

# MATHEMATICAL METHODS

## Written examination 2



## 2023 Trial Examination

### SOLUTIONS

#### SECTION 1: Multiple-choice questions (1 mark each)

1.	A	11.	A
2.	C	12.	C
3.	E	13.	E
4.	B	14.	A
5.	C	15.	B
6.	B	16.	C
7.	C	17.	D
8.	E	18.	E
9.	D	19.	E
10.	A	20.	A

**Question 1**

*Answer:* A

*Explanation:*

$$\begin{aligned} -1 &= 1 - 2x, & x &= 1 \\ 4 &= 1 - 2x, & x &= -\frac{3}{2} \end{aligned}$$

**Question 2**

*Answer:* C

*Explanation:*

$$\left(\frac{\pi}{2}, 0\right) \text{ and } (\pi, -2)$$

$$\begin{aligned} (0 - -2)^2 + \left(\frac{\pi}{2} - \pi\right)^2 \\ = 4 + \frac{\pi^2}{4} \end{aligned}$$

**Question 3**

*Answer:* E

*Explanation:*

$$\text{Range: } 1 \pm 4 \quad T = \frac{2\pi}{2} = \pi$$

**Question 4**

*Answer:* B

*Explanation:*

$$\text{dom } h^{-1} = \text{ran } h = [4, \infty)$$

$$x = 2(x - 3)^2 + 4$$

$$x - 4 = 2(x - 3)^2$$

$$\frac{x - 4}{2} = (x - 3)^2$$

$$\pm \sqrt{\frac{x - 4}{2}} = x - 3$$

$$\text{so } h^{-1}(x) = \pm \sqrt{\frac{x - 4}{2}} + 3$$

Discard negative as we want the upper section.

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### Question 5

Answer: C

Explanation:

$$0.2^2 + 0.35^2 + 0.35^2 + 0^2 + 0.1^2 = 0.295$$

### Question 6

Answer: B

Explanation:

$$\frac{y}{a} = \log_b(c - 2x)$$

$$b^{\frac{y}{a}} = c - 2x$$

$$2x = c - b^{\frac{y}{a}}$$

$$x = \frac{c - b^{\frac{y}{a}}}{2}$$

### Question 7

Answer: C

Explanation:

$$14 = \frac{2\pi}{b}, b = \frac{\pi}{7}$$

Negative cosine graph with mean value 2.5 and amplitude 2.5

$$\text{Hence } y = 2.5 - 2.5 \cos\left(\frac{\pi t}{7}\right)$$

### Question 8

Answer: E

Explanation:

Sample proportion is midpoint so  $\frac{0.428+0.612}{2} = 0.52$

### Question 9

Answer: D

Explanation:

$$y''(x) = 12x - 4, \text{ so } y''(x) = 0 \text{ at } x = \frac{1}{3}, \text{ with } y''(x) > 0 \text{ and } y''(x) < 0 \text{ on either side.}$$

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### Question 10

Answer: A

Explanation:

$$64 + 16m + n = 100$$

$$f'(x) = 3x^2 + 2mx$$

$$f'(4) = 0$$

$$48 + 8m = 0$$

$$m = -6, n = 132$$

### Question 11

Answer: A

Explanation:

$$\begin{aligned} (-5 \times a) + (0 \times 4a) + 5(1 - 5a) \\ = 5 - 30a \end{aligned}$$

### Question 12

Answer: C

Explanation:

Need same gradient but different y intercepts.

$$\begin{aligned} \frac{a}{5} = \frac{2}{a} \neq \frac{b}{5} \\ a^2 = 10 \\ a = \sqrt{10} \end{aligned}$$

$$\frac{5}{\sqrt{10}} \neq \frac{b}{2}, b \neq \sqrt{10}$$

### Question 13

Answer: E

Explanation:

$$\begin{aligned} \text{Area: } A(x) &= x\sqrt{16 - x^2} \\ A'(x) &= \sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}} \\ A'(x) &= 0 \text{ at } x = 2\sqrt{2} \\ A(2\sqrt{2}) &= 8 \end{aligned}$$

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### Question 14

Answer: A

Explanation:

$$\Pr(\hat{P} = 0) = (1 - p)^4 = \frac{16}{625} = \left(\frac{2}{5}\right)^4$$
$$\text{Hence } p = \frac{3}{5}$$
$$\Pr(\hat{P} = 1) = p^4 = \frac{81}{625}$$

### Question 15

Answer: B

Explanation:

$$\Pr(X > 265) = 0.10565$$
$$\text{So } \Pr\left(Z < \frac{265 - 250}{\sigma}\right) = 0.89435$$
$$\frac{15}{\sigma} = 1.25$$
$$\sigma = 12$$

### Question 16

Answer: C

Explanation:

$$f(x) = x^3 - 3x, \quad f'(x) = 3x^2 - 3$$
$$\text{So } x_{n+1} = x_n - \frac{x_n^3 - 3x_n}{3x_n^2 - 3}$$
$$x_{n+1} = \frac{4x_n^3 - 6x_n}{3x_n^2 - 3}$$

### Question 17

Answer: D

Explanation:

$$\text{Area} = \frac{1}{2} \times \frac{1}{2} (f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4))$$
$$= \frac{245}{16} = 15.3125$$

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### Question 18

Answer: E

Explanation:

$$\int_0^{\frac{\pi}{4}} f(x) dx = 1 \text{ so } a = 2$$

$$E(X) = \int_0^{\frac{\pi}{4}} xf(x) dx = \frac{\pi - 2}{4}$$

### Question 19

Answer: E

Explanation:

Look for first time  $\text{abs } f(x) < 0.001$

N	x	f(x)	g(x)
0	2	-10.5858	-12.7071
1	1	1.16694	-3.00499
2	0.787873	-0.97461	-5.85384
3	0.621382	-0.37007	-4.99688
4	0.547322	-0.15887	-4.63563
5	0.51305	-0.07339	-4.47441
6	0.496649	-0.03525	-4.39887
7	0.488636	-0.01727	-4.36238
8	0.484677	-0.00855	-4.34446
9	0.48271	-0.00425	-4.33558
10	0.481729	-0.00212	-4.33116
11	0.481239	-0.00106	-4.32895
12	0.480994	-0.00053	-4.32785
13	0.480872	-0.00026	-4.3273
14	0.480811	-0.00013	-4.32702
15	0.48078	-6.6E-05	-4.32689
16	0.480765	-3.3E-05	-4.32682
17	0.480757	-1.7E-05	-4.32678

### Question 20

Answer: A

Explanation:

$$\begin{aligned} \Pr(\text{both same}) &= \Pr(\text{blue, blue}) + \Pr(\text{black, black}) = \frac{u}{u+v} \times \frac{u-1}{u+v-1} + \frac{v}{u+v} \times \frac{v-1}{u+v-1} \\ &= \frac{u^2 + v^2 - u - v}{(u+v)(u+v-1)} \end{aligned}$$

## SECTION 2

## Question 1

a.  $\text{dom } f = (2, \infty)$   $\text{ran } f = R$

2 marks

b.  $f'(x) = \frac{4}{(x-2)\log_e(2)}$

1 mark

c.  $f'(6) = \frac{1}{\log_e(2)}$

$$y - 9 = \frac{1}{\log_e(2)}(x - 6)$$

$$y = \frac{1}{\log_e(2)}(x - 6) + 9$$

2 marks

d.  $f'(x) = \frac{1}{2\log_e 2}$  at  $x = 10$

1 mark

$$f(10) = 13$$

1 mark

$$\text{Distance} = \sqrt{(10 - 0)^2 + (13 - 0)^2} = \sqrt{269}$$

1 mark

e. Translation of 2 units left and one unit down followed by a dilation of factor  $\frac{1}{4}$  from the  $x$  axis

3 marks

Total 11 marks

## Question 2

a. Max temp occurs at  $T = 24.2^\circ\text{C}$  occurring at  $t = 6$  and  $t = 18$

2 marks

b.  $T = \frac{2\pi}{6} = 12$  hours

1 mark

c.  $T(t) = 22$  at  $t = 3.607, 8.393, 15.607$  and  $20.393$

$$2 \times (8.393 - 3.607) = 9.572$$

$$\frac{9.572}{24} \times 100 = 40\% \text{ of the time greater than } 22^\circ\text{C}$$

2 marks

d.

$T(t) = 23$  at  $t = 4.289, 7.711, 16.289$  and  $19.711$

$$2 \times (7.711 - 4.289) = 6.842 \text{ so } 17.158 \text{ hours}$$

$$\frac{17.158}{24} \times 100 = 71\% \text{ of the time less than } 23^\circ\text{C}$$

2 marks

e.  $4 \times (4.289 - 3.607) = 2.728 \text{ hours} = 2\text{h } 44\text{min}$

2 marks

Total 9 marks

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### Question 3

a.  $\Pr(X = 4) = \binom{10}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^6 = 0.2051$

1 mark

b.  $\Pr(X \leq 7) = \sum_{x=0}^7 \binom{10}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = 0.9453$

1 mark

c.  $\Pr(X \leq 4) = \sum_{x=4}^{10} \binom{10}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = 0.9453$

$$\Pr(4 \leq X \leq 7) = \sum_{x=4}^7 \binom{10}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = 0.7734$$

1 mark

So  $\Pr(X \geq 4 | X \leq 7) = \frac{0.7734}{0.9453} = 0.818$

1 mark

d.  $E(X) = np = 10 \times \frac{1}{2} = 5$

1 mark

$$SD(X) = \sqrt{np(1-p)} = \sqrt{10 \times \frac{1}{2} \times \frac{1}{2}} = \frac{\sqrt{10}}{2}$$

2 marks

e.  $\Pr(X = 10) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$

1 mark

Total 8 marks

### Question 4

a.  $t = 0$  when  $A = 8.1$  so  $c = 8.1$

1 mark

$A(2) = 11.043$  and  $A(10) = 8.370$

$a = 4, b = 2, c = 8$

2 marks

b. From graph see that max occurs at  $t = 2$  where  $A = 11.043 \text{ mmol/L}$

1 mark

c.  $A(t) = 10$  at  $t = 0.66106$  and  $t = 4.49464$

1 mark

$4.49464 - 0.66106 = 3.8336 \text{ min} = 3 \text{ min } 50 \text{ sec}$

1 mark

d.  $A'(t) = 4e^{-\frac{t}{2}} - 2te^{-\frac{t}{2}}$

1 mark

e. From graph of  $A'(t)$  can see that max increase occurs at  $t = 0$  and max decrease occurs at  $t = 4$

2 marks



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f. 7.9mmol/L 1 mark

g. Max is 10.659mmol/L and this occurs at  $t = 2.5$  2 marks

h.  $A(t) = B(t)$  at  $t = 3.702$  and  $11.508$  1 mark

Hannah's blood sugar is greater than Freddie's during these times  
Hence  $11.508 - 3.702 = 7.806$  min = 7min 48sec

2 marks  
Total 15 marks

**Question 5**

a.  $\int_0^4 a(4x^3 - x^4) dx = 1$  1 mark

$$a = \frac{5}{256}$$

1 mark

b.  $E(X) = \int_0^4 x f(x) dx$  1 mark

$$x = \frac{8}{3} \text{ days}$$

1 mark

c.  $\int_3^4 f(x) dx = \frac{47}{128} = 37\%$  2 marks

d.  $\int_2^3 f(x) dx = \frac{57}{128}$   
 $\int_2^4 f(x) dx = \frac{13}{16}$  1 mark

$$\frac{\frac{57}{128}}{\frac{13}{16}} = \frac{57}{104}$$

1 mark

e.  $\Pr(30 < X < 40) = 0.5249$  1 mark

f.  $\Pr(X < a) = 0.1$   
 $a = 26.02\text{mm}$  is the largest small worm.

$\Pr(X > b) = 0.15$   
 $b = 42.26\text{mm}$  is the shortest large worm. 2 marks

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- g. 75% of all worms are standard length.  
 $0.75^{20} = 0.0032$

1 mark

h.  $E(\hat{p}) = p = 0.75$

1 mark

$$SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.75 \times 0.25}{20}} = 0.0968$$

2 marks

i.  $\mu - 2\sigma = 0.75 - 2(0.0968) = 0.5564$   
 $\mu + 2\sigma = 0.75 + 2(0.0968) = 0.9436$

1 mark

In a random sample of 20 worms, the sample proportion will lie between 0.5564 and 0.9436 95% of the time.

1 mark

Total 17 marks