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MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2024

SECTION A – Multiple-choice answers

1.	D	**6.**	B	**11.**	B	**16.**	D
2.	C	**7.**	D	**12.**	C	**17.**	A
3.	A	**8.**	D	**13.**	B	**18.**	A
4.	C	**9.**	C	**14.**	B	**19.**	B
5.	D	**10.**	B	**15.**	C	**20.**	A

SECTION A – Multiple-choice solutions

Question 1

The amplitude is 2. Note that the amplitude is always positive. The period is $\frac{2\pi}{4}=\frac{\pi}{2}$.

The answer is D.

Question 2

$$\begin{aligned} d_f &= d_g \cap d_h \\ &= [-3, 1] \end{aligned}$$

The answer is C.

Question 3

$x^2-6x+k=0$ is a quadratic equation in the variable x. It will have two real solutions when the discriminant is greater than zero.

$$\begin{aligned} \Delta &> 0 \\ b^2-4ac &> 0 \quad \text{where } a=1,\ b=-6,\ c=k \\ 36-4k &> 0 \\ -4k &> -36 \\ k &< 9 \end{aligned}$$

i.e. $k \in (-\infty, 9)$

The answer is A.

Question 4

Define $f(x)$ on your CAS.

Solve $f(x)=0$ for $x>0$ to find the actual intercept.

$$\begin{aligned} x &= 2\sqrt{2} \\ &= 2.82842... \\ &= 2.8284 \text{ (correct to 4 decimal places)} \end{aligned}$$

Define $f\,'(x)$ on your CAS and use Newton's method to find an approximation.

$$x_0 = 2$$

$$x_1 = 2 - \frac{f(2)}{f'(2)} \qquad \left(\text{formula sheet i.e. } \quad x_{n+1} = x_n - \frac{f(x_n)}{f\,'(x_n)}\right)$$

$$\begin{aligned} &= 2.77976... \\ &= 2.7798 \text{ (correct to 4 decimal places)} \qquad \text{so reject option } A \end{aligned}$$

$$x_2 = 2.77976... - \frac{f(2.77976...)}{f'(2.77976...)}$$

$$\begin{aligned} &= 2.82828... \\ &= 2.8283 \text{ (correct to 4 decimal places)} \qquad \text{so reject option } B \end{aligned}$$

$$x_3 = 2.82828... - \frac{f(2.82828...)}{f'(2.82828...)}$$

$$\begin{aligned} &= 2.82842... \\ &= 2.8284 \text{ (correct to 4 decimal places)} \end{aligned}$$

The answer is C.

Question 5

Method 1

$6x + 2ay = 3$ can be rearranged to give $y = -\dfrac{3}{a}x + \dfrac{3}{2a} \quad (a \neq 0)$

$3ax + y = a$ can be rearranged to give $y = -3ax + a$

There is **no** unique solution when the gradients are equal, i.e. when

$$\begin{aligned} -\frac{3}{a} &= -3a \\ 3a^2 &= 3 \\ a &= \pm 1 \end{aligned}$$

Therefore, there **is** a unique solution when $a \neq \pm 1$, i.e. when $a \in R\backslash\{-1, 1\}$.
The answer is D.

Method 2

$$6x + 2ay = 3$$
$$3ax + y = a$$

As a matrix equation, this system can be written as

$$\begin{bmatrix} 6 & 2a \\ 3a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ a \end{bmatrix}$$

There will be no solution or infinite solutions when

$$6 - 6a^2 = 0$$
$$6\left(1 - a^2\right) = 0$$
$$a = \pm 1$$

When $a = -1$, the system becomes

$$6x - 2y = 3 \qquad (A)$$
$$-3x + y = -1 \qquad (B)$$

$(B) \times -2 \qquad 6x - 2y = 2 \qquad (C)$

Comparing equations (A) and (C) we see that there are no solutions when $a = -1$.
When $a = 1$, the system becomes

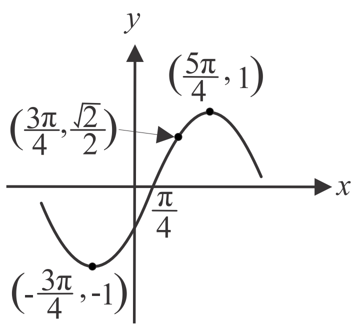
$$6x + 2y = 3 \qquad (A)$$
$$3x + y = 1 \qquad (B)$$

$(B) \times 2 \qquad 6x + 2y = 2 \qquad (C)$

Comparing equations (A) and (C) we see that there are no solutions when $a = 1$.
There will be a unique solution when $a \in R \backslash \{-1, 1\}$.
The answer is D.

Question 6

$f:[-a, a] \to R, f(x) = \sin\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right)$



For f^{-1} to exist, f must be 1 : 1.
The period of f is $2\pi \div \frac{1}{2} = 4\pi$.

So f is 1 : 1 over the interval $\left[\frac{\pi}{4} - \pi, \frac{\pi}{4} + \pi\right] = \left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$.

So the maximum value of a is $\frac{3\pi}{4}$.

You can double-check by finding the min/max points on the graph of f between say -2π and 2π,

i.e. solve $\left(\frac{d}{dx}f(x) = 0, x\right) | -2\pi < x < 2\pi$

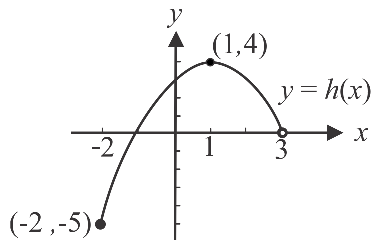
$$x = -\frac{3\pi}{4} \text{ or } x = \frac{5\pi}{4}$$

Note that the left endpoint of the domain of f cannot equal $-\frac{5\pi}{4}$ because f must be 1:1.

The answer is B.

Question 7

$h(x) = g(x-1) = 4-(x-1)^2$



Sketch the graph of h, noting that $d_h = [-2, 3)$.

$r_h = [-5, 4]$

The answer is D.

Question 8

For the graph to be continuous at $x = 0$ we require that

$$e^{a\times 0} = b - e^0$$
$$1 = b - 1$$
$$b = 2$$

For the graph to be smooth at $x = 0$, we require that

$$\frac{d}{dx}\left(e^{ax}\right) = \frac{d}{dx}\left(2 - e^{-x}\right)$$
$$ae^{ax} = e^{-x}$$

At $x = 0$, $\quad a \times 1 = 1$

$$a = 1$$

The answer is D.

Question 9

$$\int_2^3 (f(x) - 2x)dx$$
$$= \int_2^3 f(x)dx - \int_2^3 2x\, dx$$
$$= \int_2^5 f(x)dx - \int_3^5 f(x)dx - \left[\frac{2x^2}{2}\right]_2^3$$
$$= 1 - {}^{-}2 - (9-4)$$
$$= -2$$

The answer is C.

Question 10

$$y = \frac{f(x)}{g(x)}$$
$$\frac{dy}{dx} = \frac{g(x)\times f'(x) - f(x)\times g'(x)}{(g(x))^2} \quad \text{(quotient rule)}$$

At $x = 4$, $\quad \frac{dy}{dx} = \frac{-1\times{}^{-}2 - 3\times 5}{(-1)^2}$

$$= -13$$

The answer is B.

Question 11

Method 1 – trial and error and CAS (1 – Prop z Interval)

$x = 15, \quad n = 90$ (i.e. 15 out of 90 surveyed residents owned a pet)

75% confidence interval $= (0.1215, 0.2119)$
80% confidence interval $= (0.1163, 0.2170)$
So $p = 80$.
The answer is B.

Method 2 – set up and solve an appropriate equation

$\hat{p} = \dfrac{1}{6}$
The right endpoint of the interval is 0.2170. (You could also use the left endpoint.)

Solve $\dfrac{1}{6} + k\sqrt{\dfrac{\frac{1}{6} \times \frac{5}{6}}{90}} = 0.2170$ for k. (formula sheet)

$k = 1.2813$ (correct to 4 decimal places)

$\Pr(-1.2813 < Z < 1.2813) = 0.7999$ (correct to 4 decimal places)

Therefore, this confidence interval has closest to an 80% level of confidence, so $p = 80$.
The answer is B.

Question 12

Since A and B are independent

- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
- $\Pr(A | B) = \Pr(A)$

We are told that $\Pr(A) = 3\Pr(B)$ and $\Pr(A \cup B) = 0.37$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ (formula sheet)

$$0.37 = 3\Pr(B) + \Pr(B) - \Pr(A) \times \Pr(B)$$
$$0.37 = 3\Pr(B) + \Pr(B) - 3[\Pr(B)]^2$$

Let $\Pr(B) = x$.

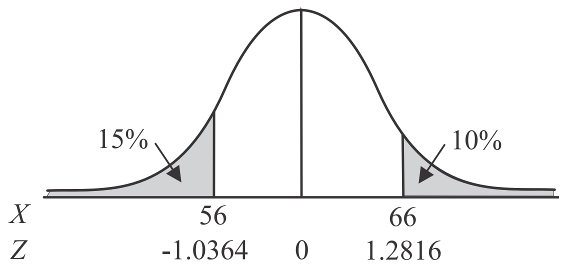
$$0.37 = 3x + x - 3x^2$$

Solving for x gives $x = 0.1$, so $\Pr(B) = 0.1$ Note that $\Pr(B)$ cannot be greater than 1.

So $\Pr(A) = 3 \times 0.1 = 0.3$

Therefore $\Pr(A | B) = \Pr(A)$
$= 0.3$
The answer is C.

Question 13



invNorm$(0.9, 0, 1)$ so $z = 1.2816$ (correct to 4 decimal places)

invNorm$(0.15, 0, 1)$ so $z = -1.0364$ (correct to 4 decimal places)

Using $z = \dfrac{x-\mu}{\sigma}$ and solving simultaneously:

$$1.2816 = \frac{66-\mu}{\sigma} \qquad (1)$$

$$-1.0364 = \frac{56-\mu}{\sigma} \qquad (2)$$

So $\mu = 60.4710\ldots$, $\sigma = 4.3140\ldots$
The closest value of the mean is 60.5.
The answer is B.

Question 14

trapezium$\left(x^2+1, 0, 3, 3\right)$, i.e. $f(x) = x^2 + 1$, $a = 0$, $b = 3$ and $n = 3$ so therefore $h = \dfrac{3-0}{3} = 1$

i	x	total
1	$0+1=1$	$f(0)+f(3)+2\times f(1)=15$
2	$1+1=2$	$15+2\times f(2)=25$

Note that $i < n$ i.e. $i < 3$ so we stop at $i = 2$.

The next instruction is: area estimate $\longleftarrow$ total $\times$ (h $\div$ 2)

So area estimate $= 25\times(1\div 2) = \dfrac{25}{2}$.

The output gives this area estimate, so the output is $\dfrac{25}{2}$.
The answer is B.

Question 15

$$\Pr(S=2|S\geq 1)=\frac{\Pr(S=2)}{1-\Pr(S=0)}$$

$$\Pr(S=2)=\frac{{}^5C_2\times{}^4C_2}{{}^9C_4}=\frac{10\times 6}{126}=\frac{10}{21}$$

and

$$1-\Pr(S=0)=1-\frac{{}^5C_0\times{}^4C_4}{{}^9C_4}=1-\frac{1\times 1}{126}=\frac{125}{126}$$

Therefore $\Pr(S=2|S\geq 1)=\frac{10}{21}\div\frac{125}{126}$

$$=\frac{12}{25}$$

The answer is C.

Question 16

$$\Pr\left(\hat{P}>\frac{1}{n}\right)=\Pr\left(\frac{X}{n}>\frac{1}{n}\right)$$

Note that $\hat{P}=\frac{X}{n}$ (formula sheet) and $X\sim\text{Binomial}\left(n=?,\ p=\frac{1}{3}\right)$

$$=\Pr(X>1)$$
$$=\Pr(X\geq 2)$$
$$\geq 0.85$$

<u>Method 1</u> – using the formula for binomial probability distribution

$$\Pr(X\geq 2)=1-[\Pr(X=0)+\Pr(X=1)]\geq 0.85$$
$$\Pr(X=0)+\Pr(X=1)\leq 0.15$$

$${}^nC_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n+{}^nC_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1}=0.15$$

$$n=8.84\ldots$$

So $n=9$

The answer is D.

<u>Method 2</u> – trial and error

If $n=6$, $\Pr(X\geq 2)=0.6488\ldots$ $\quad\text{binomCdf}\left(6,\frac{1}{3},2,6\right)$

If $n=7$, $\Pr(X\geq 2)=0.7366\ldots$ $\quad\text{binomCdf}\left(7,\frac{1}{3},2,7\right)$

If $n=8$, $\Pr(X\geq 2)=0.8049\ldots$ $\quad\text{binomCdf}\left(8,\frac{1}{3},2,8\right)$

If $n=9$, $\Pr(X\geq 2)=0.8569\ldots$ $\quad\text{binomCdf}\left(9,\frac{1}{3},2,9\right)$

The smallest value of n is 9.
The answer is D.

Question 17

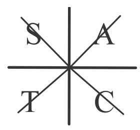
Solve $\frac{d}{dx}\left(2\tan\left(\frac{x}{2}\right)+3\right)=2$ for x.

$$x=\frac{(8k-1)\pi}{2},\ \frac{(8k+1)\pi}{2},\ \frac{(8k-3)\pi}{2},\ \frac{(8k+3)\pi}{2},\ k\in Z$$

Two of these options are given in option D but two are missing and we are asked for **all** the possible values of x, so eliminate option D.

Note that $\frac{d}{dx}\left(2\tan\left(\frac{x}{2}\right)+3\right)=\frac{1}{\cos^2\left(\frac{x}{2}\right)}$

Solve
$$\frac{1}{\cos^2\left(\frac{x}{2}\right)}=2$$
$$\cos^2\left(\frac{x}{2}\right)=\frac{1}{2}$$
$$\cos\left(\frac{x}{2}\right)=\pm\frac{1}{\sqrt{2}}$$



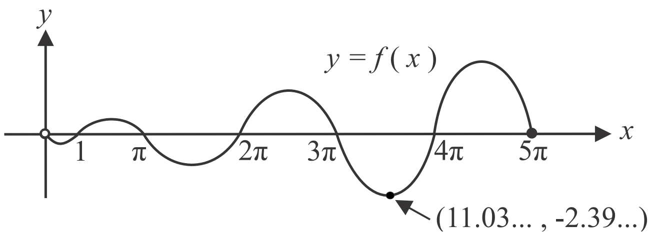
base angle is $\frac{\pi}{4}$

We require $\frac{x}{2}=\pm\frac{\pi}{4},\ \pm\frac{3\pi}{4}$ or alternatively, $\frac{x}{2}=k\pi\pm\frac{\pi}{4},\ \ k\in Z,$ because we need a solution in each quadrant. So $x=2k\pi\pm\frac{\pi}{2},\ \ k\in Z.$

The answer is A.

Question 18

Sketch the graph of $y=f(x)$.



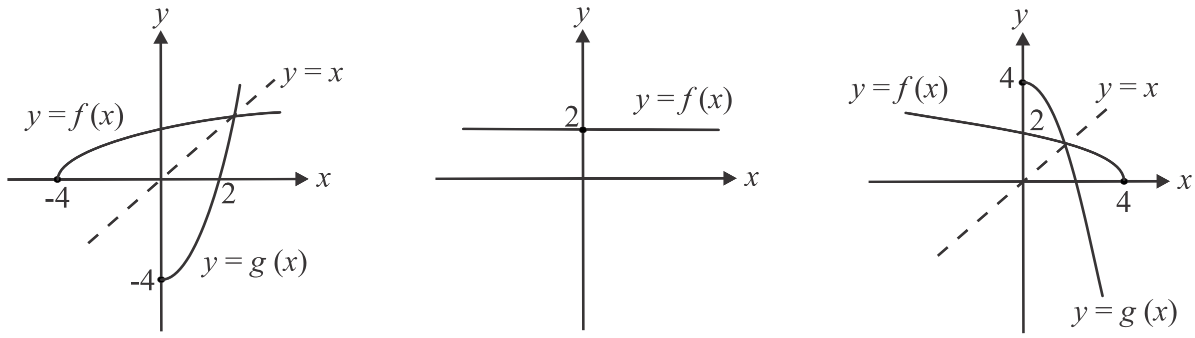
The graph of $f(x-k)$ translates the graph of $f(x)$ k units to the right when k is positive.
It translates the graph of $f(x)$ k units to the left when k is negative.
For there to be at least two stationary points with positive x-coordinates, the graph of $f(x)$ can be translated up to 11.03… units to the left. Any further left and there will be less than two stationary points with a positive x-coordinate.
In other words, how far left can we translate the graph of f so that two of its stationary points remain to the right of the y-axis?
We require $k>-11.03\ldots$
Since $-4\pi<-11.03\ldots$ i.e. $-4\pi=-12.5663\ldots$ then -4π is not a possible value of k.
The answer is A.

Question 19

A normal distribution is a continuous probability distribution so $g(x)$ cannot be negative and so $a > 0$.
Reject option A.
The standard deviation of X is half the standard deviation of Y or alternatively put, the standard deviation of Y is twice the standard deviation of X.
Therefore, the graph of f will be dilated from the y-axis by a factor of 2 to become the graph of g.
So b could equal 2.
Option B is the only option that satisfies these requirements for $a > 0$ and $b = 2$.
The answer is B.

Question 20

Given $f(0) = 2$, then $2 = \sqrt{b}$ and $b = 4$.
Sketch the graph of $f(x) = \sqrt{ax + 4}$ using sliders.
For example:



when $a = 1$ when $a = 0$ when $a = -1$

Note that when $a = 0$, f is a horizontal straight line i.e. a many:1 function, and hence does not have an inverse function so reject option B.
The gradient of g is negative (i.e. $g'(x) < 0$) for negative values of a, i.e. for $a \in R^-$.
The answer is A.

SECTION B

Question 1 (10 marks)

a. Define f on your CAS. fMax($f(x)$, x) gives $x = 3$ and $f(3) = 9$
The maximum value occurs at $(3, 9)$. **(1 mark)**

b. The stationary point is a stationary point of inflection. **(1 mark)**

c. The gradient of the graph of $f(x)$ is positive i.e. $f'(x) > 0$ for $x \in (-\infty, 0) \cup (0, 3)$. **(1 mark)**
Note that $f'(x) = 0$ at $x = 0$ and at $x = 3$.

d. Since $f(x) = 0$ for $x = 0$ and $x = 4$, if the graph of $y = f(x)$ is translated 4 or more units to the left, then it will have no positive x-intercepts and hence $f(x + h) = 0$ will have no positive solutions for $h \geq 4$. **(1 mark)**

e. Since the maximum value of f occurs at $(3,9)$ from part **a.**, then if the graph of $y = f(x)$ is translated more than 9 units downwards, it will not intersect with the x-axis and hence $f(x) + k = 0$ will have no solutions for $k < -9$. **(1 mark)**

f. Using CAS, tangentLine$(f(x), x, -1)$, the equation is $y = \frac{16}{3}x + \frac{11}{3}$. **(1 mark)**

g. Since the tangents are parallel, this second tangent has a gradient of $\frac{16}{3}$.

Solve $f'(x) = \frac{16}{3}$ for x.

$x = -1$ or $x = 2$

So $q = 2$. **(1 mark)**

h. The gradient of the parallel tangents is $\frac{16}{3}$ so the gradient of the straight line that is perpendicular is $-\frac{3}{16}$. It passes through $(0,0)$ so it's equation is $y - 0 = -\frac{3}{16}(x - 0)$

$$y = -\frac{3}{16}x$$

(1 mark)

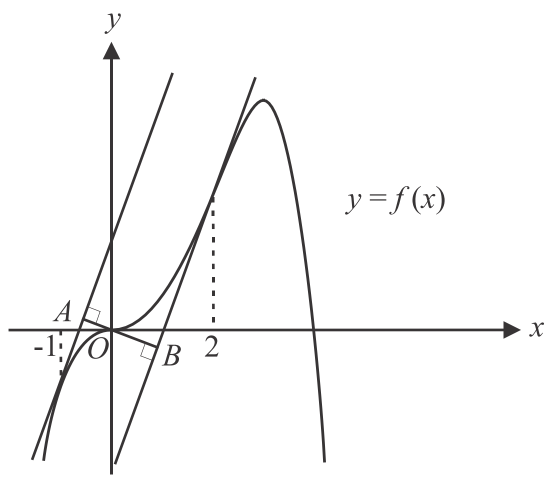
The equation of the tangent to f at $x = -1$ is $y = \frac{16}{3}x + \frac{11}{3}$ (from part **f.**)

We need to find the point of intersection of this tangent and $y = -\frac{3}{16}x$

Solve $\frac{16}{3}x + \frac{11}{3} = -\frac{3}{16}x$ for x.

$$x = -\frac{176}{265}$$

(1 mark)



A is the point $\left(-\frac{176}{265}, \frac{33}{265}\right)$ and B is the point $\left(\frac{256}{265}, -\frac{48}{265}\right)$.

The midpoint of AB is $\left(\frac{8}{53}, -\frac{3}{106}\right)$. **(1 mark)**

Question 2 (12 marks)

a. Define $f(x)$ on your CAS.

i. <u>x-intercepts</u> occur when $y = 0$

Solve $f(x) = 0$ for x. $x = -a,\ 5a$ $d_f = [-a,\ 5a]$ **(1 mark)**

ii. Halfway between $(-a, 0)$ and $(5a, 0)$ is $(2a, 0)$.

The equation of the axis of symmetry is $x = 2a$. **(1 mark)**

iii. f is strictly increasing for $x \in [-a, 2a]$. **(1 mark)**

Note that the endpoints are included.

b. Maximum height of the tunnel is $f(2a) = 2a^2$.

Width of base of tunnel is $6a$.

Solve $2a^2 < 12a$ for a. **(1 mark)**

$0 < a < 6$ or $a \in (0, 6)$ **(1 mark)**

c. $g(x) = -\frac{2}{9}(x+2.5)(x-12.5)$

$$g(0) = -\frac{2}{9} \times 2.5 \times -12.5$$

$$= -\frac{2}{9} \times \frac{5}{2} \times -\frac{25}{2}$$

$$= \frac{125}{18}$$

Note that in order to give an exact answer, it is necessary to work in fractions.

The height of the platform above the base of the tunnel is 2 metres.

So the vertical distance required is $\frac{125}{18} - \frac{36}{18} = \frac{89}{18}$ metres. **(1 mark)**

d. Let the distance from $P(x, g(x))$ to $Q(12.5, 0)$ be h.

$h = \sqrt{(x-12.5)^2 + (g(x)-0)^2}$ **(1 mark)**

Solve $\frac{dh}{dx} = 0$ for x.

$x = -0.7343\ldots$ or $x = 3.2343\ldots$

but $x \geq 0$, so $x = 3.2343\ldots$ **(1 mark)**

$h(3.2343\ldots) = 15.0087\ldots$

Maximum length of beam of light is 15.0 metres (correct to 1 decimal place) **(1 mark)**

e. Solve $\int_0^m g(x)\,dx - 2 \times 4 = \int_m^{12.5} g(x)\,dx$ for m. **(1 mark)** left side **(1 mark)** right side

$m = -8.3227\ldots$ or $m = 5.6923\ldots$ or $m = 17.6303\ldots$

But $0 \leq m \leq 12.5$ so $m = 5.69$ (correct to two decimal places) **(1 mark)**

Question 3 (15 marks)

a. Let T be a normally distributed variable with $\mu = 0,\ \sigma = 5$.

$\Pr(-2 \le T \le 2) = 0.31084\ldots$ use normCdf$(-2, 2, 0, 5)$

$= 0.3108$ (correct to 4 decimal places) **(1 mark)**

b. $\Pr(T > k) = 0.05$ or alternatively, $\Pr(T < k) = 0.95$.

Using the inverse normal function on CAS, i.e. invNorm($0.95,\ 0,\ 5$) .

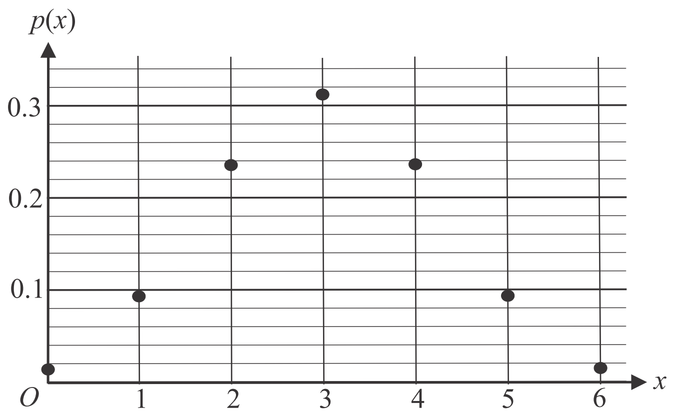
$k = 8.22426\ldots$ So $k = 8.224$ (correct to 3 decimal places) **(1 mark)**

c. i. The manager models a binomial distribution with parameters $n = 6$, $p = 0.5$.

$\Pr(\ P = 2)\ = 0.234375$ (binom Pdf(6,0.5,2) = 0.234375)

$\Pr(\ P = 3)\ = 0.3125$ (binom Pdf(6,0.5,3) = 0.3125)

$\Pr(\ P = 5)\ = 0.09375$ (binom Pdf(6,0.5,5) = 0.09375)



(1 mark) for one correct point **(1 mark)** for all points correct

ii. P is a binomially distributed variable with parameters $n = 6$, $p = 0.5$.

$\Pr(P \ge 1) = 0.984375$

$= 0.9844$ (correct to 4 decimal places) **(1 mark)**

iii. $\Pr(P \le 4 | P \ge 1)$ (conditional probability) **(1 mark)**

$$= \frac{\Pr(1 \le P \le 4)}{\Pr(P \ge 1)}$$

$$= \frac{0.875}{0.984375}$$

$= 0.8889$ (correct to 4 decimal places) **(1 mark)**

d. **i.** $0.8+m+n=1$

$$n=\frac{1}{5}-m$$ **(1 mark)**

$\mathrm{E}(X)=0\times0.8+1\times m+2\times\left(\frac{1}{5}-m\right)=\frac{2}{5}-m$

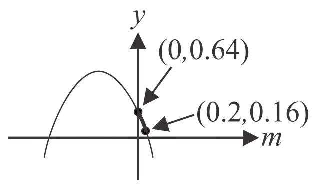
$\mathrm{E}(X^2)=0^2\times0.8+1^2\times m+2^2\times\left(\frac{1}{5}-m\right)=\frac{4}{5}-3m$ **(1 mark)**

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ (formula sheet)

$$=\frac{4}{5}-3m-\left(\frac{2}{5}-m\right)^2$$

$$=-m^2-\frac{11}{5}m+\frac{16}{25}$$ **(1 mark)**

ii. The graph of the function defining the variance is part of an inverted parabola.



From the table, $m\geq0$ and $n\geq0$ therefore $0\leq m\leq0.2$

This is the domain of the variance function.

The maximum variance therefore occurs when $m=0$.

The maximum variance is therefore $\frac{16}{25}$ or 0.64. **(1 mark)**

e. **i.** $\hat{p}=\frac{0.1752+0.2248}{2}=0.2$ **(1 mark)**

$0.2248-0.2=0.0248$ (which is the margin of error)

95% confidence interval for p is $\left(\hat{p}\pm1.96\hat{\sigma}\right)$ $\left(\text{where } \hat{\sigma}=\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}\right)$

$1.96\hat{\sigma}=0.0248$ so $\hat{\sigma}=0.0127$ (to 4 decimal places) **(1 mark)**

ii. margin of error $=1.96\sqrt{\frac{0.2\times0.8}{n}}$

We require $1.96\sqrt{\frac{0.2\times0.8}{n}}\leq0.75\times0.0248$ **(1 mark)**

$n=1777$ (to the nearest integer) **(1 mark)**

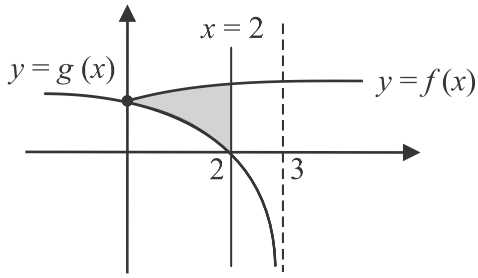
Question 4 (13 marks)

a. For g to exist, $3-x>0$, $x<3$ so $D=(-\infty, 3)$. **(1 mark)**

b. Do a quick sketch.

area required $= \int_0^2 (f(x)-g(x))dx$ **(1 mark)**

$= \log_e\left(\frac{3125}{729}\right)$ **(1 mark)**



Note, because you haven't been asked to express your answer correct to a certain number of decimal places, you must leave your answer as an exact value.

c. **i.** $h(x)=f(x)+g(x)$

$=\log_e(x+3)+\log_e(3-x)$

$=\log_e(x+3)(3-x)$ (log laws)

$=\log_e(9-x^2)$ as required **(1 mark)**

ii. $d_f=[0, \infty)$ and $d_g=(-\infty, 3)$

The intersection of these two intervals is $[0, 3)$ so $d_h=[0, 3)$. **(1 mark)**

The maximum value of $h(x)$ over this domain occurs when $x=0$.

Now $h(0)=\log_e(9)$ and as $x\to 3$ (from below), $h(x)\to -\infty$

So $r_h=(-\infty, \log_e(9)]$ **(1 mark)**

iii. $h(x)=\log_e(9-x^2)$

Let $y=\log_e(9-x^2)$

Swap x and y for inverse.

$x=\log_e(9-y^2)$

Now make y the subject.

<u>Method 1</u> – by hand

$e^x = 9 - y^2$

$y^2 = 9 - e^x$

$y = \pm\sqrt{9 - e^x}$ but $r_{h^{-1}} = d_h = [0, 3)$

So $y = \sqrt{9 - e^x}$

$h^{-1}(x) = \sqrt{9 - e^x}$ **(1 mark)**

<u>Method 2</u> - using CAS

Solve $x = \log_e(9 - y^2)$ for y.

$y = \pm\sqrt{9 - e^x}$ but $r_{h^{-1}} = d_h = [0, 3)$

So $y = \sqrt{9 - e^x}$

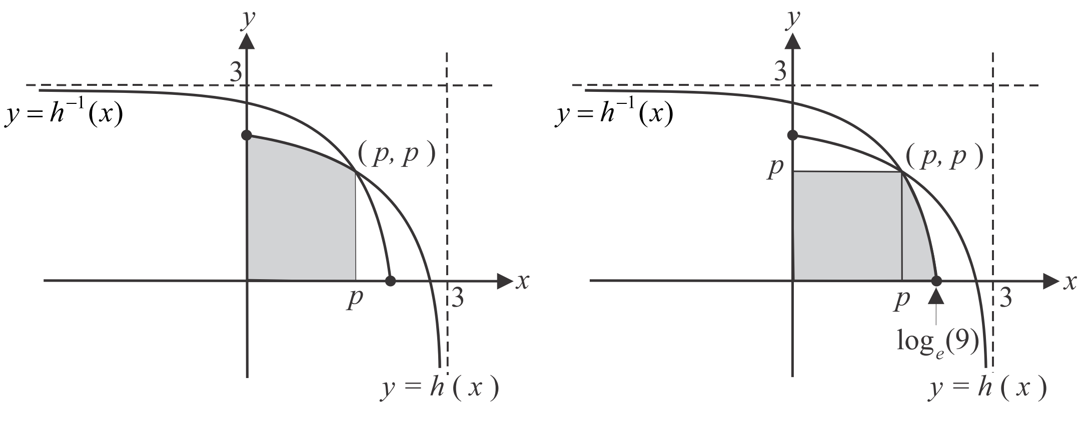
$h^{-1}(x) = \sqrt{9 - e^x}$ **(1 mark)**

d. By inspection, it is seen that the graphs of $y = h(x)$ and $y = h^{-1}(x)$ intersect on the line $y = x$, so the point of intersection is (p, p).

The average value of the function $h(x)$ over the interval $[0, p]$ is $\frac{1}{p-0}\int_0^p h(x)dx$.

(1 mark)

The graphs of $y = h(x)$ and $y = h^{-1}(x)$ are reflections of each other in the line $y = x$.
Therefore, the shaded areas shown below are equal in area.



So $\int_0^p h(x)dx = p^2 + \int_p^{\log_e(9)} h^{-1}(x)dx$

So required average value expression $= \frac{1}{p}\left(p^2 + \int_p^{\log_e(9)} h^{-1}(x)dx\right)$ **(1 mark)**

e. Solve $j_a(x)=0$ for x using CAS.

$$x = \pm\sqrt{a^2-1}$$

We require $x<0$ so $x=-\sqrt{a^2-1}$.

So Q is the point $\left(-\sqrt{a^2-1},0\right)$. **(1 mark)**

$$j_a'(x)=\frac{2x}{x^2-a^2}$$

$$j_a'\left(-\sqrt{a^2-1}\right)=\frac{-2\sqrt{a^2-1}}{-1}$$

$$=2\sqrt{a^2-1}$$ **(1 mark)**

f. Let θ be the angle that the tangent to $j_a(x)$ at Q makes with the positive branch of the x-axis.

Using our answer to part **e.**, $2\sqrt{a^2-1}=\tan(\theta)$

We require $2\sqrt{a^2-1}<\tan(45°)$ **(1 mark)**

$$2\sqrt{a^2-1}<1$$

Using CAS $-\frac{\sqrt{5}}{2}<a\le -1$ or $1\le a<\frac{\sqrt{5}}{2}$ but $a>1$ (given in question)

so $1<a<\frac{\sqrt{5}}{2}$ **(1 mark)**

Question 5 (10 marks)

a. The period of $y = \sin(x)$ is 2π.

The period of $y = \cos\left(\frac{x}{3}\right)$ is $\frac{2\pi}{1/3} = 6\pi$.

It is required that the graphs of the two functions above simultaneously repeat. Therefore, the least common multiple of 2 and 6 is required, and the least common multiple is 6.

Therefore, the graphs of the two functions simultaneously repeat after 6π.

The period of f is therefore 6π. **(1 mark)**

b. Graph f on your CAS.

Make sure that the coordinates of the points on the graph are expressed to at least 4 decimal places on your CAS.

One of the maximums occurs at (1.4184…, 1.8787…).

So the maximum value of f is 1.879, correct to 3 decimal places. **(1 mark)**

c. To obtain the graph of $y = -f(x + c)$, the graph of f is translated c units to the left (since c is positive) and then reflected in the x-axis.

The smallest positive value of c for which the graph of $y = -f(x + c)$ will coincide with the graph of f is 3π.

For example, the point $(3\pi, -1)$ on the graph of f becomes the point (0, 1) on the graph of $y = -f(x + c)$. So $c = 3\pi$. **(1 mark)**

d. Solve $\int_0^q f(x)\,dx = 0$ for q where $q > 0$ **(1 mark)**

$$q = 10.7289\ldots,\ ,\ 18.8495\ldots,\ \text{and so on}$$

The smallest possible value of q is 10.73 (correct to two decimals places). **(1 mark)**

e. The period of $y = \sin(ax)$ is $\frac{2\pi}{a}$.

The period of $y = \cos\left(\frac{x}{3a}\right)$ is $\frac{2\pi}{1/3a} = 6a\pi$.

The least common multiple is $6a\pi$.

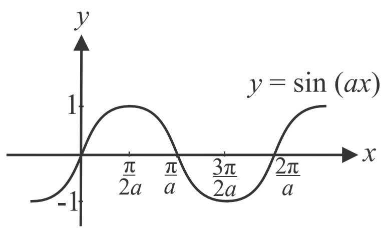
So the period of the functions f_a is $6a\pi$. **(1 mark)**

f. **i.** The maximum value of $y = \sin(ax)$

occurs at $x = \frac{\pi}{2a} + \frac{2\pi k}{a},\ k \in Z$

$$= \frac{\pi + 4\pi k}{2a}$$

$$= \frac{\pi(1+4k)}{2a}$$

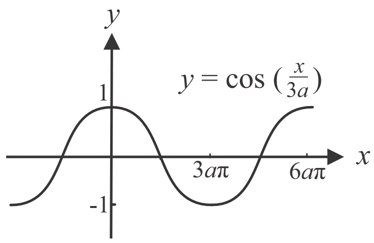


(1 mark)

ii. The maximum value of $y = \cos\left(\frac{x}{3a}\right)$

occurs at $x = 0 + 6an\pi,\quad n \in Z$

$$= 6an\pi$$



(1 mark)

g. For $f_a(x) = 2$, we would require that the maximum value of $\sin(ax)$, which is 1, occurred at the same x value as where the maximum value of $\cos\left(\frac{x}{3a}\right)$, which is also 1, occurred at.

Therefore, we would require that $\frac{\pi(1+4k)}{2a} = 6an\pi,\quad k, n \in Z$ **(1 mark)**

which leads to $a^2 = \frac{4k+1}{12n}$

Now $4k+1$ is odd and $12n$ is even therefore $\frac{4k+1}{12n}$ is never an integer for $k, n \in Z$,

therefore $\frac{4k+1}{12n}$ is never a perfect square, therefore a is never a positive integer.

Therefore, the maximum value of $f_a(x)$ cannot be 2. **(1 mark)**