2024 NHT

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Write your **student number** in the boxes above. **Letter**

Mathematical Methods
Examination 2

Question and Answer Book

VCE (NHT) Examination – Wednesday 29 May 2024

- Reading time is **15 minutes**: 10.30 am to 10.45 am
- Writing time is **2 hours**: 10.45 am to 12.45 pm

Approved materials

- Protractors, set squares, aids for curve sketching
- One bound reference
- One approved CAS calculator or CAS software and one scientific calculator

Materials supplied

- Question and Answer Book of 28 pages
- Formula Sheet
- Multiple-Choice Answer Sheet

Instructions

- Follow the instructions on your Multiple-Choice Answer Sheet.
- At the end of the examination, place your Multiple-Choice Answer Sheet inside the front cover of this book.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents pages

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Section A – Multiple-choice questions

Instructions for Section A

- Answer **all** questions in pencil on the Multiple-Choice Answer Sheet.
- Choose the **correct** response.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The period and range for the function $f(x)=\sqrt{3}\sin\left(2x-\frac{\pi}{2}\right)-\sqrt{3}$ are respectively

A. π and $\left[-\sqrt{3}, \sqrt{3}\right]$

B. 2π and $\left[-\sqrt{3}, \sqrt{3}\right]$

C. π and $\left[-2\sqrt{3}, 0\right]$

D. 2π and $\left[-2\sqrt{3}, 0\right]$

E. 4π and $\left[-2\sqrt{3}, 0\right]$

Question 2

For two independent events, A and B, it is known that $\Pr(A)=0.6$ and $\Pr(A\cup B)=0.92$.

$\Pr(B)$ is equal to

A. $\frac{4}{5}$

B. $\frac{4}{15}$

C. $\frac{8}{15}$

D. $\frac{8}{25}$

E. $\frac{1}{5}$

Do not write in this area.

Question 3

The positive real numbers a, b and c satisfy the equation

$$\log_5 a + \log_5 b = 2\log_5 c$$

Therefore, it must be true that

A. $a = 2c - b$

B. $a = c^2 - b$

C. $a = \frac{2c}{b}$

D. $a = \frac{c^2}{b}$

E. $a = \frac{b}{2c}$

Question 4

A straight line passes through the positive x-intercept of the curve of the cubic $y = x^3 - x^2 - 2x$ and also through its point of inflection.

The gradient of this line is

A. $\frac{4}{9}$

B. $\frac{2}{3}$

C. $\frac{1}{2}$

D. $-\frac{15}{7}$

E. $-\frac{20}{9}$

Question 5

When a biased coin is tossed, the probability of the coin landing on heads is 0.25. The coin is tossed 10 times and the number of heads is recorded.

The probability that exactly x heads are recorded is

A. $\binom{10}{x}(0.25)^{10-x}(0.75)^{x}$

B. $\binom{10}{x}(0.25)^{x}(0.75)^{10-x}$

C. $\binom{10}{x}(0.25)^{10-x}(0.75)^{10}$

D. $\binom{10}{x}(0.25)^{x}(0.75)^{10}$

E. $\binom{10}{x}(0.25)^{10}(0.75)^{10-x}$

Question 6

The graph of $y = \tan(ax)$ has an asymptote with equation $x = \frac{\pi}{6}$.

A possible value of a is

A. 4

B. 6

C. 9

D. 10

E. 12

Question 7

The function $f: R \rightarrow R, f(x) = a\cos\left((b-1)x\right) + b$ has a period of $\frac{\pi}{3}$ and a maximum value of 10.

Given that $b > 0$, the minimum value of the function f is

A. 6

B. 4

C. 1

D. -4

E. -10

Do not write in this area.

NHT

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Question 8

When entering a zoo, visitors are asked how many times they have previously visited.

The discrete random variable, N, with the probability distribution shown below, can be used to model the number of times each visitor has previously visited.

n	0	1	2	3	4
$\Pr(N=n)$	0.25	0.3	0.25	0.15	0.05

Given that a visitor has previously visited the zoo, what is the probability that they have previously visited more than once?

A. 0.73

B. 0.6

C. 0.55

D. 0.45

E. 0.5

Question 9

Let $f(x) = 3x^2 - 2ax + 6$, where $a \in R$.

The set of values of a such that $f(x) > 0$ for all $x \in R$ is

A. $\left\{a : a \geq 3\sqrt{2}\right\}$

B. $\left\{a : a < -3\sqrt{2}\right\} \cup \left\{a : a > 3\sqrt{2}\right\}$

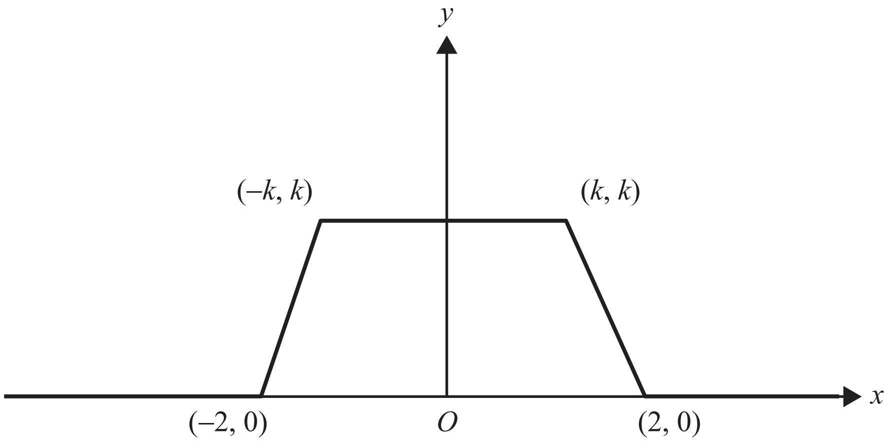
C. $\left\{a : a \leq -3\sqrt{2}\right\} \cup \left\{a : a \geq 3\sqrt{2}\right\}$

D. $\left\{a : -3\sqrt{2} < a < 3\sqrt{2}\right\}$

E. $\left\{a : -3\sqrt{2} \leq a \leq 3\sqrt{2}\right\}$

Question 10

The graph for $y = f(x)$ is shown below.



Find the value of k given that $f(x)$ is a probability density function.

A. $\sqrt{2} - 1$

B. $-\sqrt{2} - 1$

C. 1

D. $\sqrt{6} - 2$

E. $\dfrac{\sqrt{6}}{2} - 1$

Question 11

In a certain location, the number of hours, h, of daylight per day is modelled by

$$h(t) = 12 - 3\cos\left(\frac{2\pi}{365}(t - 175)\right), \text{ where } t \in Z^+$$

where t represents the day of the year. Note that $t = 1$ represents 1 January.

According to this model, during one year of 365 days, the number of days for which there are at least 11 hours of daylight is

A. 142

B. 143

C. 144

D. 221

E. 222

Question 12

Newton's method is used to estimate the x-intercept of the function

$$f:[0,\infty)\to R,\ f(x)=\log_e(2x+1)-\left(4-x^{\frac{5}{2}}\right)$$

With an initial estimate of $x_0 = 0$, the estimate for x_3 is closest to

A. 1.4717

B. 1.4718

C. 1.4752

D. 1.5628

E. 2.0000

Question 13

Suppose that we have two continuous functions, f and g, with domain R.

It is known that $g(x) < 0$ for all x and $f'(x) = g(x)$.

Given that $f(-1) = 2$ and $f(2) = -1$, what is the area enclosed by the graph of $y = g(x)$, the x-axis and the lines $x = -1$ and $x = 2$?

A. 0 square units

B. 1 square unit

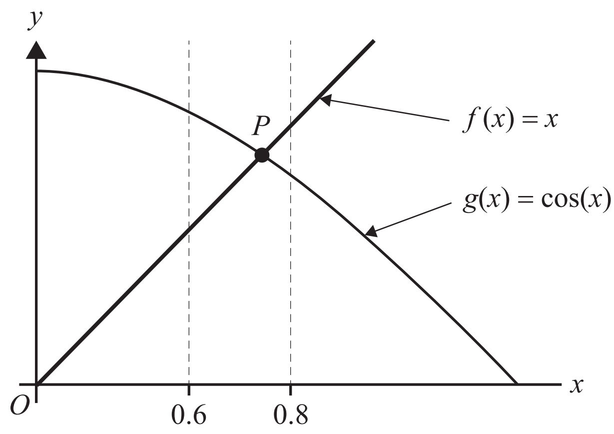
C. 2 square units

D. 3 square units

E. 4 square units

Question 14

Parts of the graphs of two functions, $f(x) = x$ and $g(x) = \cos(x)$, are shown below, where x is measured in radians.



The pseudocode below describes a simple algorithm to find the point where the two functions intersect.

```
define intersect(x0, x1):
    x ← x0
    while x < x1
        if -0.01 < cos(x) - x < 0.01 then
            return x
        end if
        x ← x + 0.01
    end while
    return "No intersection found"
```

Which of the following will the function `intersect(0.6, 0.8)` return?

A. `-0.002`

B. `0.02`

C. `0.73`

D. `0.74`

E. `No intersection found`

Question 15

The finance team at a small technology company estimates that the production cost per item is given by the rule $C(n) = 0.5n^2 - 37.5n + 1900 + \frac{1250}{n}$, where $n \in Z^+$ and n is the number of items produced.

The minimum cost per item is closest to

A. \$38.34

B. \$38.35

C. \$1229.83

D. \$1229.89

E. \$1230.05

Question 16

The values for two continuous functions, f and g, and their derivatives are given in the tables below.

	$x=0$	$x=2$
$f(x)$	2	-1
$f'(x)$	-1	2

	$x=0$	$x=2$
$g(x)$	1	-1
$g'(x)$	0	3

What is the value of $\frac{d}{dx}\big((g \circ f)(x)\big)$ at $x = 0$?

A. -3

B. -1

C. 0

D. 1

E. 2

Question 17

Suppose that for three events, A, B and C, $\Pr(A \mid B) = 1$ and $\Pr(C \mid B) = \Pr(C)$.

Which of the following statements about A, B and C **must** be true?

A. $\Pr(A) = 1$

B. $C \subseteq B$

C. $\Pr(A) \neq 1$

D. B and C are independent events and $B \subseteq A$

E. B and C are not independent events and $B \subseteq A$

Question 18

Let $f:[0, 2\pi] \to R,\ f(x)=\cos\left(ax+\frac{\pi}{6}\right)$, where $a>0$. There is a unique solution to $f(x)=1$ and a unique solution to $f(x)=-1$.

The minimum possible value of a is

A. $\frac{1}{12}$

B. $\frac{5}{12}$

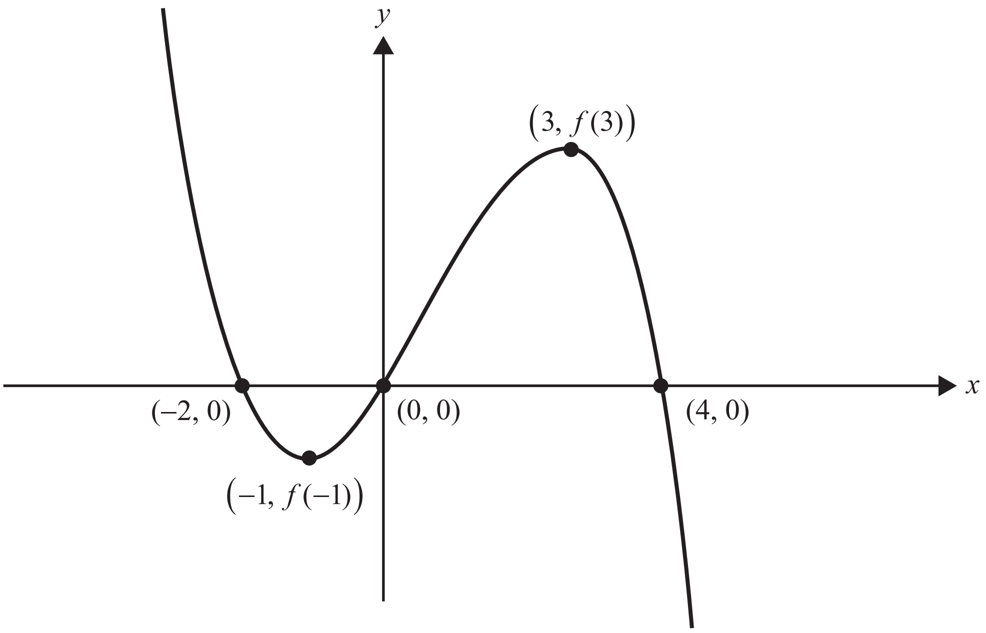
C. $\frac{5}{6}$

D. $\frac{11}{12}$

E. $\frac{17}{12}$

Question 19

Let $f: R \to R$ be a continuous and differentiable function. Part of the graph of f is shown below.
The stationary points of f are at $(-1, f(-1))$ and $(3, f(3))$.



The solution to the inequality $(x^2-x-2)f'(x)>0$ is

A. $-1<x<2$

B. $-1<x<3$

C. $2<x<3$

D. $x<-1$ or $x>2$

E. $x<-1$ or $x>3$

Question 20

Let $f : R \to R, f(x) = x^2(1 - x^2)$ and $g : [0, 1] \to R, g(x) = f(x - a)$.

Find all real values of a, such that $g(0)$ is the absolute maximum value of g.

A. $a = \pm\dfrac{\sqrt{2}}{2}$

B. $a \in \left[-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right]$

C. $a \in \left(-\infty, -\dfrac{\sqrt{2}}{2}\right] \cup \left[\dfrac{\sqrt{2}}{2}, \infty\right)$

D. $a \in \left[-\dfrac{\sqrt{2}}{2}, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \dfrac{\sqrt{2}}{2}\right]$

E. $a \in \left(-\infty, -\dfrac{\sqrt{2}}{2}\right] \cup \left[\dfrac{1}{2}, \dfrac{\sqrt{2}}{2}\right]$

End of Section A

Section B

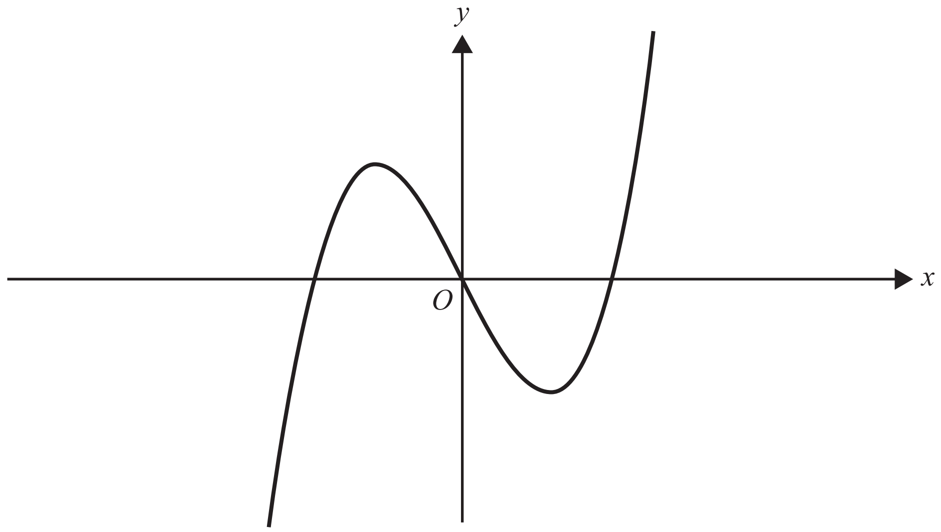
Instructions for Section B

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- In all questions where a numerical answer is required, an **exact** value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

Consider the function $f: R \to R, f(x) = x^3 - px$, where $p \in R$.

Part of the graph of f is shown below, when $p = 3$.



a. Find the values of the x-intercepts of f, when $p = 3$. 1 mark

b. Use the derivative f' to find the coordinates of the turning points of f, when $p = 3$. 2 marks

c. **i.** Find the value of p for which f would have exactly one stationary point. 1 mark

ii. Find the values of p for which f would not have any stationary points. 1 mark

d. The graph of f passes through the origin for all values of p.

i. Use calculus to show that the tangent line to f at the origin has the equation $y = -px$. 2 marks

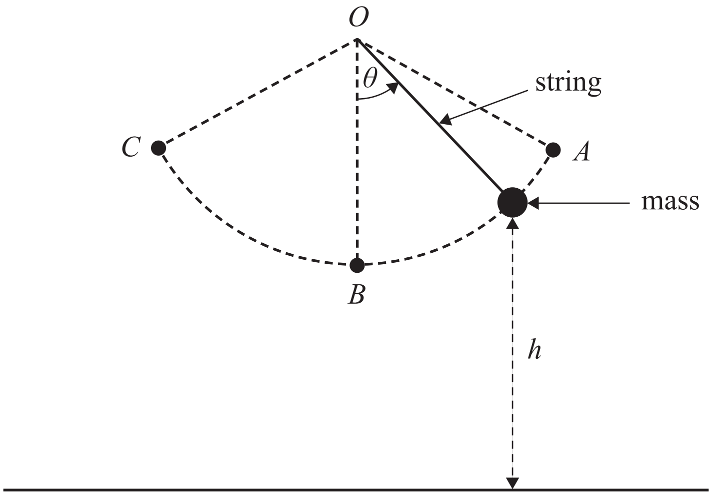
ii. Find, in terms of p, the area of the region bounded by the function f, the line $y = -px$ and the line $x = p$, where $p > 0$. 2 marks

iii. The expression for the area found in **part d.ii** also gives the area bounded by a cubic function $y = kx^3$, the x-axis and the line $x = p$, where $p > 0$.

Find all possible values of k. 1 mark

Question 2 (13 marks)

A simple pendulum consists of a mass at the end of a string of fixed length 0.5 m. The other end of the string is attached to a fixed point, O.



The mass swings back and forth in a circular arc between points A and C, each time passing through point B.

The height, h metres, of the mass above the horizontal ground is given by the function

$$h: \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \to R,\ h(\theta) = \frac{3}{2} - \frac{1}{2}\cos(\theta)$$

where θ is the angle in radians measured anticlockwise from the vertical line OB.

The mass reaches its maximum height at point A, when $\theta = \frac{\pi}{3}$, and at point C, when $\theta = -\frac{\pi}{3}$. It reaches a minimum height at point B, when $\theta = 0$.

a. Find the maximum and minimum heights of the mass, in metres, above the ground. 2 marks

b. Find, correct to two decimal places, the values of θ for which the mass has a height, h, less than 1.1 m. 2 marks

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c. The angle, θ radians, as a function of time, t seconds, is given by the function

$$\theta : [0, 30] \to R, \theta(t) = \frac{\pi}{3}\cos\left(\frac{9t}{2}\right)$$

Show that the mass is initially at point A. 1 mark

d. Determine the period of the pendulum. The period is the time in seconds taken for the mass to travel from A to C and back to A. 1 mark

e. How many times for $t \in [0, 30]$ is the mass at point A? 1 mark

f. Determine the distance, in metres, that the mass travels as it swings from point A to point C. 1 mark

Question 2 continues on the next page.

g. **i.** Find, correct to two decimal places, the height of the mass, in metres, after one second has elapsed. 2 marks

ii. State the rule for the composite function $h\big(\theta(t)\big)$. 1 mark

iii. Find, correct to two decimal places, the **average** height of the mass, in metres. 2 marks

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Question 3 (12 marks)

A small business has four people working full-time. The business rents an office, and on most days some people work from home.

On any given workday, the number of people working in the office can be modelled by the discrete random variable, N, with the following probability distribution.

n	1	2	3	4
$\Pr(N = n)$	0.3	0.4	0.2	0.1

a. Find $\mathrm{E}(N)$, the expected value for the number of people working in the office. 1 mark

b. Find the probability that all four people are working in the office, given that at least two people are working in the office. 1 mark

c. In a fortnight with ten workdays, find the probability that all four people are working in the office on at least three days.

Write your answer correct to four decimal places. 2 marks

N
H
T

d. The business owner is trying to find ways to reduce costs.

Her daily electricity costs, in dollars, can be modelled using a continuous random variable, C, which has the probability density function

$$f(c)=\begin{cases}\dfrac{3c(c-10)^2}{2500} & 0\le c\le 10\\ 0 & \text{elsewhere}\end{cases}$$

i. Identify the maximum daily electricity cost, in dollars, for this small business. 1 mark

ii. Find the mean daily electricity cost, in dollars. 1 mark

iii. Find the probability that, on a given day, the electricity cost is within two standard deviations of the mean. 2 marks

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Question 3 continues on the next page.

The business owner plans to reduce electricity costs by installing LED sensors.

She researches an LED sensor manufacturer and finds the following information.

e. In a random sample of 200 LED sensors, 30 LED sensors were found to be defective.

i. State the sample proportion of LED sensors that were found to be defective. 1 mark

ii. Use this sample proportion to find a 90% confidence interval for the population proportion of defective LED sensors, correct to three decimal places. 1 mark

f. A new sample of size n is taken, which is used to calculate a 95% confidence interval for the population proportion of defective LED sensors. This confidence interval, correct to three decimal places, is $(0.062, 0.213)$.

Using the value $z = 1.96$, determine the sample size n. 2 marks

Question 4 (13 marks)

Timothy has created a tiling pattern using transformations of the cubic function

$$f : [-4, 0] \to R, f(x) = a(x - b)^3 + c$$

where $a, b, c \in R$

On the graph below, the function f is shown in **bold**. The rest of Timothy's tiling pattern is shown with dashed lines, which are all images of transformations of f.

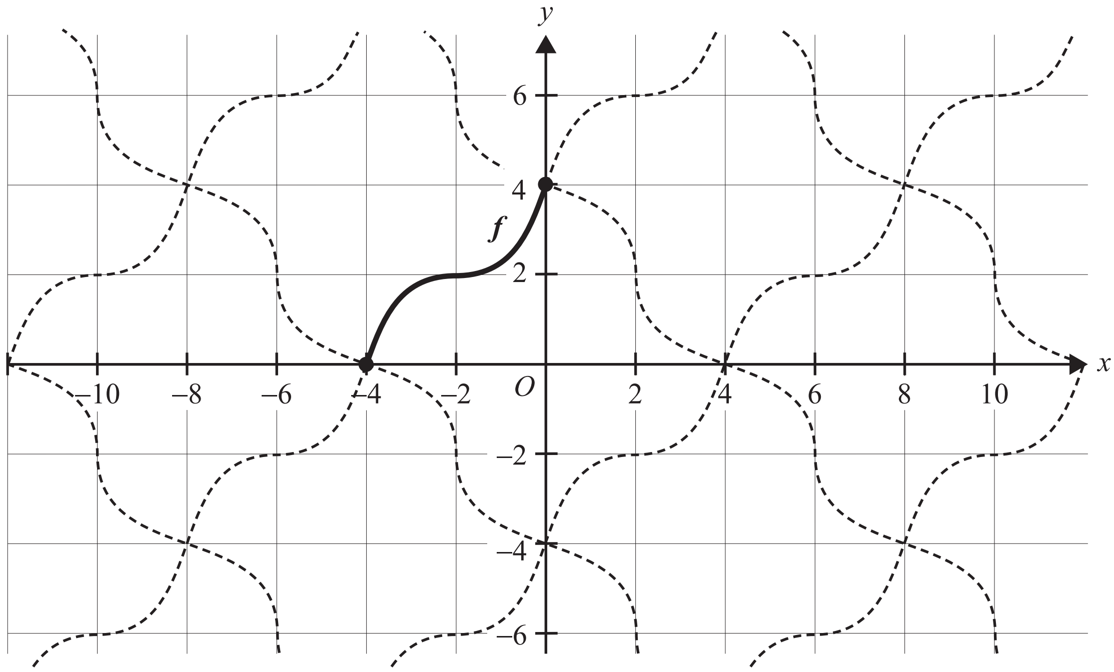

Figure 1

a. The translation of f given by $f_2(x) = f(x - 4) - 4$ describes a different edge of one of the tiles.

Trace over the edge described by f_2 on the graph in Figure 1, and label this edge f_2. 1 mark

b. Given that f has a stationary point of inflection at $(-2, 2)$ and an axial intercept at $(0, 4)$, find the values of a, b and c. 2 marks

Question 4 continues on the next page.

c. The piecewise function, p, is continuous, where the first component is f, the other component is a **translation** of f and both are edges in Timothy's tiling pattern.

Complete a possible rule and domain for the second piece of the piecewise function p. 1 mark

$$p(x) = \begin{cases} f(x) & -4 \le x \le 0 \\ \_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_ \end{cases}$$

d. Find the rule and domain for f^{-1}, the inverse function of f. 2 marks

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e. Let $m, n \in R$ and $g(x) = mf^{-1}(nx)$ be a function describing a transformation of the inverse f^{-1}.

Find the values of m and n, given that $g(x)$ describes the tile edge joining $(-4, 0)$ and $(0, -4)$, shown in bold in Figure 2. 1 mark

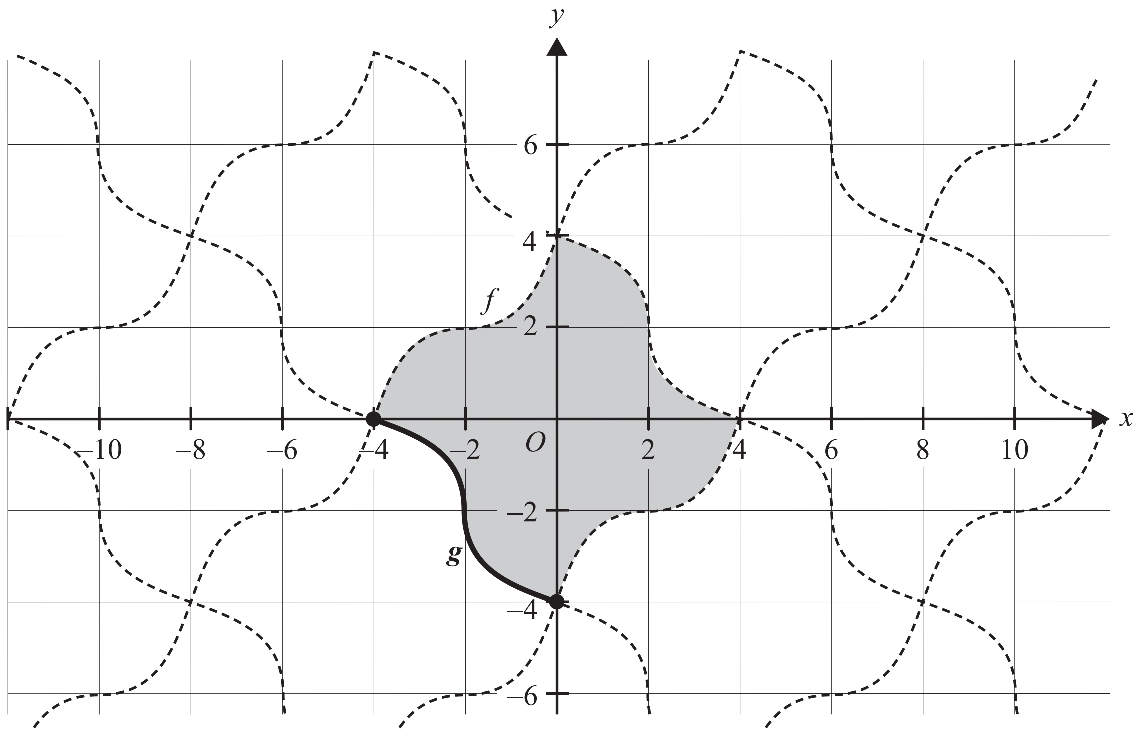

Figure 2

f. Find the area of one of the tiles. An example is shaded in Figure 2. 2 marks

g. Using the derivatives of f and g at $x = -4$, show that the angle at the left corner of the shaded tile, at the point $(-4, 0)$, is $90°$. 2 marks

Question 4 continues on the next page.

h. The family of functions defined by

$$y = g(x - 8k), \qquad k \in Z$$

gives an infinite set of edges in Timothy's tiling pattern, three of which are shown in bold on Figure 3.

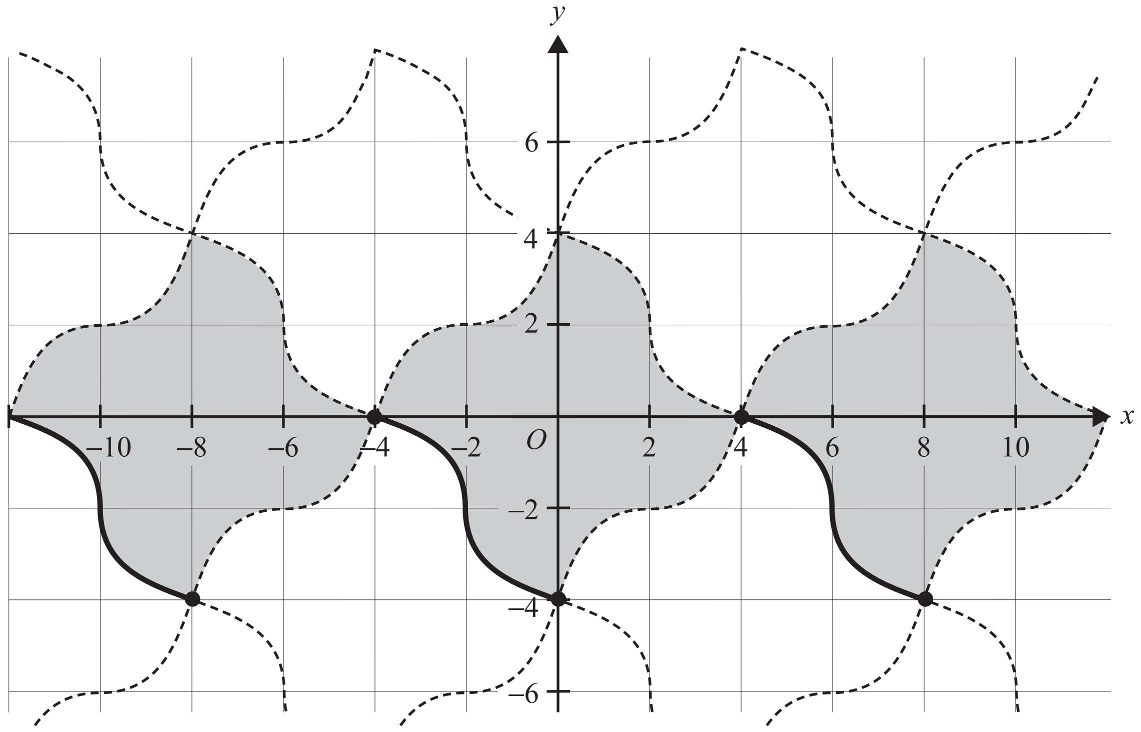

Figure 3

Three more families of functions can be used to bound an infinite set of tiles centred on the x-axis, three of which are shaded on Figure 3.

Complete the table to define three such families of functions, in terms of the given functions f or g, and a parameter $k \in Z$. 2 marks

1.	$y = g(x - 8k)$
2.	
3.	
4.	

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Question 5 (12 marks)

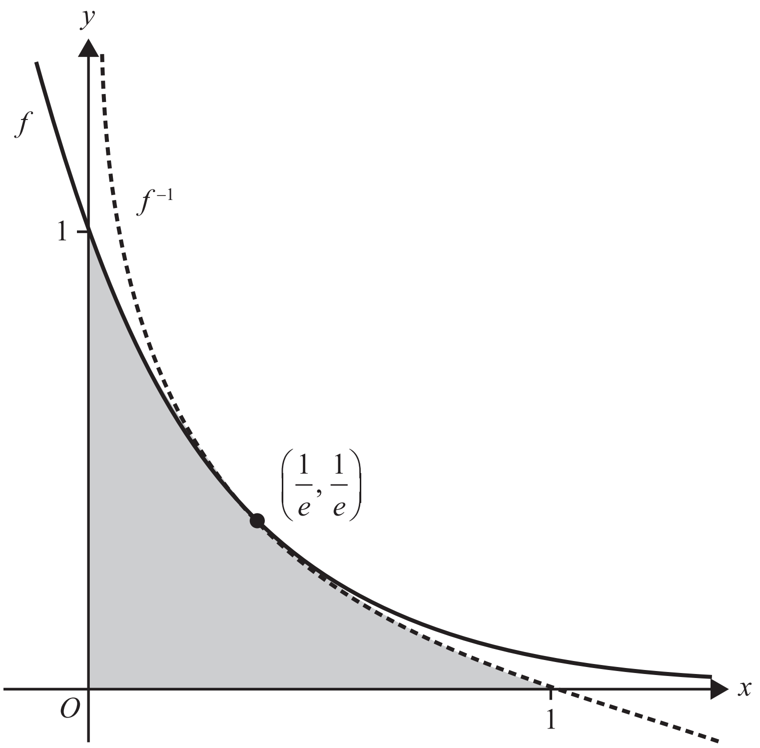
Let $f: R \to R, f(x) = e^{-ex}$.

a. Show that the rule and domain for f^{-1}, the inverse function of f, is

$$f^{-1}(x) = -\frac{1}{e}\log_e(x)$$

where $x \in (0, \infty)$ 2 marks

b. The graphs of f and f^{-1} are shown below. Their unique intersection point $\left(\frac{1}{e}, \frac{1}{e}\right)$ is also labelled.



Find the exact area of the shaded region that is bounded by the graphs of f, f^{-1} and the coordinate axes. 2 marks

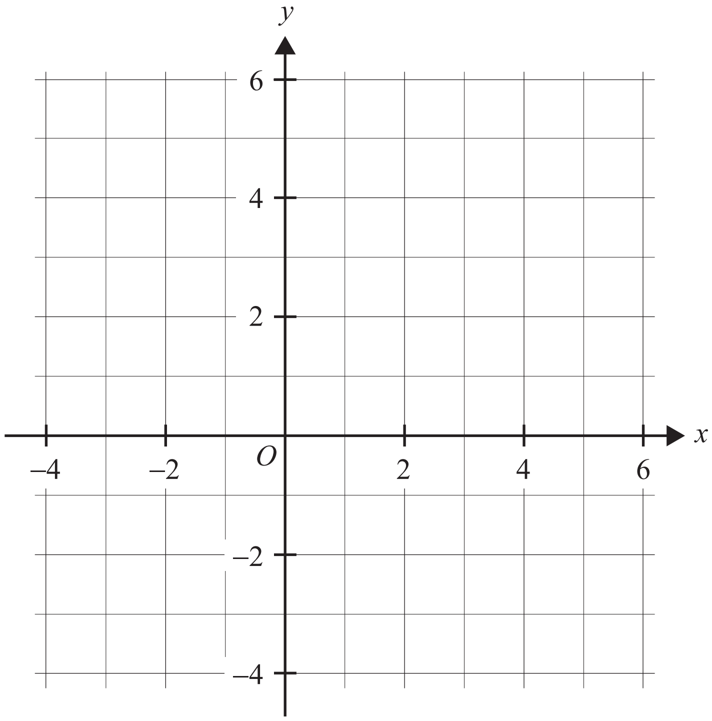
Question 5 continues on the next page.

Let $g: R \to R, g(x) = e^{ax}$, where $a \in R\backslash\{0\}$.

c. Consider the function $g_1(x) = e^{ax}$, where $a = \frac{1}{e}$.

i. Find the equation of the tangent line to the graph of $y = g_1(x)$ at $x = e$. 1 mark

ii. Sketch the graph of g_1 and its inverse on the axes provided below. Label equations of asymptotes and the coordinates of the point of intersection between the two functions. 3 marks



d. Now let $a = -4$ and consider $g_2(x) = e^{-4x}$.

i. Find the rule for ${g_2}^{-1}$, the inverse function of g_2. 1 mark

ii. Find the values of x where g_2 intersects with its inverse, correct to three decimal places. 1 mark

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e. The value of a determines the number of intersection points of the graphs of g and g^{-1}.

Complete the following table by stating the value(s) of a in each case. 2 marks

Number of points of intersection	**Value(s) of *a***
0	
1	
2	$0 < a < \frac{1}{e}$
3	

End of Question and Answer Book

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Mathematical Methods
Examination 2
Formula Sheet

You may keep this Formula Sheet.

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Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$	

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$
trapezium rule approximation	$\text{Area} \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$
binomial coefficient	$\dbinom{n}{x} = \dfrac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
binomial	$\Pr(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$	$\sigma^2 = np(1-p)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

End of Formula Sheet