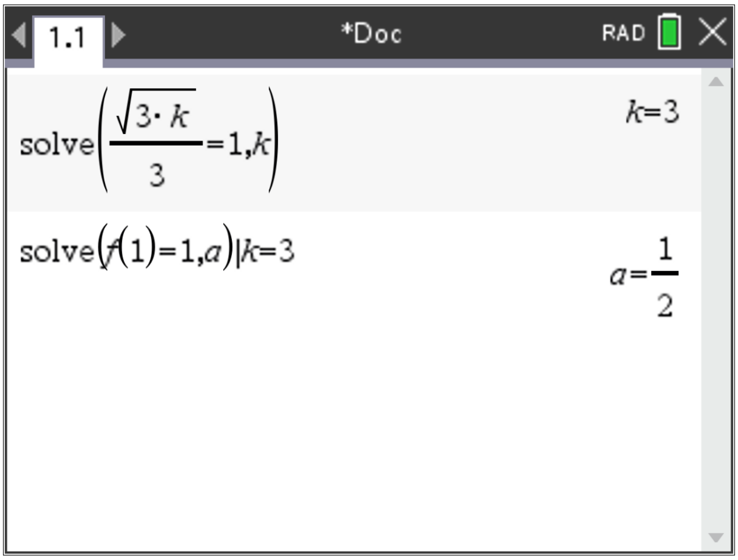
Question 5c.

Worked solution

Since a is a positive, real number, f is a negative cubic. Therefore, the maximum turning point is the stationary point with the larger x value.

Solving $\frac{\sqrt{3k}}{3}=1$ for k gives $k=3$.

Then solving $f(1)=1$ for a, given that $x=1$ and $k=3$, leads to $a=\frac{1}{2}$.



Mark allocation: 2 marks

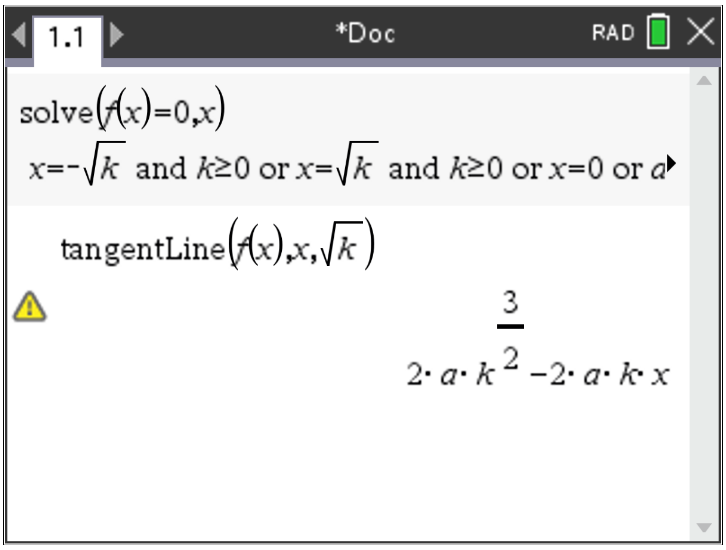
- 1 mark for $k=3$
- 1 mark for $a=\frac{1}{2}$

Question 5d.

Worked solution

The positive x-intercept is $\sqrt{k}$.

The tangent at this point is $y = -2akx + 2a\sqrt{k^3}$.



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

- *Use your CAS to find the equation of a tangent line. This will help you avoid doing a lot of unnecessary algebra by hand.*

Question 5e.

Worked solution

Solving $f(x) = -2akx + 2a\sqrt{k^3}$ for x gives $x = -2\sqrt{k}$ and $x = \sqrt{k}$.

This equation can also be solved on a CAS or by factorising both sides.

$$ax(k - x^2) = -2akx + 2a\sqrt{k^3}$$

$$-ax\left(x - \sqrt{k}\right)\left(x + \sqrt{k}\right) = -2ak\left(x - \sqrt{k}\right)$$

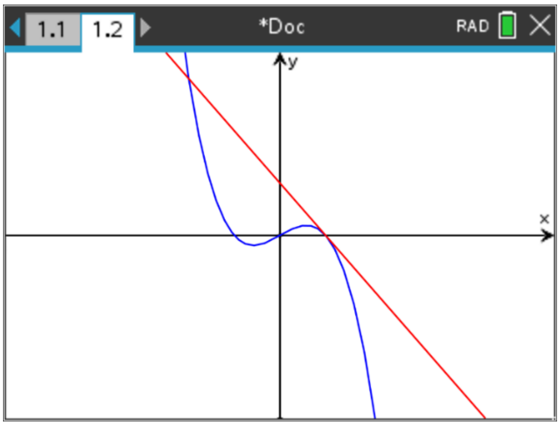
$$\left(x - \sqrt{k}\right)\left(x\left(x + \sqrt{k}\right) - 2k\right) = 0$$

$$\left(x - \sqrt{k}\right)\left(x^2 + x\sqrt{k} - 2k\right) = 0$$

$$\left(x - \sqrt{k}\right)^2\left(x + 2\sqrt{k}\right) = 0$$

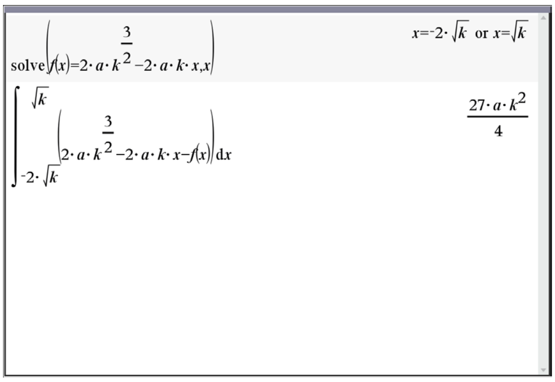
$$x = \sqrt{k} \quad \text{or} \quad x = -2\sqrt{k}$$

Since the graph of $y = f(x)$ is a negative cubic, the tangent at its positive x-intercept will be above the cubic.



Therefore, the region bounded by these two graphs is

$$\int_{-2\sqrt{k}}^{\sqrt{k}}\left(-2akx + 2a\sqrt{k^3} - f(x)\right)dx = \frac{27ak^2}{4}$$



Mark allocation: 2 marks

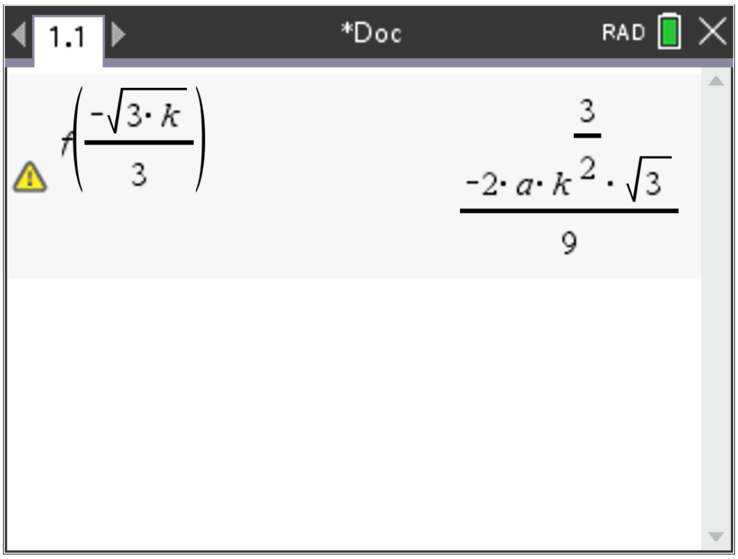
- 1 mark for finding $x = -2\sqrt{k}$, the x-value of the other point of intersection
- 1 mark for the correct answer

Question 5f.

Worked solution

A translation of $\frac{\sqrt{3k}}{3}$ units to the right

A translation of $\frac{2a\sqrt{3k^3}}{9}$ units up



Mark allocation: 1 mark

- 1 mark for the correct answer

Note: The vertical translation can be done first.

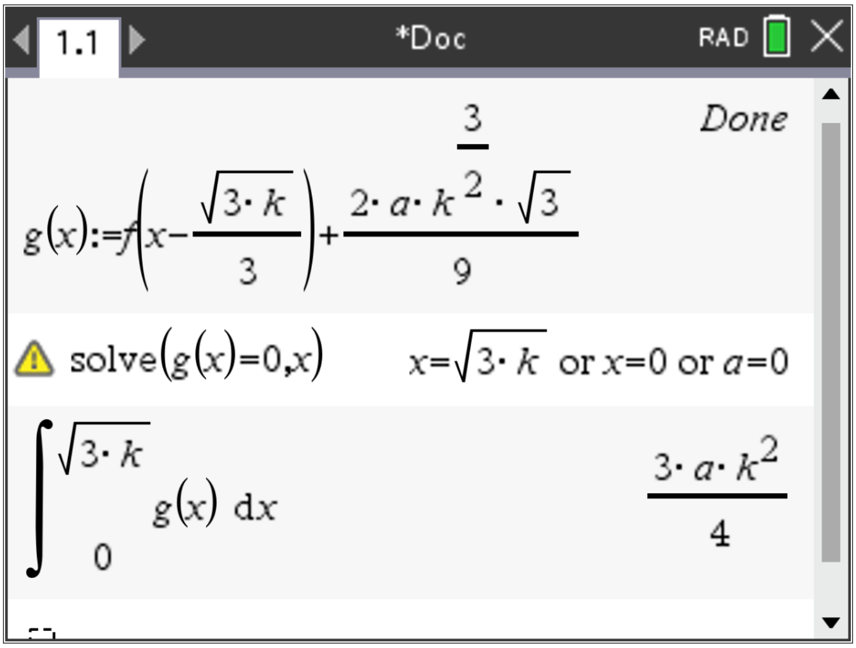
Question 5g.

Worked solution

$$g(x) = f\left(x - \frac{\sqrt{3k}}{3}\right) + \frac{2a\sqrt{3k^3}}{9}$$

$$= ax^2\left(\sqrt{3k} - x\right)$$

The x-intercepts of $y = g(x)$ are $x = 0$ and $x = \sqrt{3k}$, so the area bound by $y = g(x)$ and the x-axis is $\int_0^{\sqrt{3k}} g(x)\,dx = \frac{3ak^2}{4}$.



Mark allocation: 2 marks

- 1 mark for finding the x-intercepts of g as 0 and $\sqrt{3k}$

  Note: This mark should be awarded if these *x* values are observed as the terminals of an integral expression.

- 1 mark for the correct answer

END OF WORKED SOLUTIONS