The Mathematical Association of Victoria

Trial Examination 2023

MATHEMATICAL METHODS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A- Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The gradient of the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ at $x = \frac{\pi}{2}$ is

A. 2
B. $\frac{1}{2}$
C. 0
D. -1
E. $-\sqrt{2}$

Question 2

The period of the function $g(x) = -3\sin\left(\frac{\pi}{8}x + 1\right)$ is

A. 1
B. 3
C. 8
D. 16
E. $\frac{1}{4}$

Question 3

The largest value of a such that the inverse f^{-1} exists for the function

$f:(-\infty, a] \to R, f(x) = \frac{1}{4}x^4 - 3x^3 - 6x^2 + 20x + 10$ is

A. 1
B. -2
C. -1
D. 2
E. 10

Question 4

The maximal domain over which the graph of $h(x) = -4(x-1)^3(x-2)$ is strictly increasing is

A. $(-\infty, 1) \cup \left(1, \frac{7}{4}\right)$

B. $\left(-\infty, \frac{7}{4}\right)$

C. $\left(-\infty, \frac{7}{4}\right]$

D. $\left(\frac{7}{4}, \infty\right)$

E. $\left[\frac{7}{4}, \infty\right)$

Question 5

The simultaneous equations

$2x + ky = 3$

$-3x - y = m$

where $k, m \in R$ will have an infinite number of solutions when

A. $k \in R \setminus \left\{\frac{2}{3}\right\}$ and $m = -\frac{9}{2}$

B. $k = \frac{2}{3}$ and $m \in R \setminus \left\{-\frac{9}{2}\right\}$

C. $k = \frac{2}{3}$ and $m = -\frac{9}{2}$

D. $k \in R \setminus \left\{\frac{2}{3}\right\}$ and $m \in R \setminus \left\{-\frac{9}{2}\right\}$

E. $k \in R \setminus \left\{\frac{2}{3}\right\}$ and $m \in R$

Question 6

The graph of $f : [0, \infty) \to R, f(x) = \sqrt{x}$ has been translated 2 units to the right and 3 units down and then dilated by a factor of 4 from the *x*-axis. The transformed function, *g* could be

A. $g : [0, \infty) \to R, g(x) = 4\sqrt{x-2} - 12$

B. $g : [0, \infty) \to R, g(x) = 4\sqrt{x+2} - 3$

C. $g : [2, \infty) \to R, g(x) = 4\sqrt{x-2} - 3$

D. $g : [0, \infty) \to R, g(x) = 4\sqrt{x+2} + 3$

E. $g : [2, \infty) \to R, g(x) = 4\sqrt{x-2} - 12$

Question 7

If $\int_a^5 f(x)dx = 3$ and $\int_5^b f(x)dx = -4$ where $a < 5 < b$, then $\int_a^b \left(2f(x)+1\right)dx$ is equal to

A. -1
B. 0
C. $2+a-b$
D. $-2-a-b$
E. $-2-a+b$

Question 8

The distance between $(0,5)$ and a point $P(x,y)$ on the graph of $f(x)=(x-2)^2$ will be shortest when the x-coordinate of P is

A. 0
B. 4 only
C. $\frac{\sqrt{6}+2}{2}$ only
D. $\frac{-\sqrt{6}+2}{2}$ only
E. 4 or $\frac{-\sqrt{6}+2}{2}$ or $\frac{\sqrt{6}+2}{2}$

Question 9

The average value of $g: R\setminus\{1\} \to R, g(x) = \frac{1}{(x-1)^2}$ between $x=a$ and $x=b$ is equal to $\frac{2}{5}$ where a and b are real constants and $b > a$.
If the gradient of the tangent to g at $x=a$ equals -2, the values of a and b are respectively

A. 2 and $\frac{7}{2}$
B. $\frac{7}{2}$ and 2
C. -0.71 and 1.29
D. 2 and $\frac{1}{2}$
E. $\frac{1}{2}$ and 2

Question 10

The area bounded by the graphs of $f(x)=3x-2$ and $g(x)=(3x-2)^3$ can be found by evaluating the expression

A. $\int_{1}^{\frac{3}{2}}\left(f(x)-g(x)\right)dx$

B. $2\int_{1}^{\frac{2}{3}}\left(g(x)-f(x)\right)dx$

C. $\int_{\frac{1}{3}}^{\frac{2}{3}}\left(f(x)-g(x)\right)dx+\int_{\frac{2}{3}}^{1}\left(g(x)-f(x)\right)dx$

D. $\int_{\frac{1}{3}}^{1}\left(g(x)-f(x)\right)dx+\int_{\frac{2}{3}}^{1}\left(f(x)-g(x)\right)dx$

E. $\int_{\frac{1}{3}}^{\frac{2}{3}}\left(g(x)-f(x)\right)dx-\int_{\frac{2}{3}}^{1}\left(f(x)-g(x)\right)dx$

Question 11

Consider the following set of simultaneous equations.

$y+z=2$

$-2x-3y=8$

The general solution to these equations can be expressed as

A. $y=\lambda$, $x=-\frac{3\lambda+8}{2}$, $z=-\lambda+2$ where $\lambda\in Z$

B. $x=\lambda$, $y=-\frac{2\lambda+8}{3}$, $z=\frac{2\lambda-2}{3}$ where $\lambda\in R$

C. $y=\lambda$, $x=-\frac{3\lambda+8}{2}$, $z=-\lambda+2$ where $\lambda\in R$

D. $z=\lambda$, $y=2-\lambda$, $x=-\frac{3\lambda+14}{2}$ where $\lambda\in R$

E. $x=\lambda$, $y=-\frac{2\lambda+8}{3}$, $z=\frac{2\lambda+14}{3}$ where $\lambda\in Z$

Question 12

The maximal domain of the function $h(x)=\frac{\log_e(x-a)}{\log_e(x+a)}$, where a is a positive constant, is

A. $[-a,a]\setminus\{0\}$

B. $(-a,a)\setminus\{0\}$

C. $(0,a)$

D. $(-\infty,a)$

E. (a,∞)

SECTION A - continued

Question 13

The population of squirrels, s at time t weeks in a particular area can be modelled by the function $s:[0,\infty)\to R, s(t)=200e^{-0.01t}\sin(t)+220$.

The average rate of change between the maximum and minimum populations is closest to

A. -1.90825
B. -123.401
C. -1.90835
D. -123.408
E. 222.468

Question 14

A pseudocode to compute $\sqrt{3}$ using Newton's Method is shown below.

define $h(x)$
  return x^2-3
define $h'(x)$
  return $2x$
$x \leftarrow 2$
while $h(x)<-0.0001$ or $h(x)>0.0001$
  $x \leftarrow x-\dfrac{h(x)}{h'(x)}$
  print x, $h(x)$
end while

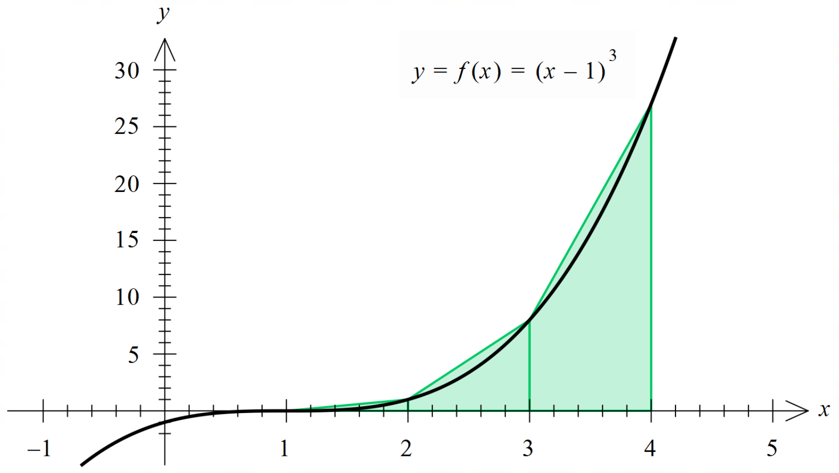
The program will stop after how many iterations?

A. 2
B. 3
C. 4
D. 5
E. 6

SECTION A - continued
TURN OVER

Question 15

The value of $\int_1^4 \left((x-1)^3\right)dx$ is approximated using a triangle and two trapeziums, with width 1 unit, as shown in the graph below.



The estimate can be found by evaluating

A. $f(2)+f(3)+\frac{1}{2}f(4)$

B. $f(2)+f(3)+f(4)$

C. $f(2)+f(3)$

D. $\frac{1}{2}\left(2f(2)+2f(3)+2f(4)\right)$

E. $\frac{1}{2}\left(f(2)+2f(3)+f(4)\right)$

Question 16

A survey found that 110 out of 400 Year 12 students study more than two hours a night. An approximate 95% confidence interval for the proportion of Year 12 students who study more than two hours per night is (0.2312, 0.3188). If the width of the confidence interval is to be decreased by 70%, the sample size must be

A. increased by a factor of $\frac{100}{9}$.

B. decreased by a factor of $\frac{9}{100}$.

C. increased by a factor of 0.7.

D. decreased by a factor of $\frac{49}{100}$.

E. increased by a factor of $\frac{100}{49}$.

Question 17

The heights of a particular type of shrub in a nursery are normally distributed with a mean of 50 cm and a standard deviation of 8 cm. The shrubs are sold according to their heights as shown in the table below.

Height (cm)	less than 40 cm	from 40 cm to 55 cm	greater than 55 cm
Cost ($)	10	20	30

If a landscape gardener, Amari, randomly chooses 50 shrubs his expected cost to the nearest cent is

A. $20.00
B. $21.60
C. $1000.00
D. $1080.17
E. $1080.20

Question 18

Consider three events *A*, *B* and *C*. *A* and *C* are mutually exclusive events and *A* and *B* are independent events.
$\Pr(A' \cap B' \cap C') = 0.1$, $\Pr(A) = 0.5$, $\Pr(B) = 0.3$ and $\Pr(C) = 0.35$.

$\Pr(B \cap C)$ equals

A. 0
B. 0.05
C. 0.1
D. 0.105
E. 0.25

Question 19

A student randomly guesses a multiple-choice test with *n* questions. Each question has the same number of alternatives and there is only one correct answer. If *X* is the number of correct answers, $\text{Var}(X) = 2$ and $\Pr(X = 2) = \dfrac{512}{2187}$.

The probability the student gets less than 2 answers correct is closest to

A. 0.0010
B. 0.0851
C. 0.6228
D. 0.3772
E. 0.1431

Question 20

The area bounded by the graph of *h* with equation $h(x) = \frac{1}{8}x^4 - x^3 + \frac{9}{4}x^2 - \frac{27}{8}$ and the line which passes through the two points of inflection of *h* is

A. 0.8
B. 1.6
C. 0
D. 1.2
E. 0.4

END OF SECTION A

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SECTION B

Instructions for Section B

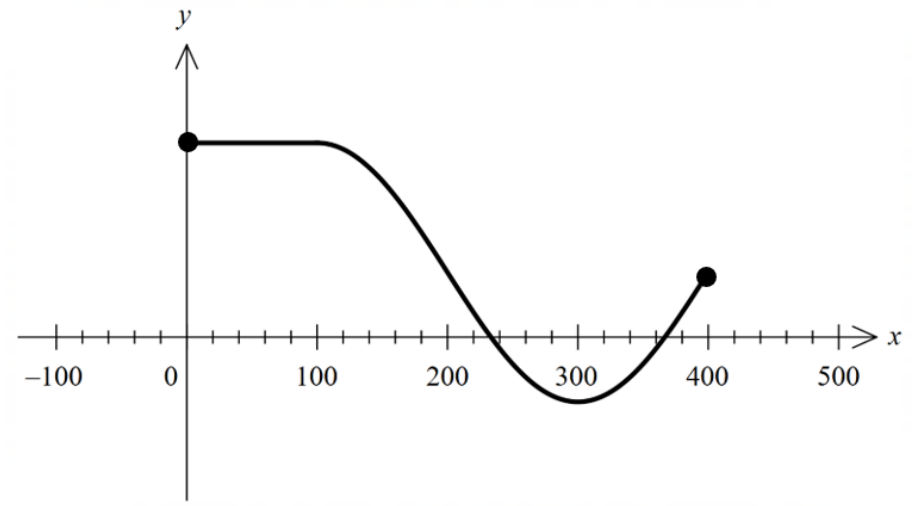
Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (12 marks)
A Victorian engineering company is proposing to build a road for the new transport hub that links the CBD to the western suburbs of Melbourne. The graph below shows a cross-section of the proposed road which will begin at a platform *k* metres above ground, run above the ground then tunnel under ground. The height above ground level is modelled by the function *f*, where

$$f(x)=\begin{cases} k & 0\le x\le 100 \\ 20\cos\left(\dfrac{\pi(x-100)}{200}\right)+10 & 100<x\le 400 \end{cases}.$$

$f(x)$ is the height above ground level, x is the horizontal distance from a point O and k is a positive real constant. All lengths are in metres.

The graph of $y=f(x)$ shown below is **continuous** at $x=100$.



a. Find the value of *k*. 1 mark

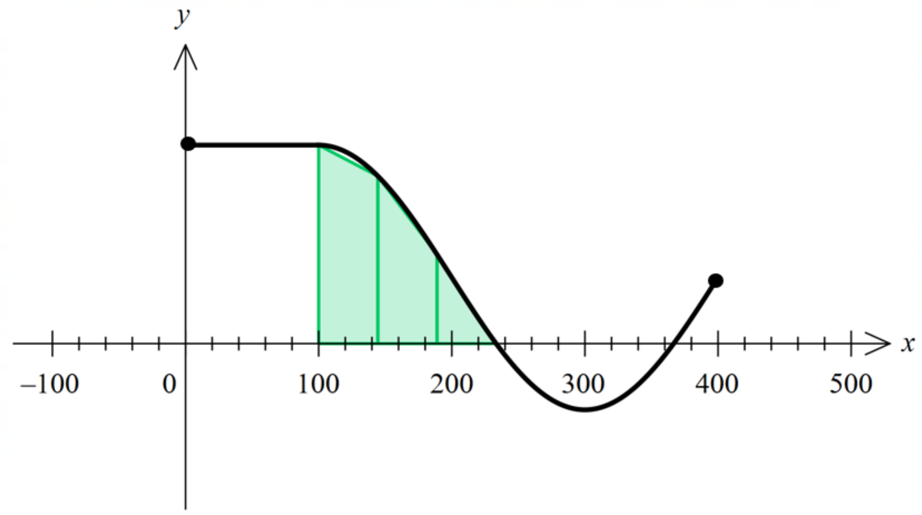
b. For what x-values does the proposed road go under ground level? Write your answer using interval notation. 1 mark

c. Show that the graph of $y = f(x)$ is **smooth** at $x = 100$. 1 mark

d. **i.** Find the coordinates of the point(s) of inflection of the curve of f. 1 mark

ii. State the maximal domain over which f is strictly decreasing. 1 mark

The engineering company uses three vertical supports of equal width for the road. The left edge of the first support is placed at $x=100$ and the right edge of the third support is placed at the x-value where the road is planned to first go underground. The cross-sections of the first two supports form trapeziums and the third a triangle as shown in the graph below.



e. Find the total area of the cross-sections. Give your answer in square metres correct to one decimal place. 3 marks

Newton's method is used to find an approximate x-intecept of f with an initial estimate of $x_0 = 150$.

f. **i.** Find the value of x_1, correct to three decimal places. 2 marks

ii. Find the horizontal distance between x_3 and the closest x-intercept of f, correct to four decimal places. 2 marks

Question 2 (12 marks)
Large plots of farmland surrounding a Victorian farmhouse have curved fences modelled by the functions f and g given by

$$f(x)=\frac{p}{x-20}-40 \text{ and } g(x)=\frac{q}{x+30}+10$$

where p and q are positive constants.
The farmhouse is at the origin and the land is in the shape of a rectangle, stretching 50 kilometres in both directions east and west from the y-axis and 80 kilometres north and south from the x-axis. A boundary fence encloses the rectangular piece of land and the curved fences stop at the boundary fence.

The graph of f has an x-intercept at $x=25$.

a. Show that $p=200$. 1 mark

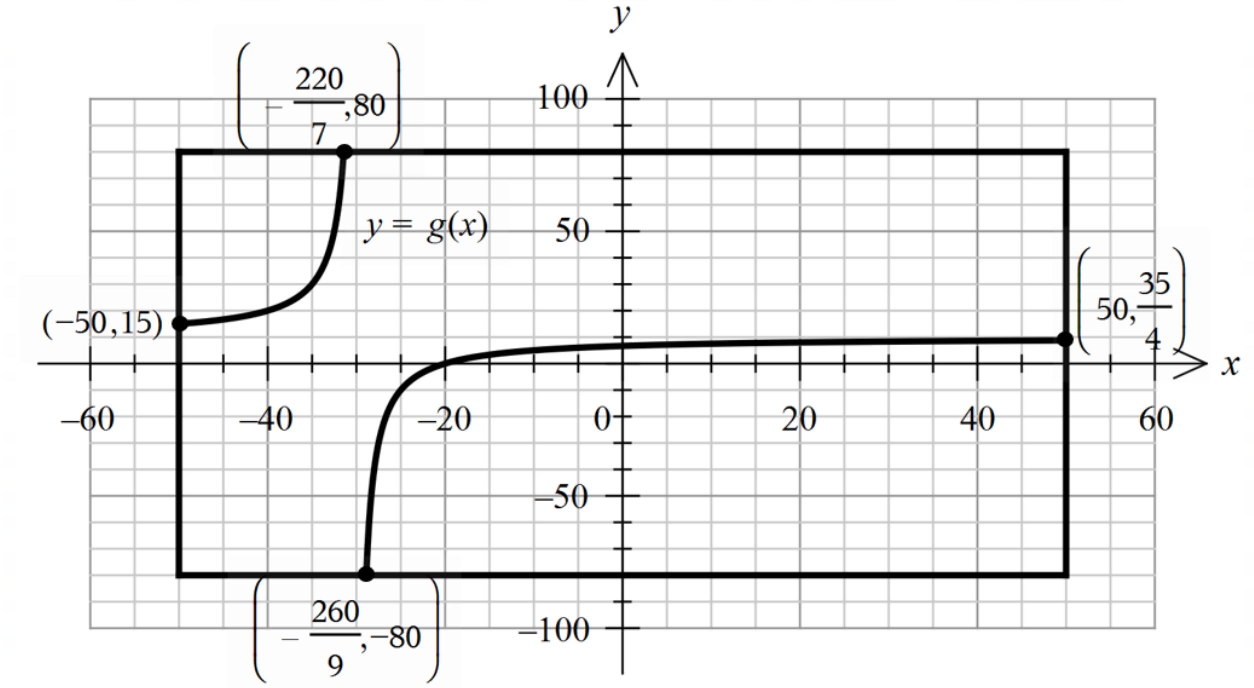
The graph of g has a gradient of 1 at $x=-20$.

b. **i.** Show that $q=-100$. 1 mark

ii. State the other x-value for which the gradient of g is equal to 1. 1 mark

c. i. Find the coordinates of the points of intersection of the graphs of f and g. 1 mark

ii. The boundary of the property and the graph of g are shown below. Sketch the graph of f, where $-50 \le x \le 50$, on the same set of axes. Clearly label the endpoints with their coordinates. 2 marks



The area bounded by the graphs of f and g and the lower limit of the farmland at $y = -80$ is to be planted with a wheat crop.

d. i. Write an expression, involving definite integrals, which when evaluated will give the area of the land that is to be cropped. 2 marks

ii. Hence, or otherwise, find the area of the farmland that is to be cropped . Give your answer correct to the nearest square kilometre. 1 mark

SECTION B - Question 2 – continued

TURN OVER

iii. What percentage of the farmland is to be cropped? Give your answer to the nearest whole number. 1 mark

The farmer decides to install two linear water pipes on his property. They are to be tangential to the curves of *f* and *g* at $x = 0$, where the water tanks are located. He wants to put a water trough for his animals where the water pipes meet.

e. Find the equations of the tangents to both *f* and *g* at $x = 0$ and show that, despite the farmer's plans, his water pipes do not meet on his property. 2 marks

Question 3 (16 marks)
Maja has 8 black socks, 10 green socks and 12 blue socks in her drawer. The socks are identical except for the colour. Maja randomly selects a pair of socks to wear on Monday morning.

a. What is the probability they are the same colour? Give your answer correct to four decimal places. 2 marks

Maja is going on a holiday for a week. She packs her bag on Monday afternoon and randomly selects 14 socks from her drawer.

b. What is the probability she will select 2 pairs of black socks, 2 pairs green socks and 3 pairs of blue socks given that she is currently wearing matching green socks? Give your answer correct to four decimal places. 1 mark

While on holidays Maja decides to go sightseeing each day. The time she spends sightseeing each day is a random variable, X hours, with probability density function, s given by

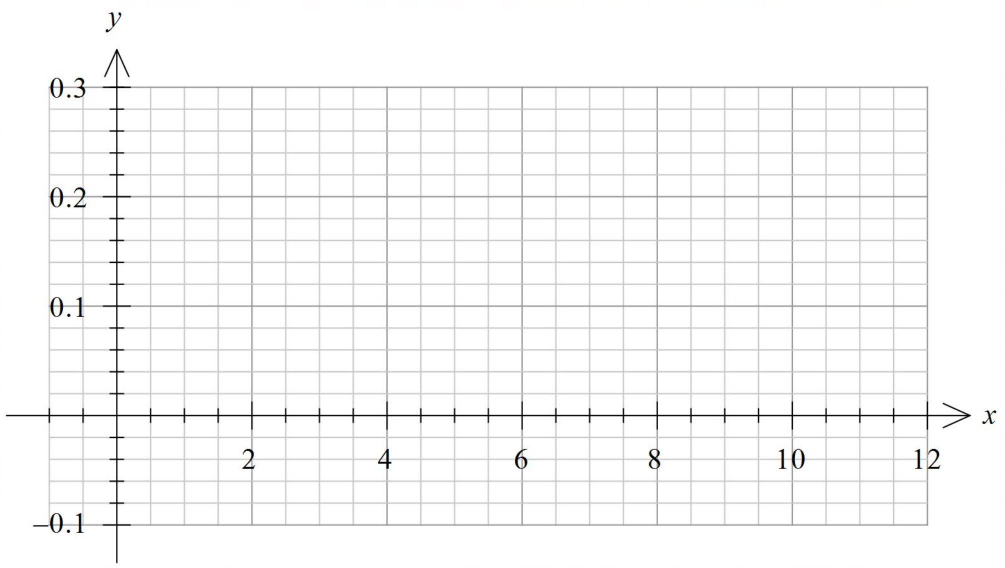
$$s(x)=\begin{cases} -\frac{1}{80}(x-2)(x-8) & 2\le x\le 6 \\ \frac{a}{x-5}+b & 6<x\le 10\text{, where } a,b\in R\text{.} \\ 0 & \text{elsewhere} \end{cases}$$

c. Find a and b if s is continuous for $2\le x\le 10$. 2 marks

Use $a = -0.11155$ and $b = 0.21155$ to answer the following questions.

d. **i.** Find the standard deviation of the number of hours Maya intends to sightsee, correct to three decimal places. 2 marks

ii. Sketch the graph of $y = s$ on the set of axes below and shade the region which represents $\Pr(X \geq \mu)$. Label all sharp points and endpoints with their coordinates. Give non-integer values correct to two decimal places. 3 marks



Maya likes to go fishing at an aquacultural farm when she is on holidays. She knows the lengths of a particular species of adult fish on the aquacultural farm are normally distributed. The lengths of 15% of the fish are more than 500 mm and 10% have lengths less 250 mm.

e. Find the mean and standard deviation for the lengths of this species of fish. Give your answers to the nearest mm. 2 marks

Maya randomly catches 30 fish and measures them. If the fish are less than 250 mm thay are considered undersized.

f. What is the probability that less than 5 of Maya's fish will be considered undersized?
Give your answer correct to four decimal places. 2 marks

In another pond on the aquacultural farm there is a different species of fish. The owner of the farm claims that only 4% of the fish are undersized. Maya catches 100 of them and finds 25 of them to be undersized.

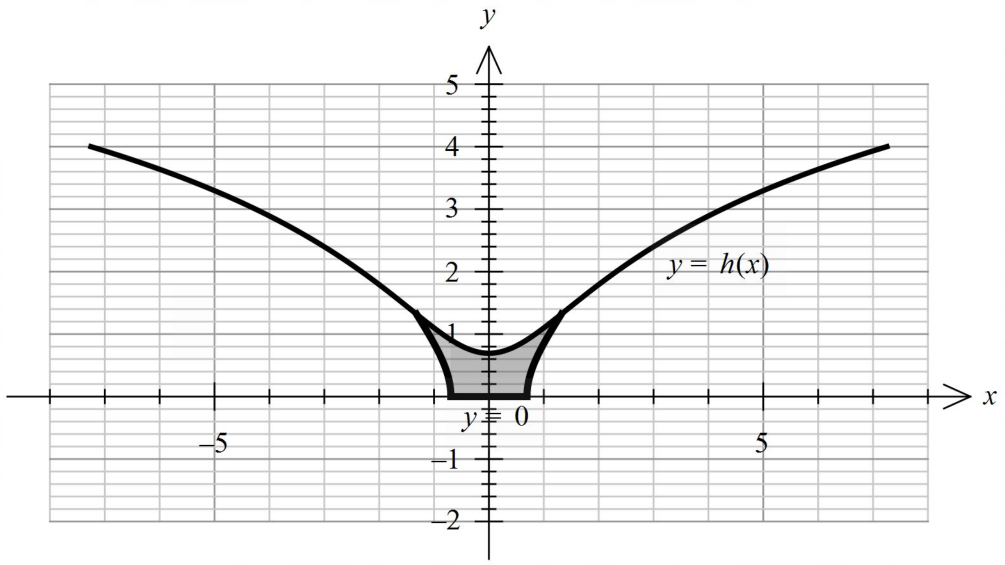
g. Find a 99% confidence interval for the proportion of undersized fish and comment on the result.
Give your answers correct to four decimal places. 2 marks

Question 4 (10 marks)
Consider the function $g:(-\infty,\infty)\to R, g(x)=\log_e(x^2+px+2)$, where p is a real constant.

a. For what values of p will g be a continuous function for all $x \in R$. 2 marks

b. When g is a continuous function, for all $x \in R$, will g^{-1} exist? Explain. 1 mark

The function $h:[a,b]\to R, h(x)=\log_e(x^2+2)$ models the side view of a punch bowl where h is the height of the bowl in decimetres above the top of a bench at $y=0$ and the positive value of x is half the width of the inside of the bowl in decimetres. The base at the bottom of the bowl is made of solid glass. The graph of h is shown below and the cross-sectional area of the base has been shaded.



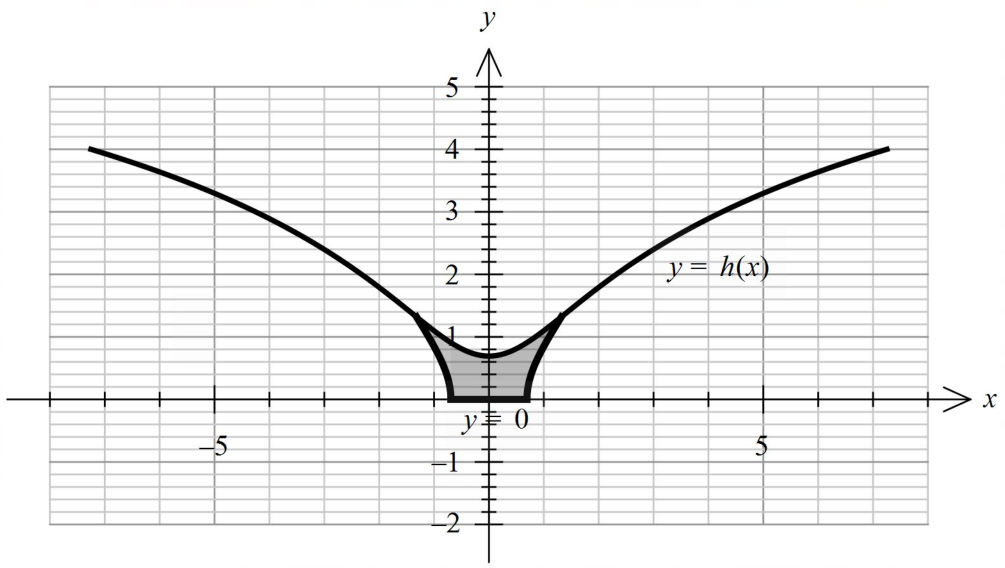
c. If the maximum height of the bowl above the bench is 4 dm, find a and b. 1 mark

SECTION B - Question 4 - continued

The bowl is in the shape of a prism with length 2 dm. Punch is poured into the bowl up to the average value of h.

d. How much punch is poured into the bowl? Give your answer in dm^3 correct to two decimal places. 3 marks

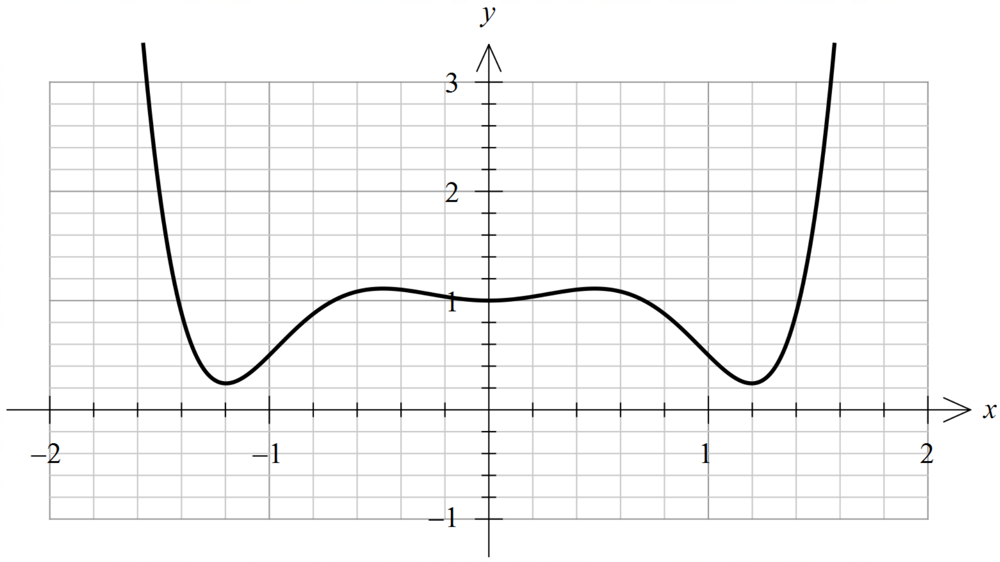
Now consider the base of the punch bowl which is made of solid glass. Its cross-sectional area is bounded by the curves of h, part of the graph of the inverse of h, the bench top and part of the graph of the reflection of the inverse of h in the y-axis as shown by the shaded area on the graph below.



e. Find the cross-sectional area of the glass base. Give your answer in dm^2 correct to two decimal places. 3 marks

Question 5 (10 marks)

Let $f: R \to R, f(x) = x^6 + ax^4 + x^2 + 1$, where a is a real constant. The graph of $y = f(x)$ when $a = -\frac{5}{2}$ is shown below.



a. Label the turning points with their coordinates correct to two decimal places. 1 mark

b. How many points of inflection does the graph have? 1 mark

c. For what values of a will $f(x) \geq 0$ for all of x? Give your answer correct to three decimal places. 2 marks

d. Solve $f'(x) = 0$ for x. Give non-integer answers in terms of a. 1 mark

SECTION B - Question 5 – continued

e. Find the value of a so that the graph of f will have 3 stationary points only. Give the coordinates of the stationary points and state their nature. 3 marks

f. Find the values of a so that the graphs of f will have no points of inflection. 2 marks

END OF QUESTION AND ANSWER BOOK