Mathematical Methods 2024

Written Examination 2

Question and Answer Book

2024 Insight Year 12 Trial Exam Paper

- **Reading time:** 15 minutes
- **Writing time:** 2 hours
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and Answer book of 27 pages
- Formula Sheet
- Answer Sheet for multiple-choice questions

Instructions

- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- You may keep the formula sheet.

Students are **NOT** permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple choice questions

Instructions for Section A

- Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
- Choose the response that is **correct** for the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The amplitude and period of the function $f(x)=5-3\cos(2\pi x)$ are, respectively

A. 3 and 2π

B. 3 and 1

C. -3 and π

D. 5 and 2π

Question 2

The graph of the cubic function $y=bx^2-ax^3$, where $a,b\in R$, has a stationary point at $x=0$ and

A. $x=-\dfrac{b}{3a}$

B. $x=-\dfrac{2b}{3a}$

C. $x=-\dfrac{b}{2a}$

D. $x=\dfrac{2b}{3a}$

Question 3

Consider the system of simultaneous equations below containing the variable, λ.

$$x + \lambda y = 3\lambda$$
$$\lambda x + 4y = 12$$

The value(s) of λ for which the system of equations has infinite solutions are

A. $\lambda \in \{-2\}$

B. $\lambda \in \{-2, 2\}$

C. $\lambda \in R \setminus \{-2\}$

D. $\lambda \in R \setminus \{-2, 2\}$

Question 4

A binomial distribution has a mean of 3 and a standard deviation of 1.5.

The number of trials in this binomial distribution is

A. 12

B. 6

C. 4

D. 3

Question 5

What is the implied domain of $y = \log_e(5 - x) + \sqrt[3]{x+1}$?

A. $(-\infty, 5)$

B. $(-1, 5)$

C. $[-1, 5)$

D. $[-1, \infty)$

Question 6

Which of the following functions does not have a horizontal asymptote?

A. $y=\dfrac{1}{x^3}$

B. $y=\dfrac{1}{(x-2)^2}$

C. $y=\dfrac{1}{x^2+1}$

D. $y=x^{\frac{1}{3}}$

Question 7

The probability distribution table for a discrete random variable X is shown below.

X	0	1	2	3
$\Pr(X=x)$	0.15	0.1	m	m^2

The mean of X is

A. 0.5

B. −1.5

C. 1.85

D. 3.85

Question 8

The function f is given by

$$f(x)=\begin{cases} x^2+1 & x<a \\ -x^2+4x+b & x\ge a \end{cases}$$

If the graph of $y=f(x)$ is smooth and continuous then

A. $a=1$ and $b=1$

B. $a=1$ and $b=-1$

C. $a=-1$ and $b=7$

D. $a=-1$ and $b=-7$

Question 9

If $\int_{-1}^{3} f(x)\,dx = 1$ and $\int_{-1}^{-3} f(x)\,dx = 7$ then the value of which of the following definite integrals is 6?

A. $\int_{-3}^{3} f(x)\,dx$

B. $\int_{-3}^{3} -f(x)\,dx$

C. $\int_{-1}^{3} x - f(x)\,dx$

D. $\int_{3}^{-3} -f(x)\,dx$

Question 10

Let $f : R \setminus \{0\} \to R,\ f(x) = \dfrac{1}{x^2}$ and $g : (-\infty, 1] \to R,\ g(x) = \sqrt{1-x} + 1$

The range of $(f \circ g)(x)$ is

A. $(-\infty, 1)$

B. $[1, \infty)$

C. R^+

D. $(0, 1]$

Question 11

Consider the following probability density function for the continuous random variable X.

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

The mean of X is located at approximately the

A. 37^{th} percentile

B. 43^{rd} percentile

C. 50^{th} percentile

D. 63^{rd} percentile

Question 12

Let $f(x)=\log_e(x+1)$.

If the sum function $f(x)+g(x)=-\log_e(1-x)$, then the rule for g is

A. $g(x)=\log_e(x^2-1)$

B. $g(x)=\log_e(1-x^2)$

C. $g(x)=\log_e\left(\dfrac{1}{1-x^2}\right)$

D. $g(x)=\log_e\left(\dfrac{1-x}{1+x}\right)$

Question 13

In a factory producing computer chips, a random sample of 120 chips is used to create a confidence interval for the proportion of defective chips.

If the width of the confidence interval needs to be halved, then, assuming that the sample proportion will be the same, how many computer chips should be randomly selected for the next sample?

A. 30

B. 60

C. 170

D. 480

Question 14

The algorithm below, written in pseudocode, estimates the value of a definite integral using the trapezium rule.

```
Inputs:   f(x), the function to integrate
          a, the lower terminal of integration
          b, the upper terminal of integration
          n, the number of trapeziums to use

Define  trap(f(x), a, b, n)
      w ← (b - a) ÷ n
      sum ← f(a) + f(b)
      x ← a + w
     For i from 1 to n - 1
          sum ←  sum + 2 × f(x)
          x ←  x + w
     EndFor
     area ← (w ÷ 2) × sum
     Return area
```

The **Return** value of the function `trap(1 + x^0.5, 1, 4, 3)` is closest to

A. 10.5

B. 15.3

C. 4.6

D. 7.6

Question 15

A cubic function has the equation $y = 2x(x-1)(x+3)$.

One tangent to the curve does not intersect the graph at any point other than the point of tangency.

The y-intercept of this tangent is

A. $-\frac{16}{27}$

B. $-\frac{26}{3}$

C. $-\frac{52\sqrt{13}}{27}$

D. -8

Question 16

X is a normally distributed random variable with a mean of 17.

If $\Pr(X > 20) = p$ then $\Pr(14 < X \leq 17)$ is

A. $1 - p$

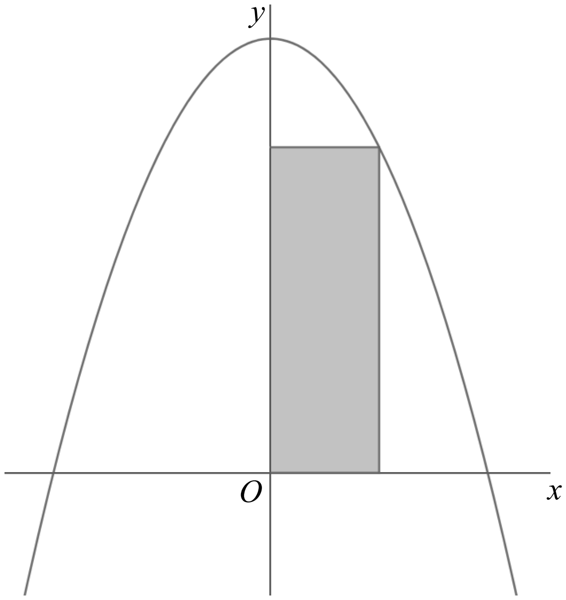
B. $\frac{1}{2} - p$

C. $1 - 2p$

D. $p + \frac{1}{2}$

Question 17

A rectangle is formed with one corner at the origin and one corner on the graph of $y = 4 - x^2$ for $0 < x < 2$, as shown in the diagram.



The maximum area of this rectangle is

A. $\dfrac{2\sqrt{3}}{3}$

B. $\dfrac{16\sqrt{3}}{9}$

C. 1

D. 2

Question 18

Let $f:(-\infty,4]\to R,\ \ f(x)=\sqrt{4-x}$ and let $g:D\to R,\ g(x)=-\log_e(x-1)$, where D is the greatest domain of g for which $(f\circ g)(x)$ is defined. The set D is

A. $(1,e^{-4}+1)$

B. $(1,e^{-4}+1]$

C. $[e^{-4}+1,\infty)$

D. $(e^{-4}+1,\infty)$

Question 19

The tangents to $y=\dfrac{1}{x^2}$ at $x=a$ and $x=-a$ are perpendicular.

The value of $a\in R^+$ is

A. $\sqrt{2}$

B. $\sqrt[3]{2}$

C. $\sqrt{3}$

D. $\dfrac{3\sqrt[3]{2}}{2}$

Question 20

Two functions, f and g, are continuous and differentiable for all $x\in R$. It is known that $f(1)=-2$, $f'(1)=\dfrac{3}{2}$, $g(1)=2$, and $g'(1)=3$.

The tangent to the graph of $y=\dfrac{[f(x)]^2}{g(x)}$ at $x=1$ is

A. $6x+y=8$

B. $y=-6x-8$

C. $9x-4y=1$

D. $y=\dfrac{9}{4}x-\dfrac{7}{4}$

This page is blank.

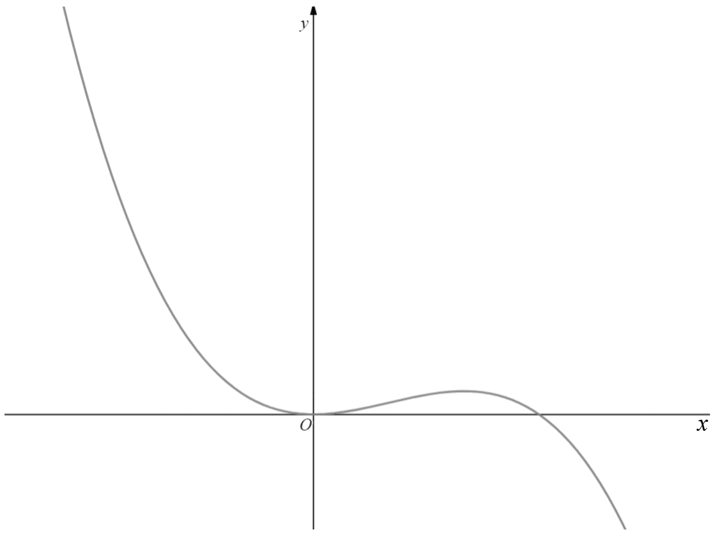
SECTION B

Instructions for Section B

- Answer **all** questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (12 marks)

Consider the function $f : R \to R,\ f(x) = x^2(1-x)$. Part of the graph of f is shown below.



a. State the coordinates of the x-intercepts of f.

1 mark

____

____

b. Find the coordinates of the stationary points of f.

1 mark

____

____

c. Find the area bounded between the graph of $y = f(x)$ and its tangent at $x = 1$.

3 marks

Consider the function $g : R \to R,\ g(x) = x^m(1-x)$ where m is a positive integer.

d. State the coordinates, in terms of m, of the maximum turning point of $y = g(x)$.

2 marks

When $m \geq 2$, the graph of $y = g(x)$ will have a non-stationary point of inflection in the interval $x \in (0,1)$.

e. Find the x-value, in terms of m, of the point on $y = g(x)$ which has a maximum gradient in the interval $x \in (0,1)$ when $m \geq 2$.

2 marks

f. Find the area bounded by the graph of $y = g(x)$ and the x-axis in terms of m.

2 marks

g. For what values of m will the tangent to $y = g(x)$ at $x = 1$ intersect with the graph of $y = g(x)$ again?

1 mark

Question 2 (14 marks)

The depth of water, in metres, of a harbour over one day is given by $D(t) = 16 - 4\cos\left(\frac{\pi}{12}t\right)$, $0 \le t \le 24$, where t is the number of hours after midnight.

a. Find the depth of the water at 4 am.

1 mark

b. Find the rate, in metres per hour, at which the water level is rising when $t = 3$.

1 mark

c. Find the average depth of the water between midnight and 8 am.

Give your answer correct to two decimal places.

2 marks

d. The water level is above x metres for a total of 3 hours during the day.

Find the value of x correct to two decimal places.

1 mark

A particular ship is only allowed to be in the harbour when the water depth is at least 19 metres.

e. What is the first value of t for which the depth of the water is 19 metres?

Give your answer correct to two decimal places.

1 mark

This ship must travel for 90 minutes after entering the harbour before it can unload its cargo. Once its cargo is unloaded, it takes 100 minutes to turn around and leave the harbour.

f. How long does the ship have to unload its cargo?

Give your answer in hours and minutes, correct to the nearest minute.

2 marks

An oceanographer believes that the depth of the water this day is better modelled by $H(t) = B - Ae^{\cos\left(\frac{\pi}{12}t\right)}$, $0 \le t \le 24$, where t is still the number of hours after midnight.

The oceanographer agrees that the depth at midnight is 12 metres and the depth at midday is 20 metres.

g. Write a set of simultaneous equations that could be used to show that

$A = \dfrac{8e}{e^2 - 1}$ and $B = \dfrac{20e^2 - 12}{e^2 - 1}$.

1 mark

h. According to the oceanographer's model, for how long during the day would the water level be above 19 metres?

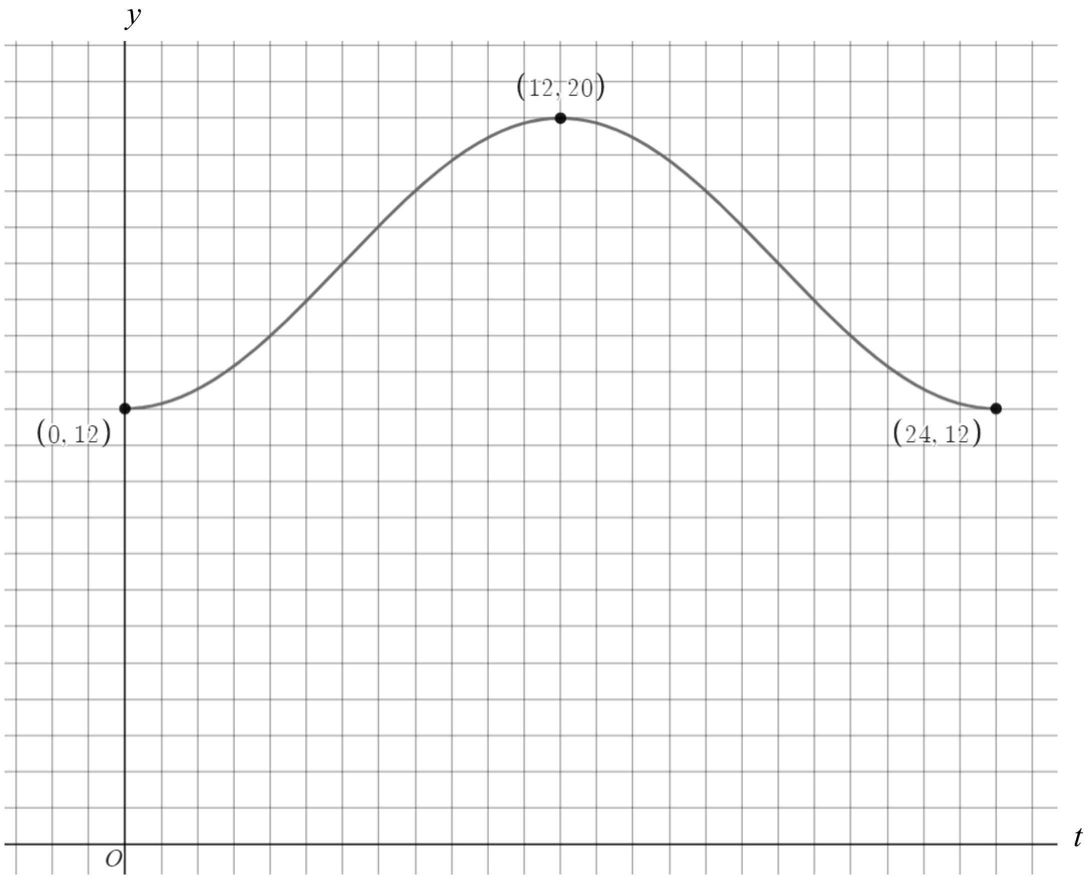
Give your answer in hours and minutes, correct to the nearest minute.

1 mark

i. The graph of $y = D(t)$ is shown below.

Sketch the graph of $y = H(t)$ on the same set of axes.

2 marks



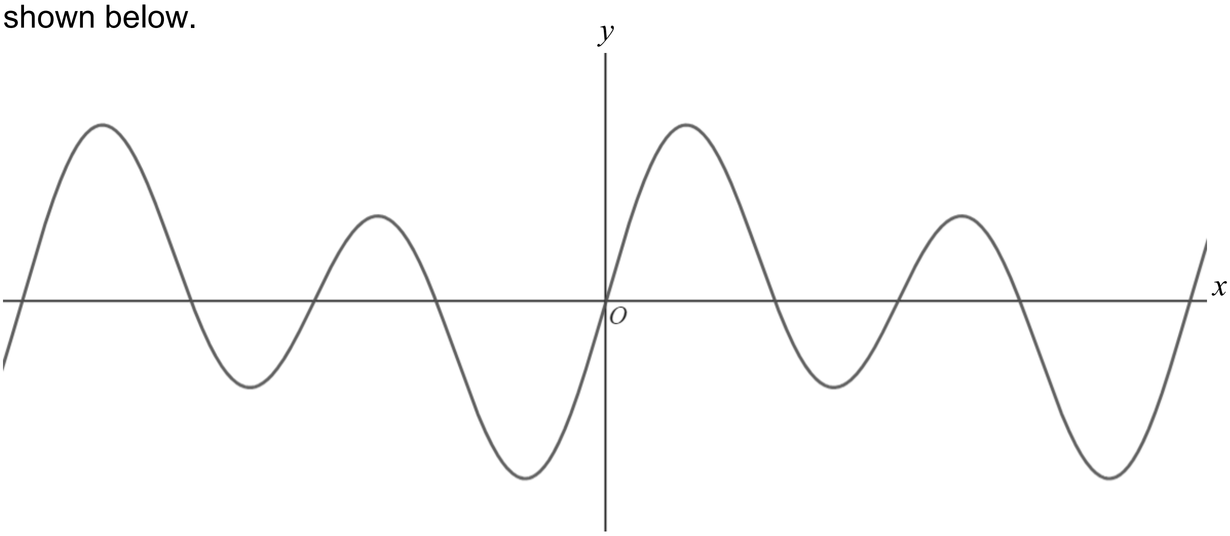
j. Find the maximum difference between the oceanographer's model for the water depth and the original model for the water depth.

Give your answer correct to two decimal places.

2 marks

Question 3 (10 marks)

Consider the function $f : R \to R,\ f(x) = \sin(x)\left(1 + 4\cos(x)\right)$. Part of the graph of $y = f(x)$ is shown below.



a. What is the period of f?

1 mark

Some of the x-intercepts of f can be found as exact values. The remaining x-intercepts cannot be found algebraically and therefore must be approximated using an algorithm.

b. State the general solution to $\sin(x) = 0$, which gives the set of x-intercepts that can be expressed exactly.

1 mark

Newton's method is used with an initial estimate of $x = 2$ to approximate the first positive x-intercept.

c. Find the values of x_1 and x_2 when Newton's method is applied to f with an initial estimate $x_0 = 2$.
Give your answers correct to three decimal places.

1 mark

d. Explain why $x_0 = 3$ would not be an appropriate initial estimate for finding the first positive x-intercept.

1 mark

The derivative of f can be written as $f'(x) = 8\cos^2(x) + \cos(x) - 4$.

e. Show that $f'(x) = 0$ whenever $\cos(x) = \dfrac{-1 \pm \sqrt{129}}{16}$.

1 mark

f. Hence, find the exact x-value of the first maximum of the graph of $y = f(x)$ which is to the right of the y-axis.

1 mark

Consider the function $g : R \to R,\ g(x) = \cos(x)\left(1 + 4\sin(x)\right)$.

g. Find the area bound between the graphs of $y = f(x)$ and $y = g(x)$ from $x = \frac{\pi}{4}$ to $x = \frac{5\pi}{4}$.

2 marks

h. The graphs of $y = g(x)$ and $y = f(ax + b)$ are the same.

Find the value of a and the smallest positive value of b.

2 marks

Question 4 (13 marks)

A soft drink company produces $600\,\text{mL}$ bottles of cola. The bottles are filled by a machine that distributes cola with a mean of $601\,\text{mL}$ and a standard deviation of $1.2\,\text{mL}$.

a. What proportion of bottles will be underfilled by this machine? That is, what proportion of bottles will receive less than $600\,\text{mL}$ of cola?

Give your answer to four decimal places.

1 mark

b. Thirty per cent of bottles are filled with at least a mL of cola.

Find the value of a, correct to two decimal places.

1 mark

Due to regulations, no more than 15% of bottles can be underfilled or the company will be fined. The company cannot change the standard deviation of the machine but can change the mean. The manager decides that in order to be confident that they will pass a test by regulators, the company should set the new mean so that the probability of any given bottle being underfilled is 10%.

c. What is the least value for the mean so that the probability of any given bottle being underfilled is 10%?

Give your answer correct to two decimal places.

2 marks

This change is made and the probability of any given bottle being underfilled is now 10%.

The regulator now visits the company to check the bottles. A random sample of 200 bottles is taken and each is checked to see if it is underfilled.

d. How many bottles in this sample are expected to be underfilled?

1 mark

e. What is the probability that more than 15% of the bottles in the sample will be underfilled?

Give your answer correct to four decimal places. Do not use a normal approximation.

2 marks

The regulator makes surprise inspections up to three times per year. The number of times, R, the regulator visits per year is described by the probability distribution table shown below.

r	0	1	2	3
$\Pr(R = r)$	0.4	0.3	0.15	0.15

f. What is the probability that the regulator visits at least twice in one year?

1 mark

g. In a two-year span, the regulator visits four times.

What is the probability the regulator visited twice in each year? Assume the number of times visited each year is independent of the number of times visited in any other year.

2 marks

The company has a problem with its labelling machine. The machine is attaching labels incorrectly to some bottles. To understand the scope of the problem, a sample of bottles is taken and a 95% confidence interval for the proportion of poorly labelled bottles is found to be $(0.2455,\ 0.4545)$.

h. What was the sample proportion, $\hat{p}$, of this sample?

1 mark

i. How many bottles were in the sample?

2 marks

Question 5 (11 marks)

Consider the function $f:[0,1] \to R,\ f(x) = a\dfrac{x}{e^x}$, where $a \in (1, \infty)$.

a. State the coordinates of the endpoints of the graph of $y = f(x)$.

1 mark

b. Find the values of a such that the graphs of $y = f(x)$ and its inverse, $y = f^{-1}(x)$, create a bounded region.

2 marks

Consider the function $g:[0,\infty) \to R,\ g(x) = k\dfrac{x}{e^{mx}}$, where $k \in (1, \infty)$ and $m \in (0, \infty)$.

c. State the coordinates of the maximum turning point of g.

2 marks

d. Find the area bound by the graphs of $y = g(x)$ and $y = x$ when $k = e$ and $m = 1$.

2 marks

The minimum gradient of g occurs when $x = \frac{2}{m}$.

e. Find an expression, in terms of k, for the value of the minimum gradient of g.

1 mark

Consider the functions $g_1 : \left[0, \frac{1}{m}\right] \to R,\ g_1(x) = k\frac{x}{e^{mx}}$ and $g_2 : \left[\frac{1}{m}, \infty\right) \to R,\ g_2(x) = k\frac{x}{e^{mx}}$, which each have the same rule as g, but different domains.

f. Find an expression, in terms of m, for the area enclosed by the graphs of $y = g_1(x)$, $y = g_1^{-1}(x)$, $y = g_2(x)$ and $y = g_2^{-1}(x)$ when the minimum gradient of g_2 is -1.

3 marks

End of Question and Answer Book