The Mathematical Association of Victoria

Trial Examination 2024

MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

a. $y = \dfrac{\log_e(x)}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \times \frac{1}{x} - \log_e(x) \times 2x}{x^4}$$

$$= \frac{x - 2x\log_e(x)}{x^4}$$

$$= \frac{1 - 2\log_e(x)}{x^3}$$ **1A**

b. $g(x) = x\tan^2(x)$

$g'(x) = x \times 2\tan(x)\sec^2(x) + \tan^2(x) \times 1$ **1M**

$$g'\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \times 2\tan\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right) + \tan^2\left(\frac{\pi}{4}\right)$$

$$= \frac{\frac{\pi}{2}}{\cos^2\left(\frac{\pi}{4}\right)} + 1$$

$$= \frac{\frac{\pi}{2}}{\frac{1}{2}} + 1$$

$$= \pi + 1$$ **1A**

Question 2

$$\log_e\left(3x^2 - 1\right) - \log_e\left(1 - 3x\right) = \log_e(2)$$

$$\log_e\left(\frac{3x^2 - 1}{1 - 3x}\right) = \log_e(2)$$ **1M**

Equate $\dfrac{3x^2 - 1}{1 - 3x} = 2$

$3x^2 - 1 = 2 - 6x$

$3x^2 + 6x - 3 = 0$

$3\left(x^2 + 2x - 1\right) = 0$ **1M**

$x^2 + 2x - 1 = 0$

$$x = \frac{-2 \pm \sqrt{4 + 4}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$=-1\pm\sqrt{2}$

eect $x=-1+\sqrt{2}$ as $1-3x>0$, $x<\frac{1}{3}$

For real solution $x=-1-\sqrt{2}$ **1A**

Question 3

$-2x+ky=m\Rightarrow y=\frac{2x+m}{k}$

$\left(1+k^2\right)x+y=2\Rightarrow y=-\left(1+k^2\right)x+2$

Equate gradients $\frac{2}{k}=-\left(1+k^2\right)$ **1M**

$1+k^2+\frac{2}{k}=0$

$k^3+k+2=0$

sing factor theorem

$(k+1)\left(k^2-k+2\right)=0$

$\Delta<0$ for k^2-k+2 giving no real solution

$k=-1$ **1M**

Substitute $k=-1$ in equations

$-2x-y=m$

$2x+y=2$

alues of k and m for which there are no solutions

$k=-1,\ m\neq-2$

Answer $k=-1,\ m\in R\setminus\{-2\}$ **1A**

Question 4

$2\sin^2(x)+\sin(x)=1$ for $x\in[-\pi,\pi]$

$2\sin^2(x)+\sin(x)-1=0$

$\left(2\sin(x)-1\right)\left(\sin(x)+1\right)=0$

$2\sin(x)=1,\ \sin(x)=-1$ **1M**

$\sin(x)=\frac{1}{2}$ for $x\in[-\pi,\pi]$ gives

$x=\frac{\pi}{6},\ \frac{5\pi}{6}$ **1A**

$\sin(x)=-1$ for $x\in[-\pi,\pi]$ gives

$x=-\frac{\pi}{2}$ **1A**

Question 5

$f:R\setminus\{2\}\to R,\ f(x)=\frac{1}{(2-x)^2}$ and $g:[-2,k)\to R,\ g(x)=2x+1$

a. For $k=1$, $g:[-2,1)\to R, g(x)=2x+1$

For $f\circ g$ to eist , test range $(g)\subseteq$ dom (f)

$[-3,3)\nsubseteq R\setminus\{2\}$

$f\circ g$ does not eist **1A**

b. For $f \circ g$ to eist

$[-3, 2k+1) \in R \setminus \{2\}$

$2k+1=2 \Rightarrow k=\frac{1}{2}$ such that $f \circ g$ eists **1A**

c. From **part b.** $k=\frac{1}{2}$

Dom $f \circ g : \left[-2, \frac{1}{2}\right)$

ule $(f \circ g)(x) = \frac{1}{(1-2x)^2}$ **1A** both rule and domain

Question 6

a. $\int_2^3 \frac{3}{2x-1}\,dx = \frac{3}{2}\int_2^3 \frac{2}{2x-1}\,dx$

$= \frac{3}{2}\left[\log_e(2x-1)\right]_2^3$ **1M**

$= \frac{3}{2}\left(\log_e(5) - \log_e(3)\right)$

in the form $a\log_e(b)$ where $a, b \in Q$

$\int_2^3 \frac{3}{2x-1}\,dx = \frac{3}{2}\log_e\left(\frac{5}{3}\right)$ **1A**

b. Two trapeziums of width $\frac{1}{2}$

Approimate area $= \frac{\frac{1}{2}}{2}\left(f(2) + 2f\left(\frac{5}{2}\right) + f(3)\right)$

$= \frac{1}{4}\left(\frac{3}{2(2)-1} + 2\left(\frac{3}{2\left(\frac{5}{2}\right)-1}\right) + \frac{3}{2(3)-1}\right)$ **1M**

$= \frac{1}{4}\left(1 + \frac{3}{2} + \frac{3}{5}\right)$

$= \frac{1}{4} \times \frac{31}{10}$

Approimate a rea $= \frac{31}{40}$ **1A**

c. $\frac{31}{40} > \frac{3}{2}\log_e\left(\frac{5}{3}\right)$

Area in **part b.** is an over estimate of the actual area because the upper straight side of each trapezia sits above the concave upward curve. **1A**

Question 7

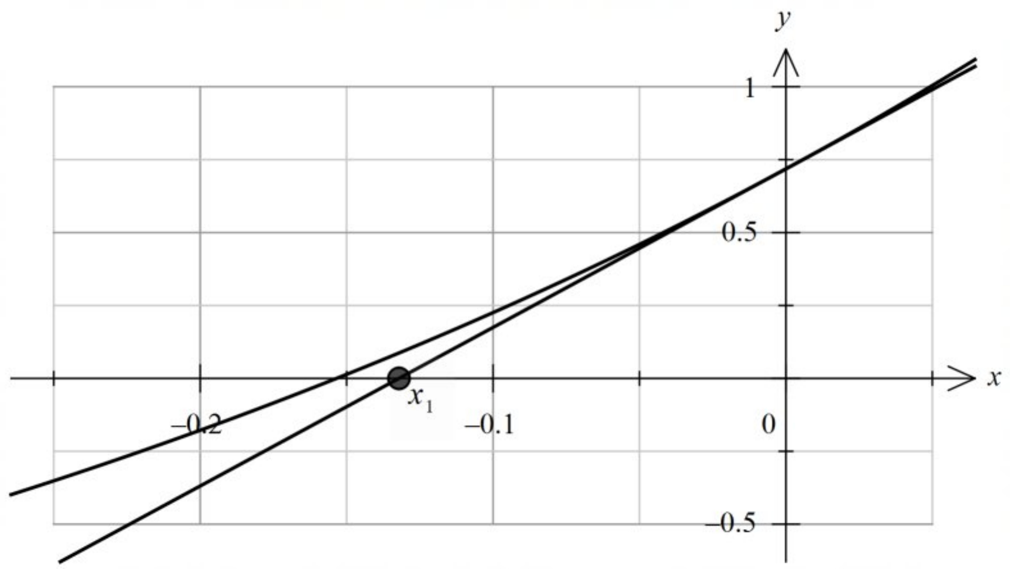
$f(x) = e^{2x+1} - 2$

a. $x_1 = x_0 - \dfrac{f(0)}{f'(0)}$

$x_1 = 0 - \dfrac{e^{2\times0+1} - 2}{2e^{2\times0+1}}$

$x_1 = -\dfrac{e-2}{2e} = \dfrac{2-e}{2e} = \dfrac{1}{e} - \dfrac{1}{2}$ **1A** or equivalent form

b. Tangent line with x_1 labelled. **1A**



c. $e^{2x+1} - 2 = 0$

$e^{2x+1} = 2$

$2x + 1 = \log_e(2)$

$x = \dfrac{\log_e(2) - 1}{2}$ **1A**

Distance $= \dfrac{2-e}{2e} - \dfrac{\log_e(2) - 1}{2}$

$= \dfrac{2 - e\log_e(2)}{2e}$

$= \dfrac{1}{e} - \dfrac{\log_e(2)}{2}$

$= \dfrac{1}{e} + \log_e\left(\dfrac{1}{\sqrt{2}}\right)$ **1H**

Question 8

$f(x) = 1 + \sqrt{2x-3}$

a. $f'(x) = \dfrac{1}{\sqrt{2x-3}} = 1$

$\frac{1}{\sqrt{2x-3}}=1$

$1=\sqrt{2x-3}$

$2x-3=1$

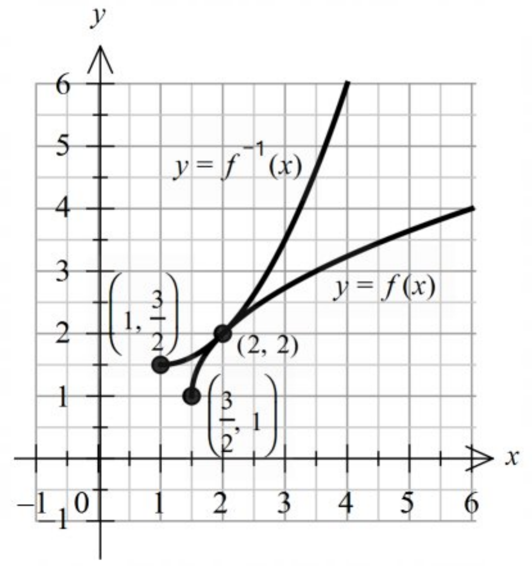
$x=2$

$(2,2)$ coordinates **1A**

b. Shape $y=f(x)$ **1A**

Shape $y=f^{-1}(x)$ **1A**

Coordinates **1A**



c. $g(x)=1+\sqrt{2x-a}=1+\sqrt{2\left(x-\frac{a}{2}\right)}$

As the graph of f touches the line $y=x$ then $a<3$.

The graphs of g will intersect the line $y=x$ twice until the coordinates of the endpoint are $(1,1)$.

If the graph of f is translated $\frac{1}{2}$ a unit left it will intersect the line $y=x$ at $(1,1)$.

$g(x)=1+\sqrt{2\left(x-\frac{3}{2}+\frac{1}{2}\right)}=1+\sqrt{2(x-1)}=1+\sqrt{2x-2}$

Hence $2\le a<3$. **1A**

OR

Solve $1+\sqrt{2x-a}=x$

$\sqrt{2x-a}=x-1$

$2x-a=(x-1)^2$

$2x-a=x^2-2x+1$

$x^2-4x+1+a=0$

$\Delta=16-4(1+a)>0$

$12-4a>0$

$a<3$

Substitute $x=1$ into $x^2-4x+1+a=0$ to find the minimum value of a.

$a=2$

Hence $2 \le a < 3$. **1A**

OR

Substitue $(1,1)$ into $g(x) = 1 + \sqrt{2x - a}$

$1 + \sqrt{2 - a} = 1$

$\sqrt{2 - a} = 0$

$a = 2$

Hence $2 \le a < 3$. **1A**

d. If the graph of f is translated $\frac{1}{2}$ a unit left then the points of intersection are at $x = 1$ and $x = 3$.

OR

Solve $1 + \sqrt{2x - 2} = x$

$2x - 2 = (x - 1)^2$

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$

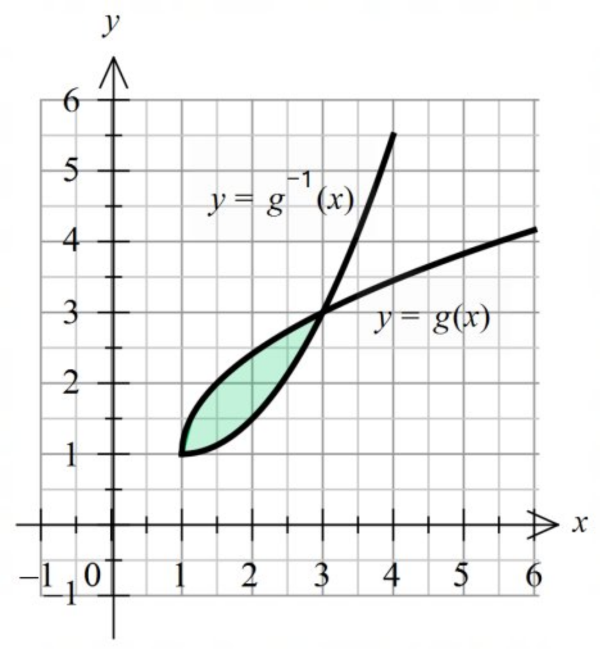
$2\int_1^3 \left(1 + \sqrt{2x - 2} - x\right) dx$ **1M**

$= 2\left[x + \frac{1}{3}(2x - 2)^{\frac{3}{2}} - \frac{1}{2}x^2\right]_1^3$ **1M**

$= 2\left(\left(3 + \frac{8}{3} - \frac{9}{2}\right) - \left(1 - \frac{1}{2}\right)\right)$

$= 2\left(-2 + \frac{8}{3}\right)$

$= \frac{4}{3}$ **1A** or equivalent form



Question 9

a. 0.02^3

$= 0.000\ 008$ or $\dfrac{1}{125\ 000}$ **1A**

b. $X \sim \text{Bi}(4, 0.3)$ **1M**

$\Pr(X = 4) = \binom{4}{3} 0.3^3 \times 0.7$

$= 4 \times 0.027 \times 0.7$

$= 0.0756$ or $\dfrac{189}{2500}$ **1A**

c. $2 \times 0.02 \times 0.98$

$= 0.0392$ or $\dfrac{49}{1250}$ **1A**

d. $0.02 n_t = 0.3 n_s$

$n_t = 15 n_s$, $n_s \in Z^+$ **1A**

e. $d(t) = \begin{cases} t-1 & 1 \le t \le 2 \\ a(t-3)^2 + b & 2 < t \le 3 \\ 0 & \text{elsewhere} \end{cases}$

$a(t-3)^2 + b = t - 1$ when $t = 2$

$a + b = 1 (1)$ **1A**

$\int_1^2 (t-1)\,dt + \int_2^3 \left(a(t-3)^2 + b\right)dt = 1$

$\left[\dfrac{t^2}{2} - t\right]_1^2 + \left[\dfrac{a}{3}(t-3)^3 + bt\right]_2^3 = 1$

$\dfrac{1}{2} + \left[\dfrac{a}{3}(t-3)^3 + bt\right]_2^3 = 1$

$\left[\dfrac{a}{3}(t-3)^3 + bt\right]_2^3 = \dfrac{1}{2}$

$b + \dfrac{a}{3} = \dfrac{1}{2} (2)$ **1M** attempt to set up the second equation

$a + b = 1 (1)$

Subtract (2) from (1)

$\dfrac{2}{3} a = \dfrac{1}{2}$

$a = \dfrac{3}{4}$, $b = \dfrac{1}{4}$ **1A**

END OF SOLUTIONS