Mathematical Methods

Written Examination 1

2024 Insight Year 12 Trial Exam Paper

Worked Solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1a.

Worked solution

$$\frac{dy}{dx} = x(-2\sin(2x)) + \cos(2x)$$
$$= \cos(2x) - 2x\sin(2x)$$

Mark allocation: 1 mark

- 1 answer mark for applying the product rule to find the derivative: $\frac{dy}{dx} = \cos(2x) - 2x\sin(2x)$

Tip

- *You may find it helpful to begin by writing down the product rule.*

Question 1b.

Worked solution

$$f'(x) = \frac{\left(e^x - 1\right)\left(\frac{1}{x}\right) - e^x \log_e(x)}{(e^x - 1)^2}$$

$$f'(1) = \frac{\left(e - 1\right)\left(\frac{1}{1}\right) - e \log_e(1)}{(e - 1)^2}$$
$$= \frac{(e-1)}{(e-1)^2}$$
$$= \frac{1}{e-1}$$

Mark allocation: 2 marks

- 1 answer mark for applying the quotient rule to find the derivative: $f'(x) = \frac{\left(e^x - 1\right)\left(\frac{1}{x}\right) - e^x \log_e(x)}{(e^x - 1)^2}$
- 1 answer mark for correctly evaluating the derivative at $x = 1$: $f'(1) = \frac{1}{e-1}$

Tip

- *When using the quotient rule, or any other differentiation rule, it can be helpful to write down the rule.*

Question 2

Worked solution

Let $\sin(x) = a$.

$2a^2 + 3a - 2 = 0$

$(2a-1)(a+2) = 0$

$a = \frac{1}{2}$ [Note that $a \neq -2$ because $-1 \leq a \leq 1$]

$\therefore \sin(x) = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Mark allocation: 2 marks

- 1 method mark for using a suitable method for solving the equation, such as factorising
- 1 answer mark for the correct values: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Tips

- *Substitution can be a useful method for producing a quadratic equation that can then be solved by factorising.*
- *Ensure you are familiar with the exact trigonometric values.*
- *Remember to consider whether answers are feasible, e.g.* $\cos(x) \neq -2$ *because* $-1 \leq \cos(x) \leq 1$.

Question 3a.

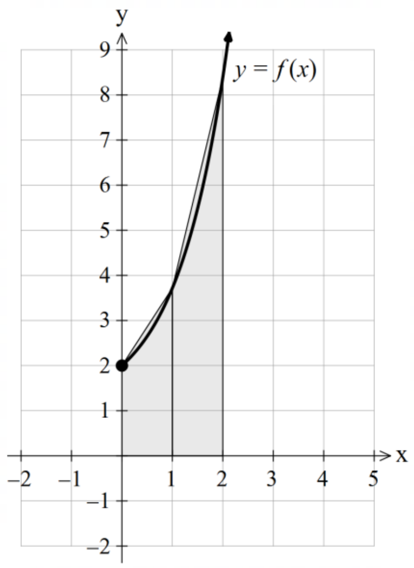
Worked solution

$$A=\frac{2-0}{4}\left[f(0)+2f(1)+f(2)\right] \quad \left(\text{or } \frac{1}{2}(1)[f(0)+f(1)]+\frac{1}{2}(1)[f(1)+f(2)]\right)$$

$$=\frac{1}{2}\left[f(0)+2f(1)+f(2)\right]$$

$$=\frac{1}{2}(e^0+1+2(e^1+1)+e^2+1)$$

$$=\frac{1}{2}(2e+e^2+5)$$



Mark allocation: 2 marks

- 1 method mark for setting up the area equation as the area of two trapeziums
- 1 answer mark for the correct answer: $\frac{1}{2}(2e+e^2+5)$

Question 3b.

Worked solution

$$f(x)=g(x)$$

$$e^x+1=4e^{-x}+1$$

$$e^x=4e^{-x}$$

$$=\frac{4}{e^x}$$

$$e^{2x}=4$$

$$2x=\log_e(4)$$

$$x=\frac{1}{2}\log_e(4)$$

$$=\log_e\left(4^{\frac{1}{2}}\right)$$

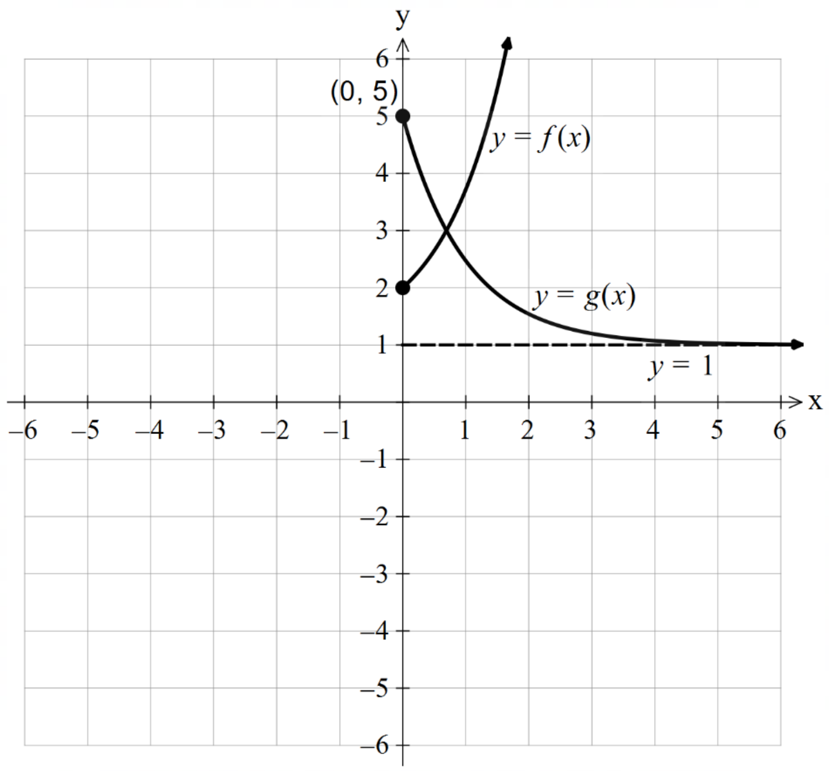
$$=\log_e(2)$$

Mark allocation: 2 marks

- 1 answer mark for obtaining $e^{2x}=4$
- 1 answer mark for the correct value of x: $\log_e(2)$

Question 3c.

Worked solution



Mark allocation: 2 marks

- 1 answer mark for the correct end point labelled with coordinates $(0, 5)$ and the correct asymptote labelled with its equation $y = 1$ (awarded regardless of the domain that the asymptote is drawn over)
- 1 answer mark for the correct shape of $y = g(x)$ with the point of intersection occurring at $0 < x < 1$

Tips

- *The value of* $\log_e 2$ *is between* 0 *and* 1 *because* $2 < e$*. Hence, on your graph the point of intersection of* $f(x)$ *and* $g(x)$ *should occur between* $x = 0$ *and* $x = 1$*.*
- *Although you were not asked to show the coordinates of the point of intersection on the graph, it is helpful to work out the* y*-coordinate of the point of intersection,* $y = e^{\log_e(2)} + 1 = 2 + 1 = 3$*, in order to sketch a more accurate graph.*

Question 4a.

Worked solution

$\text{ran } f = [-1, 3]$

Mark allocation: 1 mark

- 1 answer mark for the correct range: $[-1, 3]$

Question 4b.i.

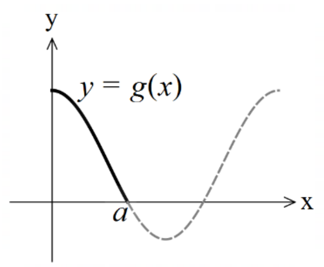
Worked solution

We need $\text{ran } g \subseteq \text{ dom } h$.

$\therefore \text{ran } g \subseteq [0, \infty)$

$\therefore 2\cos(4x) + 1 \geq 0$

Consider the graph of $g(x) = 2\cos(4x) + 1$.



a is the first positive value of x for which $2\cos(4x) + 1 = 0$.

Solve $\cos(4x) = -\frac{1}{2}$.

The required angle is in quadrant 2. The related angle in quadrant 1 is $\frac{\pi}{3}$.

$$4x = \pi - \frac{\pi}{3}$$
$$= \frac{2\pi}{3}$$
$$x = \frac{\pi}{6}$$

Therefore, $a = \frac{\pi}{6}$.

Mark allocation: 2 marks

- 1 method mark for recognising that $\text{ran } g \subseteq [0, \infty)$ or solving $2\cos(4x) + 1 = 0$
- 1 answer mark for the correct value of a: $\frac{\pi}{6}$

Tips

- *When solving inequalities, a graphical approach is often helpful.*
- *The question asks for the value of a, so ensure your final answer is stated as $a = \frac{\pi}{6}$, not $x = \frac{\pi}{6}$.*

Question 4b.ii.

Worked solution

$\text{dom } g = \left[0, \frac{\pi}{8}\right]$

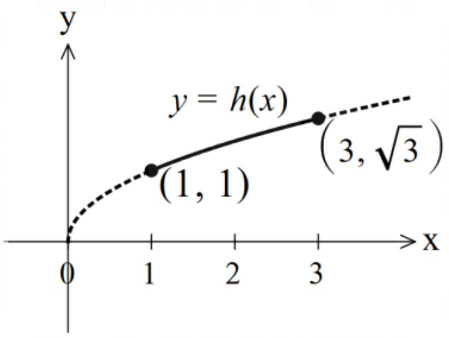
$g(0) = 2\cos(0) + 1 = 3$

$g\left(\frac{\pi}{8}\right) = 2\cos\left(\frac{\pi}{2}\right) + 1 = 1$

$\therefore$ When $\text{dom } g = \left[0, \frac{\pi}{8}\right]$, $\text{ran } g = [1, 3]$.

$\text{ran } g = [1, 3]$ becomes the input for h.

Consider the graph of h.



$h(1) = 1,\ h(3) = \sqrt{3}$

$\therefore\ \text{ran } (h \circ g) = \left[1, \sqrt{3}\right]$

Mark allocation: 2 marks

- 1 answer mark for finding the range of $g: [1,3]$
- 1 answer mark for the correct range of $(h \circ g)(x)$**:** $\left[1, \sqrt{3}\right]$

Tips

- $\text{ran } g = [1, 3]$ *becomes the input for* h*, so work out* $\text{ran } h$ *when* $x \in [1, 3]$.
- *It is not always necessary to find the rule for a composite function.* ***Part b.*** *only needed consideration of the domain and range of* g *and* h*, so the rule for* $(h \circ g)(x)$ *was not needed.*

Question 5a.

Worked solution

$f(x) = x^3 + x^2$

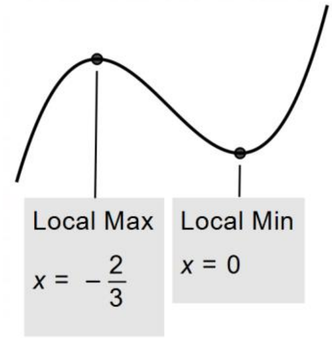
$f'(x) = 3x^2 + 2x$

Stationary points occur where $f'(x) = 0$.

$3x^2 + 2x = 0$

$x(3x + 2) = 0$

$\therefore x = 0,\ x = -\frac{2}{3}$



Since f is a positive cubic function, there is a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 0$.

Mark allocation: 2 marks

- 1 answer mark for correct x values: $x = 0,\ x = -\frac{2}{3}$
- 1 answer mark for the correct nature of each point: a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 0$

Tip

- *It is helpful to be familiar with the graphical shapes of cubic functions in order to quickly determine the nature of stationary points.*

Question 5b.

Worked solution

A point of inflection occurs where $f''=0$.

$$f''(x)=6x+2$$

$$6x+2=0$$

$$x=-\frac{1}{3}$$

Alternatively, for a cubic function with two stationary points, the point of inflection will occur half way between these points; that is, at

$$x=\frac{-\frac{2}{3}+0}{2}=-\frac{1}{3}.$$

$$f\left(-\frac{1}{3}\right)=\left(-\frac{1}{3}\right)^3+\left(-\frac{1}{3}\right)^2$$

$$=-\frac{1}{27}+\frac{1}{9}$$

$$=\frac{2}{27}$$

Therefore, the coordinates of the point of inflection are $\left(-\frac{1}{3},\frac{2}{27}\right)$.

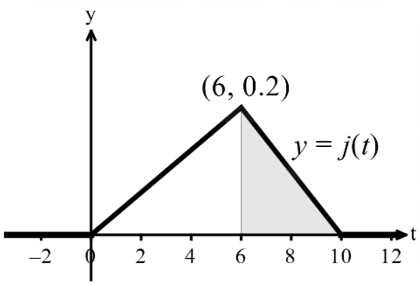
Mark allocation: 2 marks

- 1 mark for the correct x-coordinate: $x=-\frac{1}{3}$
- 1 mark for the correct y-coordinate: $y=\frac{2}{27}$

Question 6a.

Worked solution

$$\Pr(X > 6) = \frac{1}{2}\times 4\times 0.2$$
$$= 0.4$$



Mark allocation: 1 mark

- 1 answer mark for the correct probability: 0.4

Tip

- *The probability can be calculated by finding the relevant area shown on the graph.*

Question 6b.

Worked solution

$$\Pr(X \le) = \int_0^w \frac{x+1}{12}\,dx = \frac{1}{3}$$
$$\frac{1}{12}\int_0^w (x+1)\,dx = \frac{1}{3}$$
$$\frac{1}{12}\left[\frac{x^2}{2}+x\right]_0^w = \frac{1}{3}$$
$$\frac{w^2}{2}+w = 4$$
$$w^2+2w-8 = 0$$
$$(w+4)(w-2) = 0$$
$$\therefore w = -4,\ w = 2$$
$$w = 2 \text{ (since } 0 \le w \le 4)$$

Mark allocation: 3 marks

- 1 answer mark for setting up the integral expression: $\int_0^w \frac{x+1}{12}\,dx = \frac{1}{3}$ or equivalent
- 1 method mark for evaluating the definite integral, leading to $\frac{w^2}{2}+w=4$ (or any multiple of this)
- 1 answer mark for the correct answer: 2

Question 7a.

Worked solution

$g(x) = (2x-3)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}(2)(2x-3)^{-\frac{1}{2}}$

$= (2x-3)^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2x-3}}$

Mark allocation: 1 mark

- 1 answer mark for correct working leading to $g'(x) = \frac{1}{\sqrt{2x-3}}$

Tips

- *When a question asks you to 'show that' something is the case, clearly show all the steps needed.*
- *The formula for differentiating expressions of the form* $y = (ax+b)^n$ *is on the formula sheet. Alternatively, the chain rule can be used.*

Question 7b.

Worked solution

$g'(2) = \frac{1}{\sqrt{4-3}}$

$= 1$

$\therefore \tan\theta = 1$

$\theta = 45°$

Mark allocation: 2 marks

- 1 answer mark for $g'(2) = 1$
- 1 answer mark for the correct angle: $45°$

Question 7c.

Worked solution

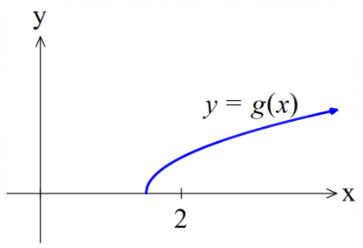
$\tan 30° = \frac{1}{\sqrt{3}}$

Therefore, the tangent angle is $\geq 30°$ when the gradient of the tangent is $\geq \frac{1}{\sqrt{3}}$, that is, when $g'(x) \geq \frac{1}{\sqrt{3}}$.

Solving $g'(x) = \frac{1}{\sqrt{3}}$ gives

$$\begin{aligned}\frac{1}{\sqrt{2x-3}} &= \frac{1}{\sqrt{3}} \\ 2x-3 &= 3 \\ x &= 3\end{aligned}$$

The graph of $g(x)$ shows that the gradient decreases as x increases. Hence, $g'(x)$ decreases as x increases.



$\text{dom } g = \left[\frac{3}{2}, \infty\right)$, thus $\text{dom } g' = \left(\frac{3}{2}, \infty\right)$

Therefore the angle is $\geq 30°$ when $\frac{3}{2} < x \leq 3$

$\therefore \frac{3}{2} < k \leq 3$

Alternative method for solving $g'(x) \geq \frac{1}{\sqrt{3}}$

$$\begin{aligned}\frac{1}{\sqrt{2x-3}} &\geq \frac{1}{\sqrt{3}} \\ 2x-3 &\leq 3 \\ x &\leq 3\end{aligned}$$

$\text{dom } g = \left[\frac{3}{2}, \infty\right)$, hence $\text{dom } g' = \left(\frac{3}{2}, \infty\right)$ if $\frac{3}{2} < x \leq 3$.

$\therefore \frac{3}{2} < k \leq 3$

Mark allocation: 3 marks

- 1 method mark for deriving $g'(x)=\frac{1}{\sqrt{3}}$ or $g'(x)\geq\frac{1}{\sqrt{3}}$
- 1 answer mark for a final answer containing $k\leq 3$ or $x\leq 3$
- 1 answer mark for the fully correct answer: $\frac{3}{2}<k\leq 3$ or $k\in\left(\frac{3}{2},3\right]$

Tips

- *Non-linear inequalities can be tricky to solve algebraically. A graphical approach is often easiest.*
- *If an algebraic approach is used for solving the inequality, you need to recognise that the fraction with the smaller denominator is actually the larger number.*
- *Consideration of the domains of g and g' is important. Since $\operatorname{dom} g=\left[\frac{3}{2},\infty\right)$, then $\operatorname{dom} g'=\left(\frac{3}{2},\infty\right)$. Once $x\leq 3$ has been obtained, consideration of $\operatorname{dom} g'$ leads to $\frac{3}{2}<x\leq 3$.*

Question 8a.

Worked solution

$$\begin{aligned}\frac{d}{dx}\left(x^2\log_e(x)\right)&=x^2\left(\frac{1}{x}\right)+2x\log_e(x)\\&=2x\log_e(x)+x\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for a working that leads to $2x\log_e(x)+x$

Tips

- *The product rule can be used to differentiate $x^2\log_e(x)$.*
- *The product rule does not have to be stated, but it is essential to include some working because the question requires you to ‘show’ how the result is obtained.*

Question 8b.

Worked solution

Note: we are ultimately going to be calculating a definite integral, so in the working below the constant of integration, $+c$, is not relevant. Hence, at each step an antiderivative, where c is zero, is used. An alternative working can be used that includes '$+c$'.

$$A=\int_1^{\frac{3}{2}}\left(x\log_e(x)+1\right)dx$$

We know from **part a.** that

$$\int\left(2x\log_e(x)+x\right)dx=x^2\log_e(x)$$

Hence

$$\int 2x\log_e(x)\,dx+\int x\,dx=x^2\log_e(x)$$

$$\int 2x\log_e(x)\,dx=x^2\log_e(x)-\int x\,dx$$

$$2\int x\log_e(x)\,dx=x^2\log_e(x)-\frac{1}{2}x^2$$

$$\int x\log_e(x)\,dx=\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2$$

$$A=\int_1^{\frac{3}{2}}\left(x\log_e(x)+1\right)dx$$

$$=\left[\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2+x\right]_1^{\frac{3}{2}}$$

$$=\frac{1}{2}\left(\frac{9}{4}\right)\log_e\left(\frac{3}{2}\right)-\frac{1}{4}\left(\frac{9}{4}\right)+\frac{3}{2}-\left(\frac{1}{2}\log_e(1)-\frac{1}{4}+1\right)$$

$$=\frac{9}{8}\log_e\left(\frac{3}{2}\right)-\frac{9}{16}+\frac{3}{2}-0-\frac{3}{4}$$

$$=\left(\frac{9}{8}\log_e\left(\frac{3}{2}\right)+\frac{3}{16}\right)\text{ square units}$$

Mark allocation: 3 marks

- 1 answer mark for a correct integral for the required area: $A=\int_{1}^{\frac{3}{2}}\left(x\log_e(x)+1\right)dx$
- 1 method mark for using the answer to **part a.** to anti-differentiate $x\log_e(x)$ or $x\log_e(x)+1$

  Possible expressions include:

  $$\int x\log_e(x)\,dx=\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2$$

  $$\int\left(x\log_e(x)+1\right)dx=\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2+x$$

  $$\int_{1}^{\frac{3}{2}}x\log_e(x)dx=\left[\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2\right]_{1}^{\frac{3}{2}}$$

  $$\int_{1}^{\frac{3}{2}}\left(x\log_e(x)+1\right)dx=\left[\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2+x\right]_{1}^{\frac{3}{2}}$$

- 1 answer mark for the correct area: $\frac{9}{8}\log_e\left(\frac{3}{2}\right)+\frac{3}{16}$

Tips

- *The use of the word 'hence' requires that you use the result of the previous part of the question, so your antiderivative should include* $x^2\log_e(x)$*. Alternative methods of integrating* $x\log_e(x)$ *should not be used.*
- *Accurate arithmetic involving fractions is an important skill often tested in the Mathematical Methods 1 exam.*

Question 9a.

Worked solution

$$\Pr(O \cap D) = \frac{2}{3} \times \frac{1}{4}$$

$$= \frac{1}{6}$$

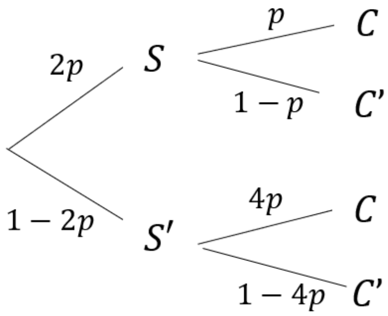
Mark allocation: 1 mark

- 1 answer mark for the correct answer: $\frac{1}{6}$

Tip

- *For independent events:* $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Question 9b.



Worked solutions

$$\text{Probability} = f(p) = \Pr(S \cap C') + \Pr(S' \cap C)$$
$$= 2p(1-p) + (1-2p)4p$$
$$= 2p - 2p^2 + 4p - 8p^2$$
$$= -10p^2 + 6p$$

The maximum occurs when $f'(p) = 0$.

$$f'(p) = -20p + 6$$
$$-20p + 6 = 0$$
$$p = \frac{3}{10}$$

Alternative methods: complete the square or note that the turning point is at $p = -\frac{b}{2a}$.

$$\text{maximum} = f\left(\frac{3}{10}\right)$$
$$= -10\left(\frac{3}{10}\right)^2 + 6\left(\frac{3}{10}\right)$$
$$= -\frac{9}{10} + \frac{18}{10}$$
$$= \frac{9}{10}$$

Mark allocation: 3 marks

- 1 method mark for obtaining the probability in terms of p: $-10p^2 + 6p$
- 1 answer mark for determining the value of p that maximises the probability: $p = \frac{3}{10}$
- 1 answer mark for correctly determining the maximum: $\frac{9}{10}$

Tip

- *Drawing a tree diagram is a useful strategy for answering questions of this type.*

Question 9c.

Worked solution

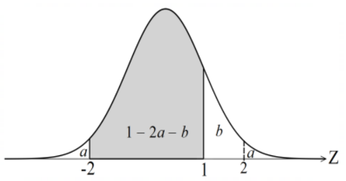
$$\Pr(V > 202 \mid V < 211) = \Pr\left(Z > \frac{202-205}{3} \mid Z < \frac{211-205}{3}\right)$$

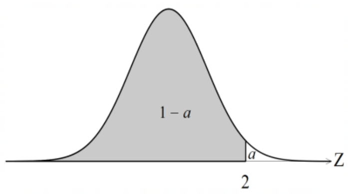
$$= \Pr(Z > -1 \mid Z < 2)$$

$$= \frac{\Pr(Z > -1 \cap Z < 2)}{\Pr(Z < 2)}$$

$$= \frac{\Pr(-1 < Z < 2)}{\Pr(Z < 2)}$$

The following diagrams will help express this in the form required.





$$\Pr(V > 202 \mid V < 211) = \frac{1-2a-b}{1-a} \quad \left(\text{or } 1 - \frac{a+b}{1-a} \text{ or } 1 + \frac{a+b}{a-1}\right)$$

Alternative method

$$\Pr(V > 202 \mid V < 211) = \frac{\Pr(-1 < Z < 2)}{\Pr(Z < 2)}$$

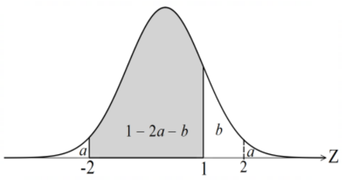
$$= \frac{1 - \Pr(Z < -1) - \Pr(Z > 2)}{1-a}$$

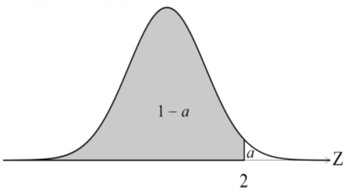
$$= \frac{1-(a+b)-a}{1-a}$$

$$= \frac{1-2a-b}{1-a}$$

Mark allocation: 2 marks

- 1 method mark for using any suitable method, such as:
  - simplified conditional probability expressed using Z:
    $\Pr(V > 202 \mid V < 211) = \dfrac{\Pr(-1 < Z < 2)}{\Pr(Z < 2)}$
  - or drawing two bell-shaped curves and representative areas (as shown below)





- 1 answer mark for the correct answer: $\dfrac{1-2a-b}{1-a}$ $\left(\text{or } 1-\dfrac{a+b}{1-a} \text{ or } 1+\dfrac{a+b}{a-1}\right)$

Tip

- *Sketching diagrams and using the symmetry properties of the normal distribution are useful techniques you can use.*

End of Worked Solutions