MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2

TSSM™
Creating VCE Success

2024 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

Explanation:

Period= $\frac{\pi}{3}$, Range= R

Question 2

Answer: C

Explanation:

For a unique solution require $\frac{p}{3} \neq \frac{2}{p-1}$
$p^2 - p - 6 \neq 0$
$(p-3)(p+2) \neq 0$
Hence $p \in R \backslash \{-2,3\}$

Question 3

Answer: C

Explanation:

$$\int_b^a f(x)\,dx = -\left(\int_a^c f(x)\,dx - \int_b^c f(x)\,dx\right)$$
$$= -(10-2) = -8$$

Question 4

Answer: C

Explanation:

Two successive stationary points are adjacent local maxima and minima. The horizontal distance is half a period $= \frac{\pi}{b}$, the vertical distance is $2a$

Hence $distance = \sqrt{(2a)^2 + \left(\frac{\pi}{b}\right)^2}$

Question 5

Answer: D

Explanation:

$$dom\,(f+g) = dom(f) \cap dom(g)$$
$$= [-5, \infty) \cap R\backslash\{2\}$$
$$= [-5, 2) \cup (2, \infty)$$

Question 6

Answer: C

Explanation:

The graph is a polynomial of order 4. $\frac{d^2y}{dx^2}$ will be of order 2. This can have up to 2 solutions, hence a maximum of two points of inflection.

Question 7

Answer: D

Explanation:

$\Pr(a,a)+\Pr(b,b)=1-\Pr(a,b)-\Pr(b,a)=1-2\Pr(a,b)$
$=1-2\times\dfrac{a}{a+b}\times\dfrac{b}{a+b-1}$
$=1-\dfrac{2ab}{(a+b)(a+b-1)}$

Question 8

Answer: B

Explanation:

Look for $f(-x)=f(x)$
$\sin\left(x+\frac{\pi}{2}\right)=\cos(x)$ which is even.

Question 9

Answer: A

Explanation:

Find when gradient of curve is equal to gradient of tangent passing through (0, 0)
$\dfrac{dy}{dx}=2x+5$
$\dfrac{f(x)-f(0)}{x-0}=2x+5$
$x^2+5x+1=2x^2+5x$
$x^2-1=0$
$x=\pm 1$

Check which passes through y, not just parallel.

At $x=1$, $y_T=7x \quad (1,7)$
At $x=-1$ $y_T=3x$ $(-1,-3)$
$y(1)=7$
$y(-1)=-3$
Hence at both $x=\pm 1$

Question 10

Answer: A

Explanation:

Use CAS

```
binomialCDf(60,80,80,0.7)/binomialCDf(50,80,80,0.7)
                                        0.2101951456
```

Question 11

Answer: D

Explanation:

$$\int_0^5 kx^2 + 3\,dx = 1$$
$$k = -\frac{42}{125}$$

Question 12

Answer: B

Explanation:

$a + a^2 + 2a + 0.4 - a^2 = 1$
$a = 0.2$

$E(X) = (0 \times 0.2) + (1 \times 0.04) + (2 \times 0.4) + (3 \times 0.36)$
$= 1.92$

Question 13

Answer: A

Explanation:

$$\sqrt{\frac{0.65 \times 0.35}{n}} = 0.1$$

$$n = \frac{91}{4}, \qquad so\ n = 23$$

Question 14

Answer: D

Explanation:

Look for graph that is a reflection through the line $y = x$.

Question 15

Answer: A

Explanation:

$$\frac{y(4) - y(0)}{4 - 0}$$
$$= 30$$

Question 16

Answer: D

Explanation:

$$\frac{1}{5-1}\int_1^5 x + \sqrt{x}\, dx = \frac{5\sqrt{5} + 17}{6}$$

Question 17

Answer: C

Explanation:

$(2)^3 - 27 = b(2)^2 - 15(2) - 1$
$b = 3$

Question 18

Answer: A

Explanation:

As $x \to 0, f(x) \to \infty$
As $x \to n, f(x) = \frac{m}{\sqrt{n^2}} = \frac{m}{n}$
Hence $\left[\frac{m}{n}, \infty\right)$

Question 19

Answer: D

Explanation:

n	x	f(x)	df(x)
0	5	-13	2
1	11.5	42.25	15
2	8.683333	7.933611	9.366667
3	7.836329	0.717417	7.672657
4	7.742826	0.008743	7.485651
5	7.741658	1.36E-06	7.483315

Question 20

Answer: B

Explanation:

$f(x) = (x+1)(x-1)(x-2)$
So area between $x = -1$ and $= 1$ and area between $x = 1$ and $x = 2$

$$\frac{1--1}{2\times 4}\left(f(-1)+2f\left(-\frac{1}{2}\right)+2f(0)+2f\left(\frac{1}{2}\right)+f(1)\right)=\frac{5}{2}$$

$$-\left(\frac{2-1}{2\times 2}\left(f(1)+2f\left(\frac{3}{2}\right)+f(2)\right)\right)=\frac{5}{16}$$

Area= $\frac{5}{2}+\frac{5}{16}=\frac{45}{16}$

$$\int_{-1}^{1} f(x)\,dx - \int_{1}^{2} f(x)\,dx = \frac{37}{12}$$

$$\frac{\frac{45}{16}}{\frac{37}{12}}\times 100 = 91.22\%$$

SECTION 2: Extended response questions

Question 1 (13 marks)

a. $f'(x) = x^3e^{-x}(4-x)$ or equivalent

1 mark

b. $f'(x) = 0$ at $x = 0, 4$

$f(0) = 0,\ \ f(4) = 256e^{-4}$
(0,0) is a local minimum
$(4, 256e^{-4})$ is a local maximum

2 marks

c. $f''(x) = x^2(x-2)(x-6)e^{-x}$ or equivalent

Hence points of inflection at $x = 2$ and $x = 6$
$(2, 16e^{-2})$ and $(6, 1296e^{-6})$

2 marks

d. $f'(x) < 0$ for $x \in (-\infty, 0) \cup (4, \infty)$

1 mark

e. Maximum at $(4, 256e^{-4})$

Hence $c < -256e^{-4}$ or $c = 0$

1 mark

f. $y_T = 27e^{-3}x$

1 mark

g. $\int_0^3 27e^{-3}x - f(x)\, dx$
$= 1.615$

2 marks

h. Distance $D(x) = 27e^{-3}x - f(x)$

$\frac{dD}{dx} = 0$ at $x = 1.13374$ and $x = 3$
$y_T(1.13374) = 1.524$
So at $(1.13, 1.52)$

3 marks

Question 2 (10 marks)

a. $V = 2x(h - 2x)(2h - x)$ or equivalent

2 marks

b. $x \in \left(0, \frac{h}{2}\right)$

2 marks

c. $V(x) = 4h^2x - 10hx^2 + 4x^3$

$\frac{dV}{dx} = 12x^2 - 20hx + 4h^2$

1 mark

d. $12x^2 - 20hx + 4h^2 = 0$

$3x^2 - 5hx + h^2 = 0$

$x = \frac{5h \pm \sqrt{25h^2 - 12h^2}}{6} = \frac{h(5 \pm \sqrt{13})}{6}$

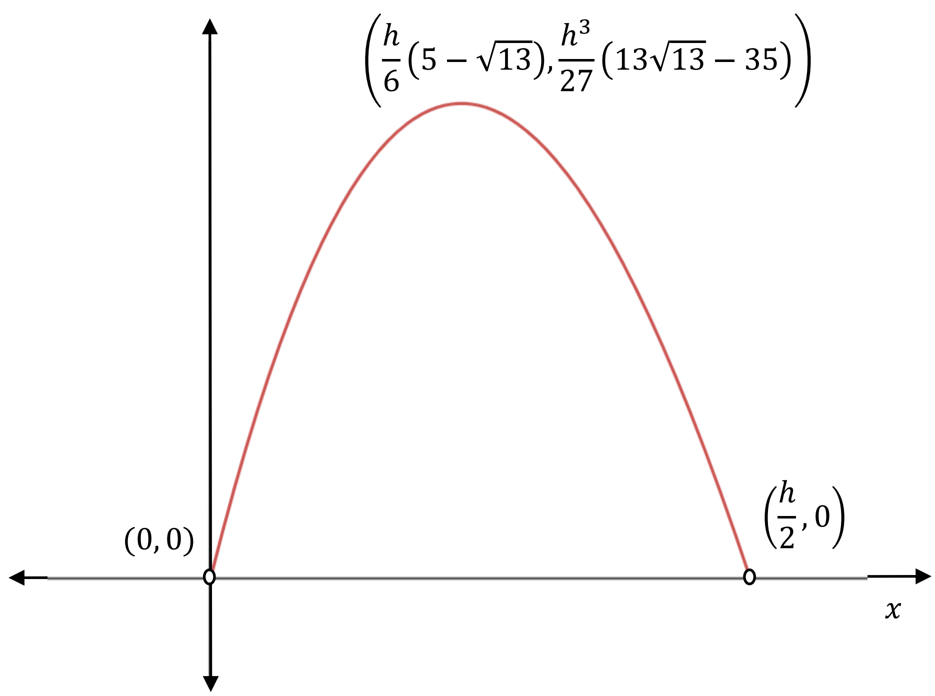
Hence $x = \frac{h}{6}(5 - \sqrt{13})$

$V\left(\frac{h}{6}(5 - \sqrt{13})\right) = \frac{h^3}{27}(13\sqrt{13} - 35)cm^3$

With dimensions $\frac{h}{3}(\sqrt{13} - 2)\ cm, \frac{h}{3}(\sqrt{13} + 7)\ cm\ and\ \frac{h}{6}(5 - \sqrt{13})\ cm$

3 marks

e.



2 marks

Question 3 (16 marks)

a. $a = \frac{1}{3},\ b = 5$

2 marks

b. $g'(x) = \frac{10}{3}x - x^2$

1 mark

c. $g'(x) = 0\ \ at\ x = 0\ and\ x = \frac{10}{3}$

So $x \in (-\infty, 0] \cup \left[\frac{10}{3}, \infty\right)$

2 marks

d. $Area = \frac{5-0}{2\times 5}\big(g(0) + 2g(1) + 2g(2) + 2g(3) + 2g(4) + g(5)\big)$
$= \frac{50}{3}$

2 marks

e. $\int_0^5 g(x)\,dx = \frac{625}{36}$

2 marks

f. Maximum occurs at $\left(\frac{10}{3}, \frac{500}{81}\right)$

$4m(n-4) = \frac{50}{9}$ and $\frac{10}{3}m\left(n - \frac{10}{3}\right) = \frac{500}{81}$
$m = \frac{25}{36}, n = 6$

2 marks

g. $\int_0^{\frac{10}{3}} \frac{1}{3}x^2(5-x)\,dx + \int_{\frac{10}{3}}^{6} \frac{25}{36}x(6-x)dx$
$= \frac{15100}{729} \approx 20.7$

2 marks

h. $h'(x) = \frac{h(6) - h\left(\frac{10}{3}\right)}{6 - \frac{10}{3}}$

$x = \frac{14}{3}, \ \ h\left(\frac{14}{3}\right) = \frac{350}{81}$

$\left(\frac{14}{3}, \frac{350}{81}\right)$

3 marks

Question 4 (21 marks)

a. $\Pr(X < 2.9) = 0.015 \quad \Pr(X > 5) = 0.091$

$\Pr\left(Z < \frac{2.9 - \mu}{\sigma}\right) = 0.015 \quad \Pr\left(Z > \frac{5 - \mu}{\sigma}\right) = 0.091$

$\frac{2.9 - \mu}{\sigma} = -2.1701 \quad \frac{5 - \mu}{\sigma} = 1.3346$

$\mu = \$4.20$ and $\sigma = \$0.60$

3 marks

b. $\Pr(X > 3.5) = 0.8783$

1 mark

c. $\Pr(X > 3.5 | X < 4.2) = \frac{\Pr(3.5 < X < 4.2)}{0.5}$

$= 0.7567$

2 marks

d. Assuming n_1 and n_2 are symmetric about μ:

$\Pr(X < n_1) = 0.3$ and $\Pr(X > n_2) = 0.3$

$n_1 = 3.89$ and $n_2 = 4.51$

2 marks

e. $\Pr(X > 6|X > 5.5) = \frac{\Pr(X>6)}{\Pr(X>5.5)}$

$\frac{0.0013}{0.0151} = 0.0892 = 9\%$

2 marks

f. $\Pr(Y > 5.5) = 0.0151$

$E(Y) = np = 2800 \times 0.0151 = 42.36$

1 mark

g. Can be done by trial and error.
$p = 0.01513$

binomialCDf(100, 8231, 8231, 0.01513)
0.9900206068
binomialCDf(100, 8230, 8230, 0.01513)
0.9899828285

So minimum number of cards is 8231.

2 marks

h. $(0.3965, 0.4835)$
There is a 95% confidence that the population proportion of coins that are gold lies between 0.3965 and 0.4835.

2 marks

i. For 95% $k = 1.95996$, for 99% $k = 2.5758$

$\frac{2.5758}{1.95996} = 1.31$

The interval would increase by a factor of 1.31

2 marks

j. $\int_0^\infty kv^3e^{-v}dv = 1$

$k = \dfrac{1}{6}$

2 marks

k. $\int_0^\infty kv^4e^{-v}dv$
$= 4$

2 marks