NAME: ____________________________

# VCE®MATHEMATICAL METHODS

## UNITS 3 & 4 Practice Examination

## Written Examination 2

**Reading time: 15 minutes** **Writing time: 2 hours**

**QUESTION AND ANSWER BOOK**

**Structure of Book**

| *Section* | *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|---|
| A | 20 | 20 | 20 |
| B | 5 | 5 | 60 |
| | | | 80 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

**Materials supplied**

- A Question and Answer Book of 29 pages.
- A double-sided page of Formulas.
- An Answer Sheet for Multiple-Choice Questions.

**Instructions**

- Write your name in the space provided above on this page.
- Write your name on the Multiple-Choice Answer Sheet.
- Unless otherwise indicated the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

**At the end of Examination**

- Place the Answer Sheet for Multiple-Choice Questions inside the front cover of this book

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

# SECTION A – Multiple-Choice Questions

## Section A – Multiple-Choice Questions

**Instructions for Section A**

Answer **all** questions in pencil on the Answer Sheet provided for Multiple-Choice Questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

### Question 1

The equation of the normal to $f: R \to R, f(x) = \log_e(x+4)$ at $x$ = -2 is:

**A.** $y = \frac{1}{2}x + \log_e(2) + 1$

**B.** $y = -2x + \log_e(2) - 4$

**C.** $y = \frac{x}{x+4} + \log_e(2) + 1$

**D.** $y = 2x - \log_e(2) + 1$

**E.** $y = \frac{-1}{2}x - \log_e(2) + 1$

### Question 2

The period and amplitude of $y = -2\sin(\pi - 3x)$ are respectively:

**A.** $3, -2$

**B.** $3, 2$

**C.** $\frac{2\pi}{3}, -2$

**D.** $\frac{2\pi}{3}, 2$

**E.** $\frac{-2\pi}{3}, 2$

**Question 3**

The coordinates of the local minimum of $f(x)=4-\frac{1}{e^{¿¿}¿}$ are:

**A.** $\left(2, 4-\frac{1}{e^4}\right)$

**B.** $(2,3)$

**C.** $\left(0, 4-\frac{1}{e^4}\right)$

**D.** $(0,0)$

**E.** No local minimum

**Question 4**

An antiderivative of $f(x)=¿$ is:

**A.** $F(x)=3a^3x^2-6a^2bx+3ab^2+c$

**B.** $F(x)=3a(ax-b)^4+cx^2$

**C.** $F(x)=\frac{3a}{4}(ax-b)^4+cx^2$

**D.** $F(x)=\frac{a^3x^4}{4}-a^2bx^3+\frac{3ab^2+c}{2}x^2+\frac{c}{2}x$

**E.** $F(x)=\frac{a^3x^4}{4}-a^2bx^3+\frac{3ab^2+c}{2}x^2-b^3x$

**Question 5**

Given $\sin(\theta)=a, \frac{\pi}{2}<\theta<\pi, \tan(\theta)$ is equal to:

**A.** $-\sqrt{1-a^2}$

**B.** $\frac{a\sqrt{1-a^2}}{a^2-1}$

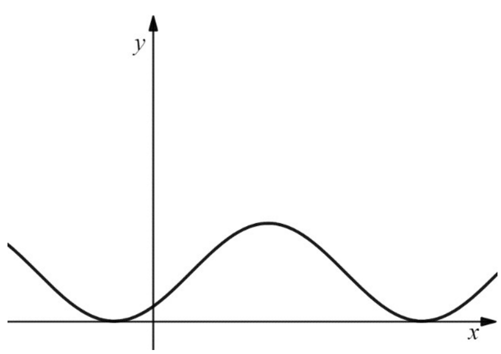
**C.** $\frac{a}{\sqrt{1-a^2}}$

**D.** $\frac{-\sqrt{1-a^2}}{a}$

**E.** $\frac{-1}{a}$

**Question 6**

The graph of $f(x)$ is shown below:




A possible equation for $f(x)$ is:

**A.** $f(x)=\sin\left(x-\frac{\pi}{4}\right)+1$

**B.** $f(x)=1-\sin\left(x-\frac{\pi}{4}\right)$

**C.** $f(x)=\cos\left(x-\frac{\pi}{4}\right)+1$

$f(x)=\sin\left(x \quad 4\right)$



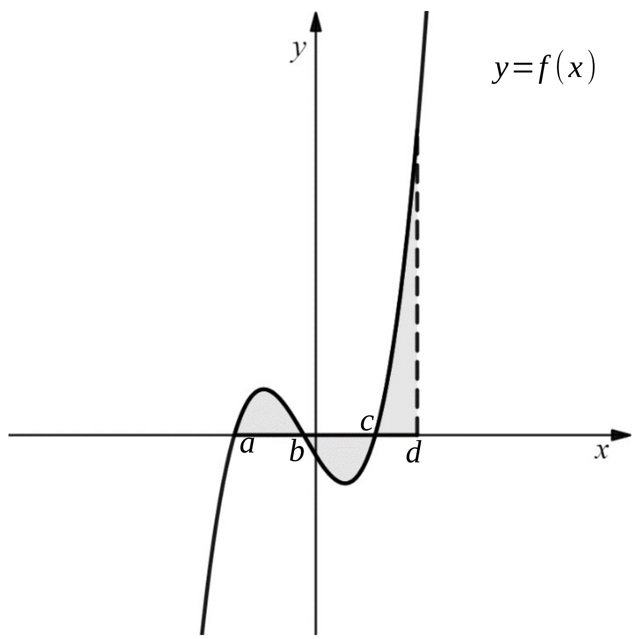
$f(x)=\sin\left(x+\frac{\pi}{4}\right)+1$

**D.**

**E.** $f(x)=1-\cos\left(x-\frac{\pi}{3}\right)$

**Question 7**

Which of the following expressions will provide the total area of the shaded region in the graph below?




**A.** $2\int_a^b f(x)\,dx+\int_c^d f(x)$

**B.** $\int_a^b f(x)dx+\int_b^c f(x)+\int_c^d f(x)$

**C.** $\int_a^b f(x)dx+\int_b^c f(x)-\int_c^d f(x)$

**D.** $\int_a^b f(x)dx-\int_b^c f(x)-\int_d^c f(x)dx$

**E.** $\int_a^b f(x)dx-\int_c^b f(x)+\int_c^d f(x)$

**Question 8**

$X$ is a random variable. $Pr(X<j)=m$ and $\Pr(X>k)=n$. Additionally, $j<k$.

The probability that $X<k$ given $X>j$ is equal to:

**A.** $\dfrac{1-m-n}{1-n}$

**B.** $1-m-n$

**C.** $\dfrac{1-m-n}{1-m}$

**D.** $m-n$

**E.** $m-n+1$

*The following information relates to Questions 9 and 10.*

A student may catch the bus or ride their bike to school. If it is raining in the morning, the probability that they will catch the bus is 0.75. However, if it is not raining, the probability that they will ride is 0.6. At this time of year there is a 40% chance that it will not be raining in the morning.

**Question 9**

What is the probability that the student will ride to school on two consecutive days?

**A.** 0.0360

**B.** 0.0801

**C.** 0.1521

**D.** 0.2116

**E.** 0.3900

**Question 10**

If it was raining three mornings in a row, what is the probability that the student rode to school on at least two of the mornings?

**A.** 0.0156

**B.** 0.1563

**C.** 0.3520

**D.** 0.6480

**E.** 0.8438

**Question 11**

A marketing company planned to conduct a survey to determine what proportion of people prefer hot dogs over all other types of fast food. They conducted an initial survey of 50 people and found that 22 of those surveyed preferred hot dogs. The company then wanted to construct a 95% confidence interval for their sample proportion with a width of 0.10. To the nearest person, how many people should they survey?

**A.** 52

**B.** 74

**C.** 265

**D.** 311

**E.** 379

**Question 12**

If ¿ and $x>0$ then:

**A.** $f(x)=2\sqrt{x}$ and $g(x)=2x$

**B.** $f(x)=\frac{1}{\sqrt{x}}$ and $g(x)=x^2$

**C.** $f(x)=e^x$ and $g(x)=e^{-x}$

**D.** $f(x)=\sin(x)$ and $g(x)=\cos(x)$

**E.** $f(x)=4\sqrt{x}$ and $g(x)=\frac{1}{2x}$

**Question 13**

$f(x)=a\sin\left(\frac{\pi x}{2}\right)$ and $g(x)=m\cos(ax)$, $\frac{3}{2}<a<3$ and $\frac{1}{4}<m<\frac{3}{4}$.

The average value of $f(x)$ between $x=0$ and $x=m$ is $\frac{12-6\sqrt{3}}{\pi}$.

The average value of $g(x)$ between $x=0$ and $x=\pi$ is 0.

The values of $a$ and $m$ respectively are:

**A.** $a=\frac{5}{2}, m=\frac{1}{2}$

**B.** $a=\frac{5}{2}, m=\frac{1}{3}$

**C.** $a=\frac{5}{2}, m=\frac{2}{3}$

**D.** $a=2, m=\frac{1}{2}$

**E.** $a=2, m=\frac{1}{3}$

**Question 14**

The following pseudo code determines if a user-inputted number, assigned to *num*, is a prime.

```
input num
a ← 0
c ← 2
t ← num
while c ≤ num/2 and a = 0
        while t ≥ 0
                if t = 0 then
                print num "is not a prime number since it is divisible by" c
                a ← 1
                end if
        t ← t - c
        end while
        c ← c + 1
        t ← num
end while
if a = 0 then
        print num "is a prime"
end if
```

What would be the values of variables $a$, $c$ and $t$ at the completion of the pseudocode for when $num = 7$?

**A.** $a=0, c=4, t=7$

**B.** $a=0, c=3, t=-1$

**C.** $a=1, c=3, t=7$

**D.** $a=1, c=4, t=-1$

**E.** $a=0, c=4, t=-1$

**Question 15**

$X$ is a normally distributed random variable and $Z$ is the standard normal variable.

If $\Pr(X < 10) = \Pr(Z > 2)$ and $\Pr(X < 30) = \Pr(Z > 1.5) - \Pr(Z > 2)$, then the mean and standard deviation of $X$ are closest to:

**A.** $\mu=50, \sigma=12$

**B.** $\mu=50, \sigma=24$

**C.** $\mu=78, \sigma=32$

**D.** $\mu=146, \sigma=100$

**E.** $\mu=146, \sigma=68$

**Question 16**

$f(x)=\sqrt{3}\sin(\pi x/2)$ and $g(x)=-\cos(\pi x/2)$. The equation that will provide the $x$-coordinates of points of intersection of $f(x)$ and $g(x)$ is:

**A.** $x=n\pi-\frac{\pi}{6}, n\in Z$

**B.** $x=2n-\frac{1}{3}, n\in Z$

**C.** $x=n\pi+\frac{\pi}{6}, n\in Z$

**D.** $x=2n\pi-\frac{\pi}{6}, n\in Z$

**E.** $x=\frac{n\pi}{2}-\frac{\pi}{3}, n\in Z$

**Question 17**

A particular pdf has the rule $f(x)=ae^{-x^2}, x \in R^{+¿¿}$. The value of $a$ is:

**A.** $a=0.3989$

**B.** $a=0.5642$

**C.** $a=1$

**D.** $a=1.128$

**E.** $a=1.414$

**Question 18**

$f(x)=x^3+5x^2-2x-24, m \leq x < n$

The inverse function, $f^{-1}(x)$, of $f(x)$ is defined when:

**A.** $m=-1, n=2$

**B.** $m=-10, n=-3$

**C.** $m=\frac{1}{2}, n=2$

**D.** $m=-3, n=1$

**E.** $m=0, n=\infty$

**Question 19**

$f(x)=4-e^x$. If $x_0=1$, then after 1 iteration, the Newton's method approximation of the $x$-intercept of $f(x)$ will be.

**A.** $\frac{4}{e}$

**B.** $\frac{4}{e^{\frac{4}{e}}}$

**C.** $4-e$

**D.** $\frac{4}{4-e}$

**E.** $\log_e(2)$

**Question 20**

Consider $f(x)=ax^2+b$, $a,b\in R$ and $a\neq 0$

Let $g(x)=6-4f(2x+1)$

If $g(1)=-86$ and $g(-1)=10$, then:

**A.** $a=-4, b=3$

**B.** $a=\frac{3}{2}, b=3$

**C.** $a=-1, b=\frac{3}{2}$

**D.** $a=\frac{3}{2}, b=-1$

**E.** $a=3, b=-4$

## SECTION B

**Instructions for Section B**

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

**Question 1** (11 marks)

A family of curves can be modelled by the equation:

$$f(x) = ae^{nx-h} + k,\ a, h, k, n \in R \text{ and } a, n \neq 0$$

**a.** **i.** What is the equation of the asymptote of $f(x)$? 1 mark

**ii.** Under what conditions does $f(x)$ have an $x$-intercept? 2 marks

**iii.** What parameter(s) affect the range of $f(x)$? 1 mark

**iv.** Determine the equation of $f^{-1}(x)$, the inverse of $f(x)$. State its domain. 2 marks

**b.** **i.** If $f'(0) = 1$, express $a$ in terms of $n$ and $h$ and hence express $f(x)$ in simplified form. 2 marks

**ii.** Hence, determine the relationship between $f(0)$ and the area bound by $f(x)$, $x = 0$ and $y = 0$ for when $k = 0$, $x < 0$ and $n > 0$. 3 marks

**Question 2** (15 marks)

Gretchen sells cars. The probability that she will sell a car within *t* days can be modelled by the probability density function:

$$f(t) = \frac{a}{t+b}, a, b \in R^{+¿¿}$$

**a.** **i.** Given the following:

$f(0) = \frac{1}{6}$

median of $f(t) = (e^3 - 1)$ days

Determine the values of the parameters *a* and *b* and express *f*(*x*) in terms of the parameter values. 3 marks

**ii.** Based upon *f*(*t*), the probability density function, what is the maximum number of days that it may take for Gretchen to sell a car? Round answer to the nearest day. 1 mark

**iii.** Gretchen had not sold a particular car by the 20th day. What is the exact probability that it will take her longer than 50 days to sell the car?

**b.** Gretchen specialises in the sale of small cars. The sale price follows a normal distribution. Let $X$ be the random variable corresponding to the sale price of a car, in thousands of dollars. The mean and standard deviation of $X$ are μ and σ respectively. $Z$ is the standard normal variable.

**i.** Given $Pr(X<3\sigma)=Pr\ (Z>1)$ and $Pr(X<\mu)=2\,Pr\ (Z<13.3)$, determine the mean and standard deviation of X. Round each to the nearest while number. 2 marks

**ii.** What is the probability that Gretchen sells two of the next three cars at more than 10% above the average sale price? Round answer to 4 decimal places. 2 marks

**iii.** What is the minimum number of cars that Gretchen would need to sell, such that the probability that not more than one sells below \$11,000 is not more than 0.05? 3 marks

**c.** Gretchen works for a large car dealership, which has many stores across the country. Achmed, the Marketing Manager, conducted a survey to measure customer satisfaction. 50 people were surveyed, and a 95% confidence interval was constructed. The margin of error (width of half of the confidence interval) was 0.12. Given a preliminary survey indicated that customer satisfaction was greater than 50%, what was the sample proportion, rounded to 2 decimal places? 2 marks

**Question 3** (11 marks)

A cylindrical water tank is to be constructed for a local town. Its capacity is 200 kL (200,000 L). The sides of the tank cost \$200/m$^2$ to manufacture, while the top and bottom cost \$100/m$^2$. Note that 1,000 L = 1 m$^3$.

**a.** Show that surface area, $S$ (m$^2$) of the water tank can be expressed according to the following equation (where $r$ is the radius of the tank in m):

$$S=\frac{400}{r}+2\pi r^2$$

2 marks

**b.** **i**. Develop an equation that provides the cost of materials for the water tank. 1 mark

**ii.** What is the radius that would provide the lowest cost for the water tank. Round answer to 2 decimal places. 2 marks

**iii.** Hence, what is the minimum cost for the water tank, rounded to the nearest dollar?
1 mark

**c.** Water must be pumped to the tank from a local reservoir. The difference in elevation between the surface of the reservoir and base of the water tank is *d* meters. The volume of water per second that the pump can deliver to the tank depends upon *d* and can be modelled by the equation:

$$V(d) = a\sqrt{k-2d}, a, k \in R \text{ and } a \neq 0$$

**i.** When the difference in elevation is 20 m, the pump can deliver $5\sqrt{15}$ L/s. However, when the difference in elevation is 40 m it can only deliver $5\sqrt{5}$ L/s. Determine the exact values of the parameters *a* and *k*.
2 marks

**ii.** What is the maximum rate that the pump can deliver to the tank?

**iii.** The water level in the reservoir varies throughout the year. This means the difference in elevation between the surface of the reservoir and the base of the tank also varies throughout the year. The equation that models the difference in elevation between the surface of the reservoir and the base of the tank is:

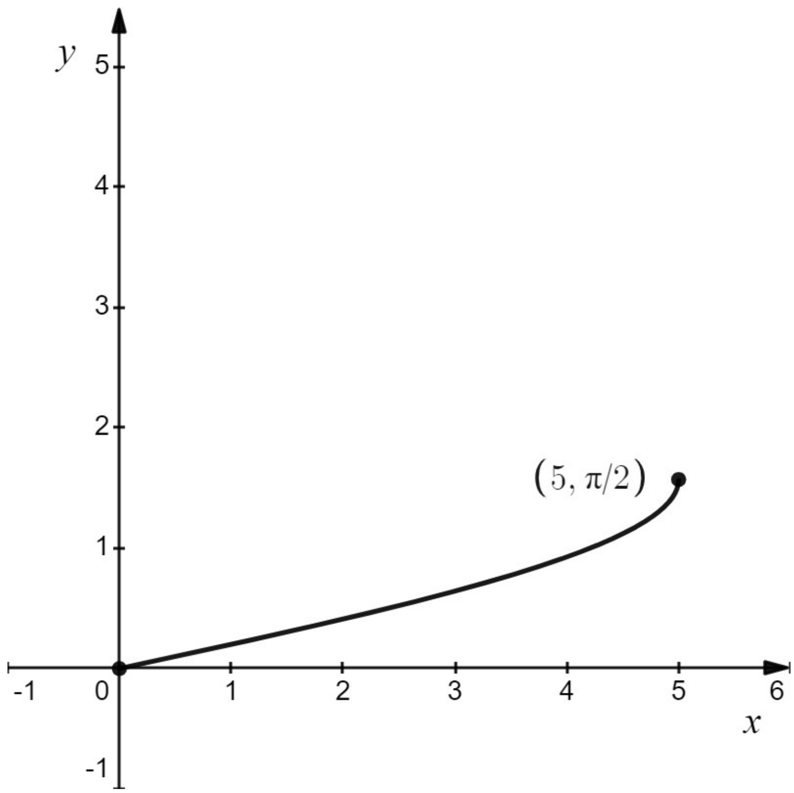
$$d(t) = 40 + 12\cos\left(\frac{\pi t}{6}\right)$$

where $t$ is in months and $t = 0$ corresponds to the $1^{st}$ of March.

Determine the percentage of the year, rounded to the nearest whole number, that the pump is unable to transfer water from the reservoir to the tank. 2 marks

**Question 4** (7 marks)

The graph of $f(x)=\sin^{-1}\left(\frac{x}{5}\right)$, the inverse of $g(x)=5\sin(x)$, is shown below for $0 \le x \le 5$.




**a.** Use the Trapezium Rule to determine an estimate of the area bound by $f(x)$, the $x$-axis and the lines $x = 0$ and $x = 5$. Use trapeziums of width 1 unit. Round estimate to 2 decimal places.

2 marks

**b.** Sketch the graph of $g(x)$ $(f^{-1}(x)¿$ on the axes above, labelling the end points. Shade in the region bound by $g(x)$ and the $y$-axis that corresponds to the areas described in Part **a**

2 marks

**c.** Write an equation involving a definite integral involving $g(x)$ that will provide the area of the region described in Part **a**.

1 mark

**d.** Hence, determine exact value of the area of the region described in part **a**. What is the percent difference, rounded to the nearest whole number, between this area and the estimate obtained using the Trapezium Rule in Part **a**?

2 marks

**Question 5** (16 marks)

A family of functions has the form $f(x) = axe^{-nx}, 0 \leq x \leq \infty$ and $a, n \in R^{+¿¿}$

**a.** Given $f(x)$ is a probability density function, rewrite $f(x)$ in a form which includes only the parameter $n$. 2 marks

**b.** Express the mean of $f(x)$ in terms of the mode of $f(x)$.

**c.** i. A particular equation for $f(x)$ is used to model the magnitude of the difference between the actual arrival time and the scheduled arrival time, in minutes, for a city's train system. A train is considered 'on time' if this difference is less than one minute.

The probability that at least two out of three trains arrive on time is 0.8968. Determine the value of $n$ for this distribution. Round the values of $n$ to the nearest whole number. 3 marks

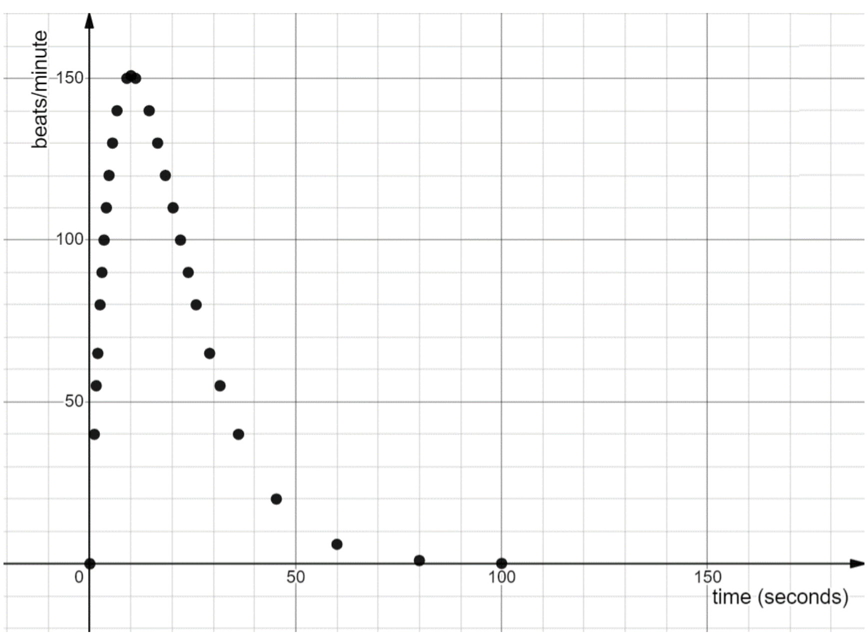
**ii.** A company operates the trains on behalf of the city council. The company receives a bonus if their P-Score is less than 1. The P-Score is the magnitude of the difference between the scheduled and actual arrival time, such that 90% of train arrivals are less than this score.

The manager of the company had a plan that would reduce its P-Score by 15%. Using the value of $n$ obtained in Part **c.i,** determine if this reduction will be sufficient to allow the company to receive the bonus. 3 marks

**iii.** What is the minimum value of $n$ in the model that would allow the company to receive the bonus? Round you answer to 2 decimal places. 2 marks

**d.** The family of curves introduced at the start of the question can also be used to model heart rate (beats/minute) in response to a sudden stress, such as a fright. When a person receives such a stress, the heart rate increases rapidly, then gradually returns to its initial value over a period of several seconds or minutes, depending upon the individual and the specific stress.

One person in a study has their heart rate monitored when, suddenly, a rubber spider was dangled from a string before their eyes. Their heart rate response to this stress is shown in the graph below. In the graph, 0 bps (beats/minute) corresponds to their resting heart rate and time = 0 corresponds to the moment the rubber spider was dangled before the person.




Using the data provided in the graph, determine the values of the parameters $a$ and $n$ in the model. Round $a$ to the nearest whole number and $n$ to 1 decimal place. If the resting heart rate of the person was 55 bps, what was their average heart rate during the first 50 seconds of the measurements? 3 marks

**END OF QUESTION AND ANSWER BOOK**

# VCE® MATHEMATICAL METHODS

## Written Examinations 1 and 2

# FORMULA SHEET

This Formula Sheet is provided for your reference.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

 2023-MME-VIC-U34-NA-EX2-QATS

**Mensuration**

| | | | |
|---|---|---|---|
| area of a trapezium | $\frac{1}{2}(x+b)h$ | volume of a pyramid | $\frac{1}{3}Ah$ |
| curved surface area of a cylinder | $2\pi rh$ | volume of a sphere | $\frac{4}{3}\pi r^3$ |
| volume of a cylinder | $\pi r^2 h$ | area of a triangle | $\frac{1}{2}bc\sin(A)$ |
| volume of a cone | $\frac{1}{3}\pi r^2 h$ | | |

**Calculus**

| | | | |
|---|---|---|---|
| $\frac{d}{dx}(x^n)=nx^{n-1}$ | | $\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c, n\neq -1$ | |
| $\frac{d}{dx}((ax+b)^n)=an(ax+b)^{n-1}$ | | $\int$ ¿¿ | |
| $\frac{d}{dx}(e^{ax})=ax^{ax}$ | | $\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$ | |
| $\frac{d}{dx}(\log_e(x))=\frac{1}{x}$ | | $\int \frac{1}{x}\,dx=\log_e(x)+c$ | |
| $\frac{d}{dx}(\sin(ax))=a\cos(ax)$ | | $\int \sin(ax)\,dx=\frac{-1}{a}\cos(ax)+c$ | |
| $\frac{d}{dx}(\cos(ax))=-a\sin(ax)$ | | $\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$ | |
| $\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}=a\,sec^2(ax)$ | | | |
| product rule | $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ | quotient rule | $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ |
| chain rule | $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$ | | |

**Probability**

| | | | |
|---|---|---|---|
| $\Pr(A¿=1-Pr\left(A'\right)$ | | $\Pr(A\cup B¿=\Pr(A¿+\Pr(B¿-\Pr(A\cap B¿$ | |
| $\Pr(A\|B)=\Pr(A\cap B¿\frac{¿}{\Pr(B)}$ | | | |
| mean | $\mu=E(X¿$ | Variance | $\operatorname{var}(X¿=\sigma^2=E(X^2¿-\mu^2$ |

| Probability distribution | | Mean | Variance |
|---|---|---|---|
| discrete | $\Pr(X=x¿=p(x)$ | $u=\Sigma\, xp(x)$ | $\sigma^2=E\left((X-\mu)^2\right)=E(X)^2-\mu^2$ |
| continuous | $\Pr(a<X<b¿=\int_a^b f(x)\,dx$ | $\mu=\int_{-\infty}^{\infty} xf(x)\,dx$ | $\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$ |

**Sample proportions**

| | | | |
|---|---|---|---|
| $\hat{P}=\frac{X}{n}$ | | Mean | $E(\hat{P})=p$ |
| Standard deviation | $sd(\hat{P})=\sqrt{\frac{p(1-p)}{n}}$ | approximate confidence interval | $\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ |

**VCE® MATHEMATICAL METHODS**
**UNITS 3 & 4 Practice Examination 2**

## Multiple-Choice Answer Sheet

| **Student Name:** |
|---|

Shade the letter that corresponds to each correct answer.

| **Question** | | | | | |
|---|---|---|---|---|---|
| **1** | A | B | C | D | E |
| **2** | A | B | C | D | E |
| **3** | A | B | C | D | E |
| **4** | A | B | C | D | E |
| **5** | A | B | C | D | E |
| **6** | A | B | C | D | E |
| **7** | A | B | C | D | E |
| **8** | A | B | C | D | E |
| **9** | A | B | C | D | E |
| **10** | A | B | C | D | E |
| **11** | A | B | C | D | E |
| **12** | A | B | C | D | E |
| **13** | A | B | C | D | E |
| **14** | A | B | C | D | E |
| **15** | A | B | C | D | E |
| **16** | A | B | C | D | E |
| **17** | A | B | C | D | E |
| **18** | A | B | C | D | E |
| **19** | A | B | C | D | E |
| **20** | A | B | C | D | E |

# VCE® MATHEMATICAL METHODS

## Unit 3 and 4 Practice Examination

## Written EXAMINATION 2

# ADVICE FOR TEACHERS


2023-MME-VIC-U34-NA-EX2-QATS

# IMPORTANT SECURITY ADVICE FOR EXAMINATION TASKS

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# Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

## Section A: Multiple-Choice Answers – 20 marks

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1.** | **B** | **6.** | **A** | **11.** | **E** | **16.** | **B** |
| **2.** | **D** | **7.** | **D** | **12.** | **B** | **17.** | **D** |
| **3.** | **B** | **8.** | **C** | **13.** | **E** | **18.** | **C** |
| **4.** | **E** | **9.** | **D** | **14.** | **A** | **19.** | **A** |
| **5.** | **B** | **10.** | **B** | **15.** | **E** | **20.** | **E** |

## Section A: Multiple-Choice Solutions

**Question 1: B**

Use Normal Line function on the CAS.

By hand:

$f(-2)=\log_e(-2+4)$

¿ $\log_e(2)$

$f'(x)=\dfrac{1}{x+4}$

$f'(-2)=\dfrac{1}{2}$

$f'_N=-2$

$y-y_1=m(x-x_1)$

$x_1=-2,\ y_1=\log_e(2), m=-2$

$y-\log_e(2)=-2(x--2)$

$y=-2x+\log_e(2)-4$

**Question 2: D**

The period and amplitude of $y=-2\sin(\pi-3x)$ are respectively:

$Period=\dfrac{2\pi}{n}$

¿ $\dfrac{2\pi}{3}$Amplitude = 2

**Question 3: B**

Using CAS, solve $f'(x)=0$ for $x$ then substitute the $x$-value into $f(x)$ to find the corresponding $y$-coordinate.

Turning point at (2, 3)

By hand method 1:
Recognise that $f(x)$ has the same structure as the pdf for the Normal distribution. The untranslated Normal distribution pdf has a local maximum at $x = 0$. $f(x)$ has been reflected about the $x$-axis and therefore is has a local minimum. It has also been translated right 2 units. Therefore, the $x$-coordinate of the local minimum is 2. The $y$-coordinate is $f(2)$.

$f(2)=4-\frac{1}{e^0}$

¿3

Turning point at (2, 3)

By hand method 2:

$f'(x)=\frac{2(x-2)}{e^{(x-2)^2}}$

Solve $f'(x)= 0$ for $x$.

$\frac{2(x-2)}{e^{(x-2)^2}}=0$

$2(x-2)=0$

$x=2$

$f(2)=4-\frac{1}{e^0}$

¿3

Turning point at (2, 3)

**Question 4: E**

An antiderivative of $f(x)=$¿ is:

Using CAS, integrate $f(x)$ and compare with options. If none of the options match, expand the antiderivative then compare with remaining options.

By hand:

$F(x)=\int$ ¿¿

¿ $\frac{1}{4a}(ax-b)^4+\frac{c}{2}x^2$

Expand and gather like terms.

$F(x)=\frac{1}{4a}(a^2x^2-2abx+b^2)$¿

¿ $\frac{1}{4a}(a^3x^3-3a^2bx^2+3ab^2x-b^3)(ax-b)+\frac{c}{2}x^2$

¿ $\frac{1}{4a}(a^4x^4-4a^3bx^3+6a^2b^2x^2-4ab^3+b^4)+\frac{c}{2}x^2$

¿ $\frac{a^3x^4}{4}-a^2bx^3+\frac{3ab^2+c}{2}x^2-b^3x$

Note: the constant involving equation parameters is not required for ‘an’ antiderivative.

**Question 5: B**

Using CAS, evaluate:

$\tan(\theta)=\tan(\sin^{-1}(a))\vee\frac{\pi}{2}<\theta<\pi$

¿ $\frac{a\sqrt{1-a^2}}{a^2-1}$

By hand:

$\sin^2(\theta)+\cos^2(\theta)=1$

$\cos^2(\theta)=1-\sin^2(\theta)$

¿ $1-a^2$

$\cos(\theta)=\sqrt{1-a^2}$

¿ $-\sqrt{1-a^2}$ since cos is negative in the second quadrant.

$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$

¿ $-\frac{a}{\sqrt{1-a^2}}$

None of the options match. Therefore, rationalise the denominator.

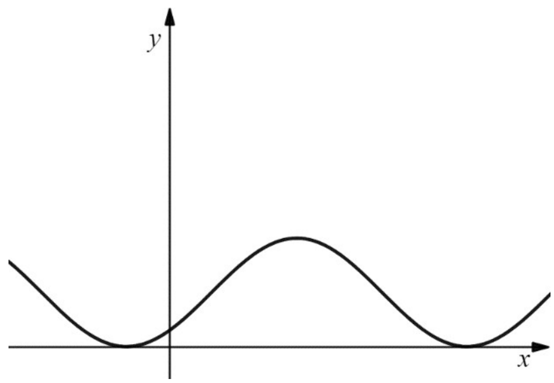
$\tan(\theta)=\frac{-a}{\sqrt{1-a^2}}\times\frac{\sqrt{1-a^2}}{\sqrt{1-a^2}}$

¿ $-\frac{a\sqrt{1-a^2}}{1-a^2}$

¿ $\frac{a\sqrt{1-a^2}}{a^2-1}$

**Question 6: A**

The graph of $f(x)$ is shown below:




Check value of $f(x)$ for $x = 0$ to eliminate options. According to graph, $0 < f(0) < \sim 1/2$.

$$f(0)=\sin\left(0-\frac{\pi}{4}\right)+1=1-\frac{\sqrt{2}}{2} \quad 0.3\, \textit{Option A is feasible on this basis.}$$

$$f(0)=1-\sin\left(0-\frac{\pi}{4}\right)=1+\frac{\sqrt{2}}{2} \quad 1.3\, \textit{Option B is not feasible on this basis.}$$

$$f(0)=\sin\left(0+\frac{\pi}{4}\right)+1=1+\frac{\sqrt{2}}{2} \quad 1.3\, \textit{Option C is not feasible on this basis.}$$

$$f(0)=\cos\left(0-\frac{\pi}{4}\right)+1=1-\frac{\sqrt{2}}{2} \quad 0.3\, \textit{Option D is feasible on this basis.}$$

$$f(0)=1-\cos\left(0-\frac{\pi}{3}\right)=\frac{1}{2}\, \textit{Option E is feasible on this basis.}$$

Therefore, Options B and C can be eliminated.

According to transformations, Option A has been translated vertically 1 unit and horizontally $\pi/4$ units. This should place the turning point located at $(\pi/2, 1)$ in the untransformed sin graph to $(3\pi/4, 2)$ in the transformed graph. Option A seems consistent with this.

According to transformations, Option D has been translated vertically 1 unit and horizontally $\pi/4$ units. This should place the turning point located at $(0, 1)$ in the untransformed cos graph to $(\pi/4, 2)$ in the transformed graph. Option D seems inconsistent with this since the turning point is more than $\pi/4$ units to the right.

According to transformations, Option E has been reflected about the $x$-axis, translated vertically 1 unit and horizontally $\pi/3$ units. This should place the turning point located at (0, 1) in the untransformed cos graph to ($-\pi/3$, 0) in the transformed graph. Option E is therefore definitely inconsistent with graph.

Therefore, the graph could be that of Option A.

**Question 7: D**

Which of the following expressions will provide the total area of the shaded region in the graph below?

The first and last sections are above the $x$-axis while the middle section is below. This means an adjustment needs to be made to the definite integral to ensure the middle section is positive. Therefore, the following sum of definite integrals would provide the total area of the shaded region.

$$Area=\int_a^b f(x)\,dx-\int_b^c f(x)\,dx+\int_c^d f(x)\,dx$$

Option A can be eliminated since it cannot be assumed that the area of this region is equal to the area of the region between $b$ and $c$.

Options B and C can be eliminated since they will not convert the section below the $x$-axis to a positive area. Option E reverses the terminals and multiplies the definite integral by -1 for the section below the $x$-axis. This will have the effect of producing a negative result since the two adjustments will cancel each other out. The first pair of definite integrals in Option D are identical to those shown above. However, two adjustments have been applied to the third definite integral in Option D, which will cancel each other out, resulting in a positive area. Option D is therefore correct.

**Question 8: C**

Conditional probability problem.

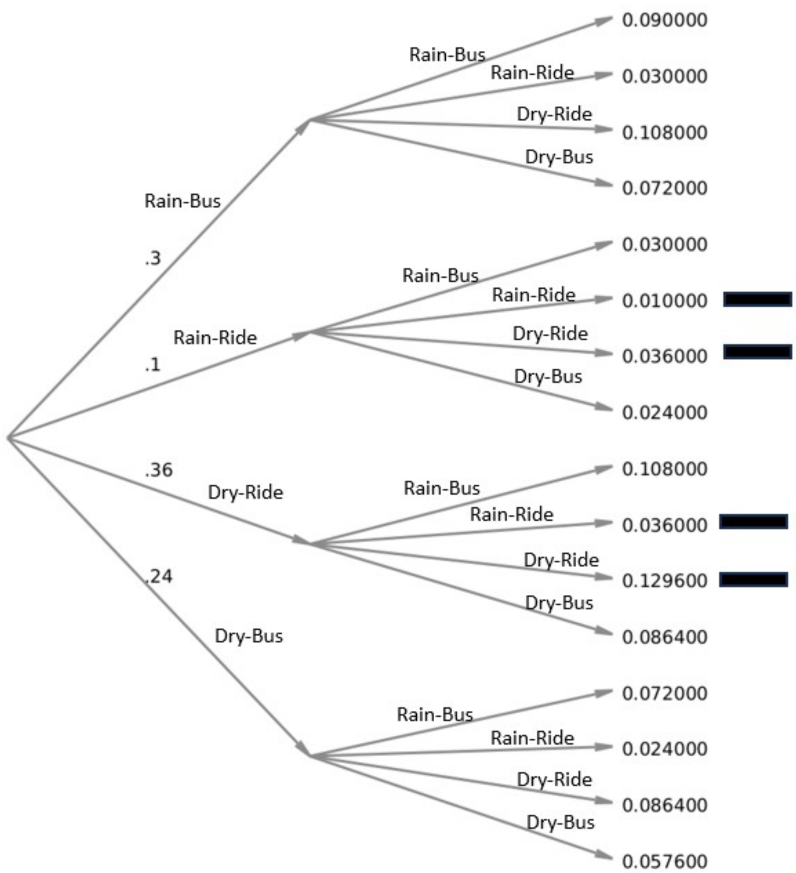
$$Pr(X<k \vee X>j)=\frac{Pr\ (X<k \cap X>j)}{Pr\ (X>j)}$$

$$Pr(X<k \cap X>j)=1-n-m\,,Pr(X>j)=1-m$$

$$\therefore Pr(X<k \vee X>j)=\frac{1-n-m}{1-m}$$

**Question 9: D**

A tree diagram can be used to determine the required probability.




The pathways that yield riding on both days are indicated by the solid bar on the diagram. Summing these probabilities gives 0.2116.

Alternatively, the probability can be determined by considering the feasible pathways. The outcomes are independent.

$$Pr(\text{Ride} \cap \text{Rain}) \times Pr(\text{Ride} \cap \text{Rain}) = 0.25 \times 0.4 \times 0.25 \times 0.4$$
$$Pr(\text{Ride} \cap \text{Rain}) \times Pr(\text{Ride} \cap \text{Dry}) = 0.25 \times 0.4 \times 0.6 \times 0.6$$
$$Pr(\text{Ride} \cap \text{Dry}) \times Pr(\text{Ride} \cap \text{Rain}) = 0.6 \times 0.6 \times 0.25 \times 0.4$$
$$Pr(\text{Ride} \cap \text{Dry}) \times Pr(\text{Ride} \cap \text{Dry}) = 0.6 \times 0.6 \times 0.6 \times 0.6$$

Summing these probabilities gives 0.2116.

**Question 10: B**

This is a Binomial distribution problem.

$$Pr\left(R \geq 2 \middle| p = 0.25 \wedge n = 3\right) = 0.1563$$

**Question 11: E**

$$\hat{p} = \frac{22}{50} = \frac{11}{25}$$

If width of confidence interval is 0.1 then $Z\sigma = 0.05$

$$Z\sigma = Z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$$

For a 95% confidence interval, $Z = 1.96$. This can be obtained using inverse Normal function on the CAS with area = 0.025, $\mu = 0$ and $\sigma = 1$ (then use right hand side tail for a positive $Z$ value).

$$1.96\sqrt{\frac{\frac{11}{25}\left(1-\frac{11}{25}\right)}{n}} = 0.05$$

$$\sqrt{\frac{\frac{11}{25}\left(1-\frac{11}{25}\right)}{n}} = 0.02551$$

$$\frac{0.2464}{n} = 0.000651$$

$$n = 378.6 \quad 379$$

**Question 12: B**

Evaluate each option using CAS.

**Question 13: E**

Set up a pair of equations using given information.

$$\frac{1}{m}\int_0^m f\left(x\right)dx = \frac{12-6\sqrt{3}}{\pi}$$

$$\frac{1}{\pi}\int_0^\pi g\left(x\right)dx = 0$$

Solve for *a* and *m* using CAS, applying the given restriction on *a* and *m*.

$a=2, m=\frac{1}{3}$

**Question 14: A**

The pseudo code works by systematically checking if each integer from to 2 up to half of the inputted number is a factor of the inputted number. The inner **while** loop subtracts the potential factor from the number until the result is zero, which indicates that it is a factor, or until the result is less than zero, which indicates that it is not a factor. If the integer being tested is not a factor, the outer **while** loop repeats with the integer to be tested set to the next value (ie to 3 then four and so on). If the integer being tested exceeds half of the value of the inputted number the **while** loops cease and the pseudo completes, concluding that the number is a prime. If an integer repeatedly subtracted from the inputted number results in zero, the pseudo code concludes that this integer is a factor and then ceases. This pseudo code will work, but it is not necessarily the most efficient algorithm, since it will test more integers that necessary before it concludes that a number is a prime.

The variable *num* is equal to 7 at the beginning and does not change. 7 is a prime number. If *num* is not a prime, $a = 1$. If it is a prime, it remains equal to its initial value of 0. The variable $c$ has an initial value of 2. Since it is not a factor of *num*, its value is increased by 1. Since 3 is not a factor of *num* and since 3 is less than or equal to *num*/2, $c$ is again increased by 1. Since 4 is greater than *num*/2, the pseudo code completes the final step after the while statements and before ending. The variable $t$ is initially set to the value of the inputted number (7). It then has the first integer (2) repeatedly subtracted from it until the result is less than zero. At this point it is then returned to the value of the inputted number and the process is repeated for the next integer (3). The number 3 will be subtracted from $t$ until the result is negative (since it won't equal 0 since 3 is not a factor of 7), after which time it is reset to the value of the inputted number. The next integer is greater than half of the inputted number and so the **while** loops cease. The final value of $t$ is therefore 7, the value of the inputted number. The value of $a$, $c$ and $t$ are not changed in this last step.

Therefore, at the completion of the pseudo code for *num* = 7 the values of the variables are:

$a = 0$, $c = 4$ and $t = 7$.

**Question 15: E**

$By\,symmetry, Pr(X<10)=Pr\ (Z\leftarrow 2)$

$Z=\frac{x-\mu}{\sigma}$

$-2=\frac{10-\mu}{\sigma}$ 1.

$Pr(Z>1.5)-Pr(Z>2)=0.066807-0.02275$

¿ $0.044057$

Use CAS to determine $k$, the value of $Z$, such that $\Pr(Z > k) = 0.044057$.

$k = 1.70543$

$By\, symmetry, Pr(X < 30) = Pr\ (Z \leftarrow 1.70543)$

$-1.70543 = \frac{30-\mu}{\sigma}$ 2.

Solve **1.** and **2.** Using CAS for μ and σ.

$\mu = 146, \sigma = 68$ ¿ *nearest whole number*.

By hand:

$-2 = \frac{10-\mu}{\sigma}$ 1.

$-1.70543 = \frac{30-\mu}{\sigma}$ 2.

From **1.**

$-2\sigma = 10 - \mu$

$\theta = 0.5\mu - 5$

Substitute $0.5\mu - 5$ for σ into **2.**

$-1.70543 = \frac{30-\mu}{0.5\mu - 5}$

$8.52715 - 0.852715\mu = 30 - \mu$

$0.147285\mu = 21.4729$

$\mu = 145.79$

$\therefore \sigma = 0.5 \times 145.79 - 5$

¿ $67.9$

$\mu = 146, \sigma = 68$ ¿ *nearest whole number*.

**Question 16: B**

$f(x) = \sqrt{3}\sin(\pi x/2)$ and $g(x) = -\cos(\pi x/2)$.

$\sqrt{3}\sin(\pi x/2) = -\cos(\pi x/2)$

$$\frac{\sqrt{3}\sin\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)} = -1$$

$$\sqrt{3}\tan\left(\frac{\pi x}{2}\right) = -1$$

$$\tan\left(\frac{\pi x}{2}\right)=\frac{-1}{\sqrt{3}}$$

$$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=\frac{-\pi}{6}$$

$$\frac{\pi x}{2}=n\pi+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$¿\, n\pi-\frac{\pi}{6}$$

$$\pi x=2n\pi-\frac{\pi}{3}$$

$$x=2n-\frac{1}{3}, Z\in Z$$

**Question 17: D**

A particular pdf has the rule $f(x)=ae^{-x^2}, x\in R^{+¿¿}$. The value of $a$ is:

Since $f(x)$ is a pdf:

$$\int_0^\infty a e^{-x^2}\,dx=1$$

Use CAS to solve for $a$.

$a=1.128$

**Question 18: C**

Determine which of the domains allows $f(x)$ to pass the horizontal line test.

Determine $x$- coordinates of turning points by solving $f'(x) = 0$.

$x \quad -3.5$ and $x \quad 0.2$

Since $f(x)$ is a positive cubic, the turning point with the greater $x$- coordinate is a local minimum, and the turning point with the lesser $x$-intercept is a local maximum.

Check domains in the options for which satisfies the horizontal line test.

**Question 19: A**

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$x_0=1$

$f'(x)=-e^x$

1st iteration

$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$

¿ $1-\frac{4-e}{-e}$

¿ $\frac{-e-4+e}{-e}$

¿ $\frac{4}{e}$

**Question 20: E**

$g(x)=6-4\left(a(2x+1)^2+b\right)$
¿ $-16ax^2-16ax-4a-4b+6$
Solve for *a* and *b* using CAS given:
$g(1)=-86$
$g(-1)=10$,

## Section B: Extended Answer Solutions – 60 marks

**Question 1 (11 marks)**

A family of curves can be modelled by the equation:

$$f(x)=ae^{nx-h}+k, a,h,k,n\in R \text{ and } a,n\neq 0$$

**a.** **i.** What is the equation of the asymptote of $f(x)$? **1 mark**

$y=k$

- **1 mark** for correct equation of the asymptote.

**ii.** Under what conditions does $f(x)$ have an $x$-intercept? **2 marks**

Solve $f(x) = 0$

$ae^{nx-h}+k=0$

$e^{nx-h}=\frac{-k}{a}$

$nx-h=\log_e\left(\frac{-k}{a}\right)$

$nx=\log_e\left(\frac{-k}{a}\right)+h$

$x=\frac{1}{n}\log_e\left(\frac{-k}{a}\right)+\frac{h}{n}$

The $x$-intercept will exist when:

$\frac{-k}{a}>0$

Therefore, $x$-intercept exists when:

$k>0\wedge a<0$

$k<0\wedge a>0$

- **2 marks** (1 mark for each set of conditions expressed in a valid form)

**iii.** What parameter(s) affect the range of $f(x)$? **1 mark**

$k$ will affect the range since it determines vertical translation of the graph.

$a$ will affect the range since it may cause reflection about the $x$-axis if it is less than 0.

- **1 mark** for inclusion of both $a$ and $k$, and not any other parameters.

**iv.** Determine the equation of $f^{1}(x)$, the inverse of $f(x)$. State its domain.

**2 marks**

$Let\ y=f(x)$

Let $x = y$ and $y = x$ and rearrange to make $y$ the subject. The rearrangement can be done using the CAS.

By hand:

$x=ae^{ny-h}+k$

$x-k=ae^{ny-h}$

$x-k=ae^{ny-h}$

$\frac{x-k}{a}=e^{ny-h}$

$ny-h=\log_e\left(\frac{x-k}{a}\right)$

$ny=\log_e\left(\frac{x-k}{a}\right)+h$

$y=\frac{1}{n}\log_e\left(\frac{x-k}{a}\right)+\frac{h}{n}$

$\therefore f^{-1}(x)=\frac{1}{n}\log_e\left(\frac{x-k}{a}\right)+\frac{h}{n}$

The domain of the inverse is equal to the range of $f(x)$.

The range of $f(x)$ is $(k,\infty)$ for when $a > 0$.

The range of $f(x)$ is $(-\infty,k)$ for when $a < 0$.

Therefore:

The domain of $f^{-1}(x)$ is $(k,\infty)$ for when $a > 0$.

The domain of $f^{-1}(x)$ is $(-\infty,k)$ for when $a < 0$.

- **1 mark** for a valid equation of the inverse.
- **1 mark** for inclusion of the domain presented in a valid form.

**c.** **i.** If $f'(0) = 1$, express $a$ in terms of $n$ and $h$ and hence express $f(x)$ in simplified form.

**2 marks**

$f'(x)=ane^{nx-h}$

Solve $f'(0)=1\ for\ a$ using CAS or by hand.

Buy hand:

$an\,e^{n(0)-h}=1$

$\frac{an}{e^{h}}=1$

$an=e^{h}$

$a=\frac{e^{h}}{n}$

$f(x)=a\,e^{nx-h}+k$

$\therefore f(x)=\frac{e^{h}}{n}e^{nx-h}+k$

¿ $\frac{1}{n}e^{nx-h+h}+k$

¿ $\frac{e^{nx}}{n}+k$

- **1 mark** for *a* expressed in terms of *n* and *h*.
- **1 mark** for $f(x)$ expressed in simplified form.

**ii.** Hence, determine the relationship between $f(0)$ and the area bound by $f(x)$, $x = 0$ and $y = 0$ for when $k = 0$, $x < 0$ and $n > 0$. **3 marks**

$Area=\int_{-\infty}^{0}\frac{e^{nx}}{n}dx$

¿ $\left[\frac{e^{nx}}{n^{2}}\right]_{-\infty}^{0}$

¿ $\frac{1}{n^{2}}-0$

¿ $\frac{1}{n^{2}}$

$f(0)=\frac{e^{n(0)}}{n}\vee k=0$

¿ $\frac{1}{n}$

$\therefore f(0)=n\times\text{Area}$

- **1 mark** for determination of relevant area in terms of n.
- **1 mark** for determination of $f(0)$.
- **1 mark** for provision of the relationship between $f(0)$ and the specified area.

**Question 2 (15 marks)**

Gretchen sells cars. The probability that she will sell a car within $t$ days can be modelled by the probability density function:

$$f(t)=\frac{a}{t+b},0<a,b<10$$

**a.** **i.** Given the following:

$f(0)=\frac{1}{6}$

median of $f(t)=(e^3-1)$ days

Determine the values of the parameters $a$ and $b$ and express $f(x)$ in terms of the parameter values.

$3\,marks$

$f(0)=\frac{a}{0+b}=\frac{1}{6}$

$6a=b1.$

$\int_0^{e^3-1}\frac{a}{t+b}\,dt=\frac{1}{2}$

$\left[a\log_e(t+b)\right]_0^{e^3-1}=\frac{1}{2}$

$a\log_e(e^3-1+b)-a\log_e(b)=\frac{1}{2}2.$

Solve system of equations using CAS.

$a=\frac{1}{6},b=1$

$\therefore f(t)=\frac{1}{6(t+1)}$

- **1 mark** for a valid system of equations.
- **1 mark** for a valid solution to system of equations.
- **1 mark** for a $f(t)$ rewritten with correct values of $a$ and $b$.

**ii.** Based upon $f(t)$, the probability density function, what is the maximum number of days that it may take for Gretchen to sell a car? Round answer to the nearest day. **1 mark**

Solve $\int_t^{t_{max}} f(t)\,dt=1$

$$\therefore \int_{t}^{t_{\max}} \frac{1}{6(t+1)}\,dt = 1$$

Solve using CAS.

$t_{\max} = e^{6} - 1$ days

= 402 days to the nearest day.

- **1 mark** for correct number of days.

**iii.** Gretchen had not sold a particular car by the 20th day. What is the exact probability that it will take her longer than 50 days to sell the car?

Conditional probability problem.

$$Pr(t>50 | t>20) = \frac{Pr\ (t>50 \cap t>20)}{Pr\ (t>20)}$$

¿ $$\frac{Pr\ (t>50)}{Pr\ (t>20)}$$

¿ $$\frac{1-\int_{0}^{50} f(t)\,dt}{1-\int_{0}^{20} f(t)\,dt}$$

¿ $$\frac{1-\int_{0}^{50} \frac{1}{6(t+1)}\,dt}{1-\int_{0}^{20} \frac{1}{6(t+1)}\,dt}$$

Evaluate using CAS.

$$Pr(t>50 | t>20) = \frac{\log_e(51)-6}{\log_e(21)-6}$$

- **1 mark** for a valid consideration of conditional probability.
- **1 mark** for correct answer.

**b.** Gretchen specialises in the sale of small cars. The sale price follows a normal distribution. Let $X$ be the random variable corresponding to the sale price of a car, in thousands of dollars. The mean and standard deviation of $X$ are $\mu$ and $\sigma$ respectively. $Z$ is the standard normal variable.

**i.** Given $Pr(X<3\sigma)=Pr\ (Z>1)$ and $Pr(X<\mu)=2\,Pr\ (Z<13.3)$, determine the mean and standard deviation of X. Round each to the nearest while number. **2 marks**

$Pr(X<3\sigma)=Pr\ (Z>1)$

$\therefore Z=\frac{x-\mu}{\sigma}$

$\therefore -1=\frac{3\sigma-\mu}{\sigma}$

$-\sigma=3\sigma-\mu$

$\mu=4\sigma\,1.$

$Pr(X<\mu)=2\,Pr\ (X<13.3)$

$2\,Pr(X<13.3)=\frac{1}{2}$

$Pr(X<13.3)=\frac{1}{4}$

Determine corresponding $Z$ value using CAS.

$Z=-0.6745$

$Z=\frac{x-\mu}{\sigma}$

$-0.6745=\frac{13.3-\mu}{\sigma}\,2.$

Solve system of equation using CAS.

By hand:

Substitute $4\sigma$ for $\mu$ into **2.**

$-0.6745=\frac{13.3-4\sigma}{\sigma}$

$-0.6745\sigma=13.3-4\sigma$

$3.3255\sigma=13.3$

$\sigma=4$

$\therefore \mu=16$

- **1 mark** for a valid system of equations for determining $\mu$ and $\sigma$.
- **1 mark** for correct values of $\mu$ and $\sigma$ rounded to the nearest whole numbers.

**ii.** What is the probability that Gretchen sells two of the next three cars at more than 10% above the average sale price? Round answer to 4 decimal places. **2 marks**

The average sale price is \$16 k.
10% more than 16 is 17.6
$Pr(X > 17.6) = 0.3446\,(determine\ using\ CAS)$
Pr($X$ > 17.6) for two of the next three sales is a Binomial distribution problem.
$n = 3,\ p = 0.3446,\ lower\ bound = 2,\ upper\ bound = 2$

$Pr(X > 17.6)\ for\ 2\ of\ the\ next\ 3\ cars = 0.2335$¿ $4\ decimal\ places.$

- **1 mark** for correct probability for $X$ > 10% of mean.
- **1 mark** for correct probability that 2 of 3 sales will be greater than 10% of the mean, rounded to 4 decimal places.

**iii.** What is the minimum number of cars that Gretchen would need to sell, such that the probability that not more than one sells below \$11,000 is not more than 0.05? **3 marks**

Determine Pr($X$ < 11) using CAS with mean of 16, standard deviation of 4.
$Pr(X < 11) = 0.1057$
A car will either sell below 11,000 dollars, or it will not. Therefore, Binomial distribution problem.
Let $B$ be a random variable corresponding to the number of cars sold below \$11,000 from $n$ cars sold.
$n = ?,\ p = 0.1057,\ lower\ bound = 0,\ upper\ bound = 1$
$\therefore Pr$ ¿B = 0) + Pr($B$ = 1) = 0.05
$\text{nCr}(n, 0)\, p^0 (1-p)^n + \text{nCr}(n, 1)\, p^1 (1-p)^{n-1} = 0.05$
$(0.8943)^n + 0.1057\, n (0.8943)^{n-1} = 0.05$
Solve for $n$ using CAS.
$n = 42.97.$
Check which way $n$ value must be rounded.
If $n$ = 43, $Pr$ ¿$B$ = 0) + Pr($B$ = 1) =0.0499, which is less than 0.05.
If $n$ = 42, $Pr$ ¿$B$ = 0) + Pr($B$ = 1) =0.0547, which is more than 0.05.
Therefore, minimum number of cars is 43.
An alternative approach would be trial-and-error using the Binomialcdf function on the CAS.

- **1 mark** for correct p value for sale price less than \$11,000
- **1 mark** for a valid approach involving the Binomial distribution.
- **1 mark** for the correct minimum number of cars sales.

**c.** Gretchen works for a large car dealership with many stores across the country. Achmed, the marketing manager, conducted a survey to measure customer satisfaction. 50 people were surveyed, and a 95% confidence interval was constructed. The margin of error (width of half of the confidence interval) was 0.12. Given a preliminary survey indicated that customer satisfaction was greater than

50%, what was the sample proportion, rounded to 2 decimal places? **2 marks**

Margin of error = half of width of confidence interval.

$\therefore 0.12 = Z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$95\%\ confidence\ intervale\ so\ Z = 1.96, n = 50$

$0.12 = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{50}}$

Solve for $p$ using CAS. Alternatively:

$\sqrt{\frac{\hat{p}(1-\hat{p})}{50}} = 0.06122$

$\frac{\hat{p}(1-\hat{p})}{50} = 0.003748$

$\hat{p}(1-\hat{p}) = 0.1874$

$\hat{p}^2 - \hat{p} + 0.1874 = 0$

$\hat{p} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1, b = -1, c = 0.1874$

$\hat{p} = \frac{1 \pm \sqrt{1 - 0.7496}}{2}$

¿$0.75\ \text{or}\ 0.25$

However, since it is known that $\hat{p} > 0.5, \hat{p} = 0.75$.

- **1 mark** for a valid equation relating given information to $\hat{p}$.
- **1 mark** for correct value of $\hat{p}$ rounded to 2 decimal places.

**Question 3 (11 marks)**

A cylindrical water tank is to be constructed for a local town. Its capacity is 200 kL (200,000 L). The sides of the tank cost \$200/m$^2$ to manufacture while the top and bottom cost \$100/m$^2$.

**a.** Show that surface area, $S$ (m$^2$) of the water tank can be expressed according to the following equation (where $r$ is the radius of the tank in m): **2 marks**

$$S = \frac{400}{r} + 2\pi r^2$$

Consider the volume of the tank:

$V = \pi r^2 h$

$200 = \pi r^2 h$

$h = \frac{200}{\pi r^2}$

Consider the surface area of the tank:

$S = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r \times \frac{200}{\pi r^2}$

$= 2\pi r^2 + \frac{400}{r}$

$= \frac{400}{r} + 2\pi r^2$

- **1 mark** for developing an expression for *h* using the tank volume.
- **1 mark** for showing that when this expression for *h* is substituted into the formula for the surface area of a cylinder, the required equation results.

**b.** **i**. Develop an equation that provides the cost of materials for the water tank. **1 mark**

The sides of the tank cost $200/m$^2$ and the base/top cost $100/m$^2$.

Let $C$ be the cost of the tank.

$C(r)$ = cost of sides x area of sides + cost of base/top x area of base/top

¿ $100 \times 2\pi r^2 + 200 \times \frac{400}{r}$

¿ $200\pi r^2 + \frac{80000}{r}$

- **1 mark** for a valid equation for the cost.

**ii.** What is the radius that would provide the lowest cost for the water tank. Round answer to 2 decimal places. **2 marks**

Solve $C'(r) = 0$ for $r$ using CAS.

$$r = \frac{2 \times 5^{\frac{2}{3}}}{\pi^{\frac{1}{3}}}$$

¿ $3.99\, m$ to 2 decimal places.

- **1 mark** for a valid calculus or graphical approach.
- **1 mark** for the correct radius rounded to 2 decimal places.

**iii.** Hence, what is the minimum cost for the water tank, rounded to the nearest dollar? **1 mark**

Substitute $r = 3.99$ into the cost function developed above.

$$C(3.99) = 200\pi(3.99)^2 + \frac{80000}{4.99}$$

¿ $\$30{,}053$

- **1 mark** for the correct cost rounded to the nearest dollar.

**c.** Water must be pumped to the tank from a local reservoir. The difference in elevation between the surface of the reservoir and base of the water tank is $d$ meters. The volume of water per second that the pump can deliver to the tank depends upon $d$ and can be modelled by the equation:

$$V(d) = a\sqrt{k - 2d}, a, k \in R \text{ and } a \neq 0$$

**i.** When the difference in elevation is 20 m, the pump can deliver $5\sqrt{15}$ L/s. However, when the difference in elevation is 40 m it can only deliver $5\sqrt{5}$ L/s. Determine the exact values of the parameters *a* and *k*. **2 marks**

$V(20)=a\sqrt{k-40}=5\sqrt{15}$ 1.

$V(40)=a\sqrt{k-80}=5\sqrt{5}$ **2.**

Solve system of equations using CAS. Alternatively:

$\frac{1.}{2.}$ to eliminate $a$

$$\frac{a\sqrt{k-40}}{a\sqrt{k-80}}=\frac{5\sqrt{15}}{5\sqrt{5}}$$

$$\frac{\sqrt{k-40}}{\sqrt{k-80}}=\sqrt{3}$$

$$\frac{k-40}{k-80}=3$$

$$k-40=3k-240$$

$$2k=200$$

$$k=100$$

Substitute $k$ = 100 into **1.**

$$a\sqrt{k-40}=5\sqrt{15}$$

$$a\sqrt{100-40}=5\sqrt{15}$$

$$a\sqrt{60}=5\sqrt{15}$$

$$2a\sqrt{15}=5\sqrt{15}$$

$$2a=5$$

$$a=\frac{5}{2}$$

- **1 mark** for a valid system of equations using given information.
- **1 mark** for correct values of *a* and *k*.

**ii.** What is the maximum rate that the pump can deliver to the tank?

$$V(d)=\frac{5}{2}\sqrt{100-2d}$$

As *d* decreases, rate increases. Maximum rate will therefore occur when $d$ = 0.

$$V(0)=\frac{5}{2}\sqrt{100-2(0)}$$

¿ $25\,L/s$

- **1 mark** for the correct maximum rate.

**iii.** The water level in the reservoir varies throughout the year. This means the difference in elevation between the surface of the reservoir and the base of the tank also varies throughout the year. The equation that models the difference in elevation between the surface of the reservoir and the base of the tank is: **2 marks**

$$d(t)=40+12\cos\left(\frac{\pi t}{6}\right)$$

Where $t$ is in months and $t = 0$ corresponds to the 1st of March.

Determine the percentage of the year, rounded to the nearest whole number, that the pump is unable to transfer water from the reservoir to the tank.

$d$ is a function of $t$ as specified by the equation provided. But volume that can be pumped per second is a function of $d$. Therefore, the volume that can be pumped is a function of $t$.

$V(t)=\frac{5}{2}\sqrt{100-2d(t)}$

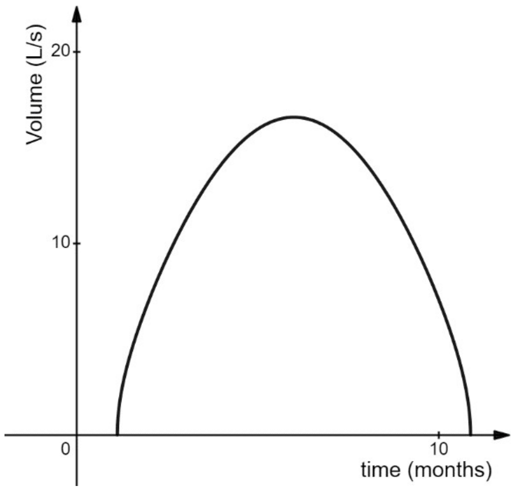
¿ $\frac{5}{2}\sqrt{100-2\left(40+12\cos\left(\frac{\pi t}{6}\right)\right)}$

¿ $\frac{5}{2}\sqrt{20-24\cos\left(\frac{\pi t}{6}\right)}$

Since the square root function only accepts numbers greater than 0, water cannot be pumped when:

$20-24\cos\left(\frac{\pi t}{6}\right)<0$

The graph of $V(t)$ for one year is shown below. This could be generated on the CAS. Note that no water is able to be pumped at the start and end of the year.




Solve using the CAS. Water cannot be pumped when:

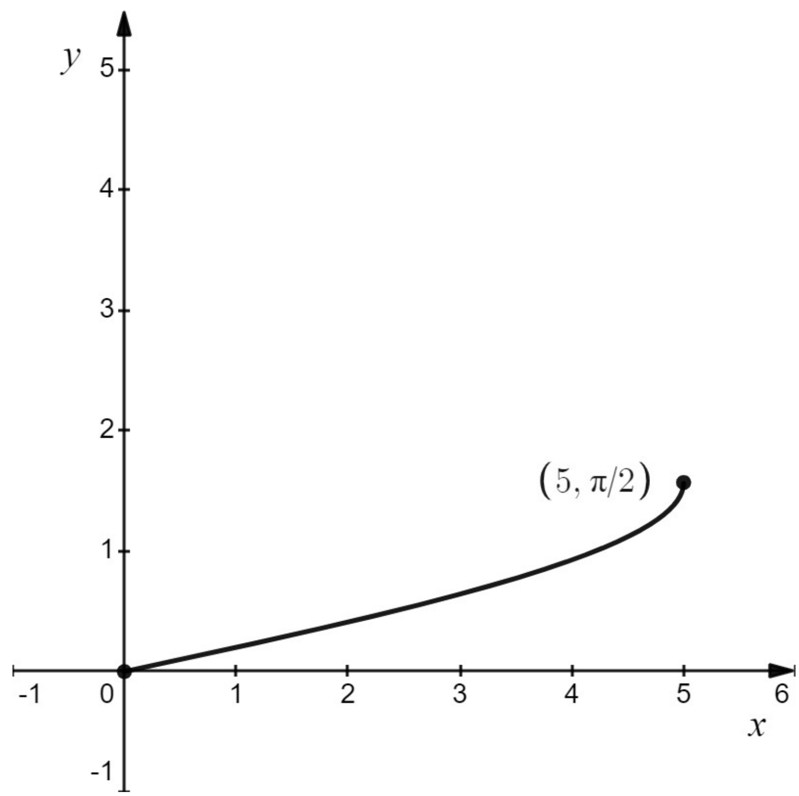
$0 < t < 1.12 \cup 10.88 < t < 12$

Therefore, for 2.24 months of the year water cannot be pumped to the tank. This is 18.7% of the year, or 19% rounded to the nearest whole number.

- **1 mark** for a valid equation relating *V* to *t*.
- **1 mark** for the correct percentage, rounded to the nearest whole number.

**Question 4 (7 marks)**

The graph of $f(x) = \sin^{-1}\left(\frac{x}{5}\right)$, the inverse of $g(x) = 5\sin(x)$, is shown below for $0 \le x \le 5$.




**a.** Use the Trapezium Rule to determine an estimate of the area bound by $f(x)$, the $x$-axis and the lines $x = 0$ and $x = 5$. Use trapeziums of width 1 unit. Round estimate to 2 decimal places. **2 marks**

$$\text{Area (estimate)} = \frac{1}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right] \times 1$$

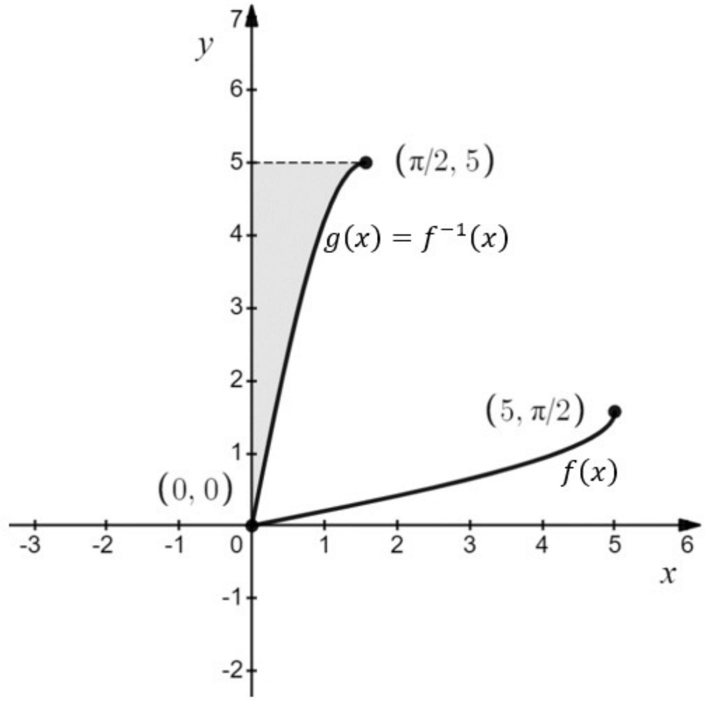
$$¿\frac{1}{2}\left[0 + 0.4027 + 0.8230 + 1.2870 + 1.8546 + 1.5708\right]$$

$¿2.97$ square units to 2 decimal places.

- **1 mark** for a valid approach for application of the Trapezium Rule.
- **1 mark** for correct estimate of area rounded to 2 decimal places.

**b.** Sketch the graph of $g(x)$ $(f^{-1}(x)¿$ on the axes above, labelling the end points. Shade in the region bound by $g(x)$ and the $y$-axis that corresponds to the areas described in Part **a**. **2 marks**

The graph of the inverse will be a reflection of $f(x)$ about the line $y = x$. The coordinates of the endpoints will be the reverse of the endpoint of $f(x)$.




- **1 mark** for accurate sketch of inverse of $g(x)$ (correct shape and position and endpoints labelled).
- **1 mark** for correct region shaded.

**c.** Write an equation involving a definite integral involving $g(x)$ that will provide the area of the region described in Part **a**. **1 mark**

Area required is the area bound by $y = 5$, $g(x)$ and the lines $x = 0$ and $x = \pi/2$.

$$Area = \int_0^{\frac{\pi}{2}} 5 - 5\sin(x)\,dx$$

- **1 mark** for a definite integral that will provide the required area.

**d.** Hence, determine exact value of the area of the region described in part **a**. What is the percent difference, rounded to the nearest whole number, between the estimate obtained using the Trapezium Rule in Part **a** and the actual area? **2 marks**

$$Area = \int_0^{\frac{\pi}{2}} 5 - 5\sin(x)\,dx$$

Evaluate using CAS. Alternatively:

$\text{Area} = \left[5x + 5\cos(x)\right]_0^{\frac{\pi}{2}}$

¿ $\left(\frac{5\pi}{2}+0\right)-(0+5)$

¿ $\frac{5\pi-10}{2}\,\textit{square units}$

2.85

$\%\,\text{difference} = \frac{2.97-2.85}{2.85} \times 100$

$= 4\%$

- **1 mark** for correct area.
- **1 mark** for the correct percentage difference rounded to the nearest whole number.

**Question 5 (16 marks)**-

A family of functions has the form $f(x) = ax\,e^{-nx}, 0 \le x \le \infty$ and $a, n \in R^{+\text{¿¿}}$

**a.** Given $f(x)$ is a probability density function, rewrite $f(x)$ in a form which includes only the parameter $n$. **2 marks**

$$\int_0^{\infty} ax\,e^{-nx}\,dx = 1$$

Evaluate using CAS and express $a$ in terms of $n$ (note that n >0).

$a = n^2$

$\therefore f(x) = n^2 x\,e^{-nx}, 0 \le x \le \infty$ and $a, n \in R^{+\text{¿¿}}$

- **1 mark** for a valid application of the properties of a pdf.
- **1 mark** for $f(x)$ expressed in terms of the parameter $n$.

**b.** Express the mean of $f(x)$ in terms of the mode of $f(x)$.

Determine the mean:

$$\mu = \int_0^\infty x f(x)\,dx$$

$$¿\int_0^\infty n^2 x^2 e^{-nx}\,dx$$

Evaluate using CAS

$$\mu = \frac{2}{n}$$

Determine the mode:

Examination of a graph of $f(x)$ shows that the mode occurs at the turning point of $f(x)$

Solve $f'(x) = 0$ for $x$

$$f'(x) = (n^2 - n^3 x) e^{-nx}$$

$$(n^2 - n^3 x) e^{-nx} = 0$$

$$n^2 - n^3 x = 0$$

$$n^3 x = n^2$$

$$x = \frac{1}{n}$$

$$\therefore \text{mode} = \frac{1}{x}$$

$$\therefore \mu = 2 \times \text{mode}$$

- **1 mark** for determination of the mean.
- **1 mark** for determination of the mode.
- **1 mark** for expressing mean in terms of the mode.

**c.** **i.** A particular equation for $f(x)$ is used to model the magnitude of the difference between the actual arrival time and the scheduled arrival time, in minutes, for a city's train system. A train is considered 'on time' if this difference is less than one minute.

The probability that at least two out of three trains arrive on time is 0.8968. Determine the value of $n$ for this distribution. Round the values of $n$ to the nearest whole number.

A train is either on time or it is not. This is a Binomial distribution problem. Let $X$ be a random variable representing the number of trains that are on time from $n$ trains. The probability that $X = 2$ or $X = 3$ given $n = 3$ is known. Use the Binomial distribution formula to relate the given probability to the probability, $p$, that a particular train arrives on time. **3 marks**

$$Pr(X=2 \vee X=3 \vee n=3)=nCr(3,2)\,p^2(1-p)+p^3(1-p)^0$$
$$0.8968=3p^2(1-p)+p^3$$
Solve for $p$ using CAS.
$$p=0.8008$$

$$\therefore Pr(x<1)=0.8008$$
$$\int_0^1 n^2 x e^x \, dx=0.8008$$
Solve for $n$ using CAS.
$$n=3$$

- **1 mark** for application of the Binomial distribution to find of trains arriving on time.
- **1 mark** for valid definite integral relating *f*(*x*), suitable terminals, and *f*(*x*) to the probability calculated using the Binomial distribution.
- **1 mark** for the correct value of *n* rounded to the nearest whole number.

**ii.** A company operates the trains on behalf of the city council. The company receives a bonus if their P-Score is less than 1. The P-Score is the magnitude of the difference between the scheduled and actual arrival time, such that 90% of train arrivals are less than this score.

The manager of the company had a plan that would reduce its P-Score by 15%. Using the value of *n* obtained in part **c.i,**, determine if this reduction will be sufficient to allow the company to receive the bonus.

Determine the time by which 90% of trains would arrive. **3 marks**

$$0.9=\int_o^m f(x)\,dx$$

$$¿\int_0^m 9xe^{-3x}\,dx$$
Solve for $m$ using CAS.
$$m=1.2966 \text{ minutes}$$
Thus, 90% of trains arrive within 1.2966 minutes of the scheduled arrive time. Hence, the P-Score for the company is 1.2966.

If the P-Score was reduced by 15% it would be 1.102.

Since the P-Score is greater than 1 the plan would not reduce it sufficiently enough for company to receive the bonus.

- **1 mark** for correct calculation of the company's current P-Score.
- **1 mark** for correct calculation of the planned P-Score.

- **1 mark** for the correct conclusion regarding whether the planned reduction to the P-Score will allow the company to receive the bonus.

**iii.** What is the minimum value of $n$ in the model that would allow the company to receive the bonus? Round you answer to 2 decimal places. **2 marks**

Construct a definite integral for determination of the P-Score that would allow for the bonus.

$$0.9=\int_{o}^{1} f(x)\,dx$$

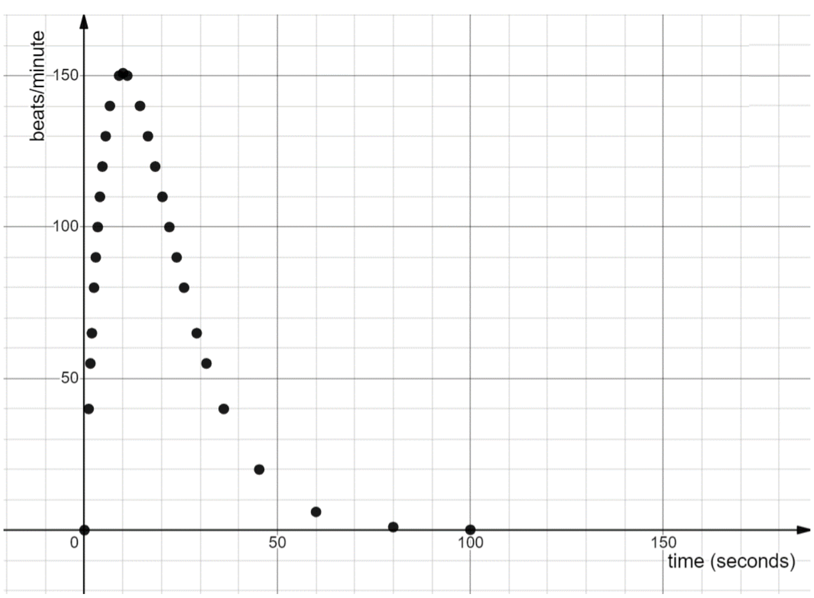
$$¿\int_{0}^{1} n^2 x e^{-nx}\,dx$$

Solve for $n$ using CAS.
$n=3.89$ rounded to 2 decimal places.

- **1 mark** for a valid definite integral relating $f(x)$, $n$ and the minimum conditions necessary for the company to receive the bonus.
- **1 mark** for the correct value of $n$ rounded to decimal places.

**d.** The family of curves introduced at the start of the question can also be used to model heart rate (beats/minute) in response to a sudden stress, such as a fright. When a person receives such a stress, the heart rate increases rapidly, then gradually returns to its initial value over a period of several seconds or minutes, depending upon the individual and the specific stress.

One person in a study had their heart rate monitored when, suddenly, a rubber spider was dangled from a string before their eyes. Their heart rate response to this stress is shown in the graph below. In the graph, 0 bps (beats/minute) corresponds to their resting heart rate and time = 0 corresponds to the moment the rubber spider was dangled before the person.



Using the data provided in the graph, determine the values of the parameters *a* and *n* in the model. Round *a* to the nearest whole number and *n* to 1 decimal place. If the resting heart rate of the person was 55 bps, what was their average heart rate during the first 50 seconds of the measurements? **3 marks**

$f(x)=ax\,e^{-nx}$

Select 2 data points whose coordinates can be reasonably accurately read from the graph.
Thus:
$f(20)=110$
$f(45)=20$
Solve this system of equations using CAS.
$a=41$ rounded to the nearest whole number.
$n=0.1$ rounded to 1 decimal place.

$f(x)=41\,x\,e^{-0.1\,x}$
$f$ ¿

¿ $\frac{1}{50}\int_{0}^{50} 41\,x\,e^{-0.1\,x}\,dx$

¿ 79 rounded to the nearest whole number.

Therefore, the average heart rate of the person in the first 50 seconds after the initial fright is:
79 + resting heart rate = 79 + 55 = 134 bpm

- **1 mark** for system of equations using data from the graph.
- **1 mark** for correct value of *a* and *n,* rounded to the nearest whole number and 1 decimal place respectively.
- **1 mark** for correct average heart rate.