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WITH CENTRES ALL OVER AUSTRALIA

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 SOLUTIONS

WEEKS 19 - 27

NOTES TO YEAR 12 MATHS METHODS TUTORS:

This document is for staff use only. No student is to view the contents within. The purpose of this solutions book is for tutors to FACILITATE the learning of students. It is not to be regarded as a means to "Spoon-feed" answers. You may, provide solutions after students have attempted the questions, as a means of giving feedback to their responses.

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NQT EDUCATION TUITION WORKBOOK

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HOW TO USE THIS BOOKLET

WELCOME TO VCE STUDIES AT NQT EDUCATION. Let us tell you a little about our classes and what you can do to maximise your learning with us.

NQT Education currently offers classes in the following VCE subjects:
- VCE English for years 11 & 12
- VCE Mathematical Methods for years 11 & 12

NQT Education's VCE curriculum follows closely in line with the Victorian Curriculum and Assessment Authority (VCAA's) Study Designs so that what you are learning topics in line with what you are studying at school. However, given that each school is different and it is likely you may be covering Areas of Study different to that of your peers, the material covered in NQT classes may be pre-taught or revisional in nature.

The work is divided into weeks and each cover sheet outlines clearly the Area of Study you will be undertaking as well as the key Outcomes for the different Areas of Study. It is important that you stick to the allocated weeks in this book and you are encouraged to complete all activities for homework if unable to complete all tasks in lesson.

VCE English at NQT Education

It is highly likely that your classmates are studying different text(s) from you. It is also likely your tutor may not be necessarily familiar with the texts you are studying. HOWEVER, the focus of VCE English classes at NQT is about gaining essential skills that will help you prepare for your SACs, assessment tasks and / or exam(s).

At NQT Education, we understand that in order to achieve your very best at VCE English, you will need to develop and hone your writing and analytical skills and with the help of our worksheets and your tutor's expertise, you should be able to achieve your very best. Ensure that you bring in any relevant work, texts, notes, assessment tasks, draft SACs, sample exams, etc. to supplement your studies. You are also strongly encouraged to bring in any drafts or writing tasks for your tutor to look over as they will also be able to provide invaluable advice and feedback.

VCE Mathematical Methods at NQT

It is essential that you bring in your CAS calculator each week as well as your notebook as there will be substantial workings out that will need to be completed in addition to the work within this book. Each week, there is clearly explained theory, definitions of key terms as well as worked examples. This is then followed up by series of activities that progress in difficulty to allow you ample practice in new topics and concepts. Again, your tutor is there to help should y0u also require assistance with your own VCE Mathematical Methods coursework.

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 19

Topic: Integration

Introduction to integration
Polynomial Function
Integrating function forms
Specific anti-differentiation

What you need to know about Mathematical Methods: Unit 3 & 4

Students are expected to apply techniques, routines and processes involving rational and real arithmetic, algebraic manipulation, equation solving, graph sketching, differentiation and integration.

It is assumed students taking the program are familiar with determining the equation of a straight line, basic factorisation, Pythagoras theorem, indentifying and manipulation of quadratic and exponential functions and sketching graphs of basic functions. Basic concepts of probability are also assumed.

There are four study areas you need to satisfactorily complete in order to accomplish Unit 3 & 4:

AREA OF STUDY 1

Polynomial and power functions

- Define key features of functions and ability to manipulate them
- Power, exponential, logarithmic, circular and modulus functions
- Transformations of functions
- Graphing polynomial, sum, difference, product, composite and inverse functions
- Applications of hybrid functions

AREA OF STUDY 2

Algebra

- Algebra of various functions
- Logarithm and exponent laws
- Simultaneous equations
- General solutions, finding approximate or exact solutions within a restricted domain

AREA OF STUDY 3

Calculus

- Determining the original function from its derivative and anti-derivative
- Derivative properties of single and combined functions
- Applications of differentiation to curve sketching
- Limiting values
- Properties of anti-derivatives and definite integrals

AREA OF STUDY 4

Probability

- Discreet and continuous random variables
- Central measures and standard deviations
- Construction of probability density functions
- Binomial and normal distributions
- Conditional probability

For all NQT lessons, bring your own Graphics Calculator to each and every class, as well as a notebook/exercise book to be used throughout the year. Also you should bring your Mathematical Methods textbook as an additional resource for your learning.

Introduction to Integration

Integration, also known as anti-differentiation is the reverse process to differentiation. Generally we use to find the equation $f(x)$ when the $f'(x)$ is known. This can be represented such that the equation known is $f(x)$ and the anti-differentiation of the equation is $F(x)$, usually denoted by a capital letter.

Polynomial Function

Let's look at an example

Consider the function $4x^3 + 1, 4x^3 + 2, \dots \dots, 4x^3 + n$.

$f(x)$	$f'(x)$
$4x^3 + 1$	$12x^2$
$4x^3 + 2$	$12x^2$
$4x^3 + 5$	$12x^2$

You will realise that the constant of the function is dropped when differentiating. In a case where the process of anti-differentiation is introduced, we must consider a constant since the gradient function does not provide information about the location of the curve.

So $\int 12x^2\, dx = 4x^3 + c$

This **indefinite integral** expression says the integration of $12x^2$ in terms of x denoted by dx is $4x^3 + c$, where c is a constant.

Thus the general rule to find the anti-derivative of the polynomial expression of ax^n is

$\int ax^n\, dx = a \int x^n = \frac{ax^{n+1}}{n+1}$+c , given that a and c are constants and $n \geq 0$

Some important facts:

In a case where integrating a constant

$$\boldsymbol{\int a\ dx = ax + c}$$

A fraction can be written but not necessary as

$$\boldsymbol{\int \frac{1}{a}\ dx = \int \frac{dx}{a}}$$

Anti-derivatives are linear such that the integrations of functions can be separated

$$\boldsymbol{\int \big(f(x) \pm g(x)\big)\ dx = \int f(x)\ dx\ \pm\ \int g(x)\ dx}$$

Note that the integrated function can be express as a capital letter, in this case F:

$$\boldsymbol{F'(x) = f(x)\ such\ that\ \int f(x) = F(x) + c}$$

Work Example

$$Find\ \int (4x^5 + \frac{x}{3} + \sqrt{x} + 1)\ dx$$

Solution

Step 1. Since anti-derivatives are linear, the terms can be expressed individually.

$$\rightarrow\ \int 4x^5\ dx + \int \frac{x}{3} dx + \int \sqrt{x}\ \ dx + \int 1\ dx = 4\int x^5\ dx + \frac{1}{3}\ \int x\ dx + \int x^{\frac{1}{2}}\ dx + \int 1\ dx$$

Step 2. Use the polynomial expression integration rule.

$$\rightarrow\ \boldsymbol{\left(\frac{4x^6}{6}\right) + \left(\frac{\frac{1}{3}x^2}{2}\right) + \frac{\frac{x^3}{2}}{\frac{3}{2}} + x + c = \left(\frac{2x^6}{3}\right) + \left(\frac{x^2}{6}\right) + \frac{x^3}{3} + x + c}$$

You may wonder that each integration leaves a constant, however one general 'c' can represent the summation of all constants.

Check by differentiating the answer and you should end up with the same expression!

Integrating function forms

When integrating the function, it is easier to expand brackets then solve using the polynomial expression for integration.

For example

$$\int \frac{(2x^2+1)(x^2-1)}{x^2}\,dx$$

$$\int \frac{2x^4-x^2-1}{x^2}\,dx = \int 2x^2-1-x^{-2}\,dx = \frac{2x^3}{3} - x + x^{-1} + c$$

There are integrating function forms that you will need to remember for simplicity.

To integrate a linear function $ax+b$ simply raise the power of n, increase the power by one and divide the new power and the coefficient of x

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, where\ n \neq -1$$

For example

$\int (3x-4)^5\,dx = \frac{(3x-4)^6}{3(6)} + c = \frac{1}{18}(3x-4)^6 + c$

Thus the solution is

$\frac{1}{18}(3x-4)^6 + c$, where c is the constant.

Specific Anti-Differentiation

When we integrate a function we have a constant value c. Additional information is needed to solve for the constant and therefore we can describe the specific function.

Example

Find the equation of the curve y that passes through the point $(2,2)$ given that $\frac{dy}{dx} = 2x^3 + \frac{2}{x^2}$

Solution

In general $y = \int \frac{dy}{dx} dx$

$$y = \int (2x^3 + \frac{2}{x^2})\, dx$$

$$= \int 2x^3 dx + \int \frac{2}{x^2} dx = \frac{x^4}{2} - \frac{2}{x} + c$$

Given $y = \frac{x^4}{2} - \frac{2}{x} + c$ substitute $(2,2)$ into the equation

$$2 = \frac{16}{2} - \frac{2}{2} + c = 8 - 1 + c \qquad \therefore c = \frac{2}{7}$$

The specific equation is $y = \frac{x^4}{2} - \frac{2}{x} + \frac{2}{7}$

It is likely you will encounter worded problems that require you to integrate a function with hidden information in the text. Such as

- The particle is at rest or stationary implies velocity is zero.
- Constant acceleration or velocity means the derivative of acceleration or velocity is zero respectively.

Decrypting texted information will become more apparent through practice!

Testing Understanding

1. Solve

$a. \int x^3\, dx$

x^4/4+c

$b. \int \frac{x}{3}\, dx$

1/3+c

$c. \int x^2 + \frac{1}{x^3} - 5x\, dx$

x^3/3 -2/x^2 -5x^2/2+c

$d. \int 4\, dx$

4x+c

$e. \int (x^3)^2\, dx$

x^7/7+c

$f. \int (2cx)^3\, dx$

2c^3x^4+c

$g. \int (x-1)^2\, dx$

(x-1)^3/(3) +c

$h. \int 4x^{3/2}\, dx$

(8/5) x^(5/2) +c

$i \int \frac{dx}{x^6}$

-5/x^5+c

$j. \int 4\sqrt{x}\, dx$

2/x^(-1/2) +c

$k. \int \frac{1}{x^2} - \frac{1}{x^5} dx$

-1/x+4/x^4+c

$l. \int x^m x^{m-2}\, dx$

x^(2m-1)/(2m-1) +c

$m. \int (x-1)(x-1)\, dx$

(x-1)^3/3+c

$n. \int \frac{x^4-x+1}{x^4} dx$

x+2/x^2 -3/x^3+c

$o. \int (2x-3)^2\, dx$

(2x-3)^3/(6)) +c

$p. \int (6x-3)^{\frac{2}{3}}\, dx$

(6x-3)^(5/3) /(10) +c

$q. \int \frac{1}{(3x-2)^2} dx$

(3x-2)/(-3) +c

$r. \int 4\left(3x - \frac{1}{2}\right)^{-2}\, dx$

4(3x-1/2)^-1/(-3) +c

2. Find the function if it passes through $(1,2)$ if the derivative is

$\boldsymbol{a}.\int \boldsymbol{x(2-x)\ dx}$

x^2 -x^3/3+c

x^2-x^3+4/3

$\boldsymbol{b}.\int \sqrt[3]{\boldsymbol{x-1}}\ \boldsymbol{dx}$

4(x-1)^(4/3)/3+c

4(x-1)^(4/3)/3+2

$\boldsymbol{c}.\int \boldsymbol{(ax-b)^3\ dx}$

(ax-b)^4/4a +c

2=(a-b)^4/4a+c

2-(a-b)^4/4a=c

therefore

y=(ax-b)^4/4a +2-(a-b)^4/4a = [(ax-b)^4+2-(a-b)^4]/4a

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 20

Topic: Integration

Integrating the exponential function
Integrating trigonometric function
Integrating 1/x

Integrating the exponential function

We know that the derivative of e^x is itself and simply the integration of e^x as well. In mathematical terms,

$$\int e^x\,dx = e^x + c$$

In order to integrate an exponential function whose index is a linear expression, divide the function by the coefficient of x

$$\int e^{kx+a}\,dx = \frac{e^{kx+a}}{k} + c$$

Here are some examples

Solve

a. $\int e^{10x}\,dx$

b. $\int e^{-4x+1}\,dx$

c. $\int e^x(1-e^{-2x})\,dx$

Solution

a. $\int e^{10x}\,dx$

Using the above technique, divide by the coefficient of x

$$\int e^{10x}\,dx = \frac{e^{10x}}{10} + c$$

Solution

b. $\int e^{-4x+1}\,dx$

Using the above technique, divide by the coefficient of x

$$\int e^{-4x+1}\,dx = \frac{e^{-4x+1}}{-4} + c$$

Solution

c. $\int e^x(1-e^{-2x})\,dx$

First simplify the equation $\int e^x(1-e^{-2x})\,dx = \int e^x - e^{-x}\,dx$

Imply the integration separation property. $\int e^x - e^{-x}\,dx = \int e^x\,dx - \int e^{-x}\,dx$

Using the above technique, divide the coefficient $\int e^x\,dx - \int e^{-x}\,dx = e^x + e^{-x} + c$

Note c is the combination of constants!

Integrating trigonometric function

In previous topics we know by now the derivative of the trigonometric functions, sine, cosine and tan.

Also, the integrating a derivative will gives us the function that we differentiated!

$\frac{d}{dx}(\sin(x)) = \cos(x)$

$$\int \frac{d}{dx}(\sin(x)) = \int \cos x \, dx$$

Thus $\int \cos x \, dx = \sin x + c$

Consequentially $\int \sin x \, dx = -\cos x + c$

Let the students try to solve for the other trigonometric function. *Go through it together

Similarly, $f(x) = \sin kx$ and $f'(x) = kcos\ kx$

$\int f'(x)\, dx = f(x) + c_1$

$\int kcos\ kx\ dx = \sin kx + c_1$ given that k and c are constant

$k \int cos\ kx\ dx = \sin kx + c_1$

Therefore $\int cos\ kx\ dx = \frac{1}{\mathrm{k}} \sin kx + c$

Note the constant combines as a simple term $c = \frac{1}{k} c_1$

In general

$$\int cos\ (kx + a)\ dx = \frac{1}{\mathrm{k}} \sin(kx + a) + c$$

$$\int sin\ (kx + a) dx = -\frac{1}{\mathrm{k}} \cos(kx + a) + c$$

Given $k\ and\ c$ are coefficient/constants.

Examples

Solve

a. $\int 3sin2x\, dx$ b. $\int \cos(\frac{-x}{3} + 1)\ dx$ c. $\int \sin x \cos x\, dx$

Solution

a. $\int 3sin2x\, dx$

Using the formula, divide by the coefficient of x

$$\int 3sin2x\, dx\ = -\frac{3}{2}\cos 2x + c$$

Solution

b. $\int \cos(\frac{-x}{3} + 1)\ dx$

Using the formula, divide by the coefficient of x

$$\int \cos(\frac{-x}{3} + 1)\ dx\quad = -\frac{1}{3}\sin\left(-\frac{x}{3} + 1\right) + c$$

Solution

c. $\int \sin x \cos x\, dx$

First simplify the equation using trigonometry rules, consider $\sin 2\mathrm{x} = 2\sin x \cos x$

$So\ \int \sin x \cos x\, dx = \int \frac{1}{2}\sin 2x\, dx$

Using the formula, divide the coefficient $\int \frac{1}{2}\sin 2x\, dx = -\frac{1}{4}\cos 2x + c$

Integrating $\frac{1}{x}$

You may think integrating $\frac{1}{x}$ is simply raising the power by one but that is not the case. You will see that

$\frac{1}{x}$ is x^{-1} if we raise the power then it will become x^0 which is just 1.

However we investigated that $\frac{d}{dx}\log_e x = \frac{1}{x}$

Since

$\int f'(x)\, dx = f(x) + c$, we integrate both sides of the above equation.

$$\int \frac{d}{dx}\log_e x \; dx = \int \frac{1}{x} dx$$

$$\int \frac{1}{x} dx = \log_e |x| + c$$

Note the $\log_e |x|$ is restricted to fit throughout the domain of $\frac{1}{x}$

Similarly, $f(x) = \log_e g(x)$, then $f'(x) = \frac{g'(x)}{g(x)}$

In general

$$\int \frac{g'(x)}{g(x)}\, dx = \log_e|g(x)| + c$$

where $g(x)$ is a function and c is a constant

Also,

$$\int \frac{a}{ax+b} dx = \log_e|ax+b| + c$$

where a, b and c are constants

Examples ***(Tutor should go through c* question)***

Solve

a. $\int -\frac{2}{x}\,dx$ b. $\int \frac{3}{2x+5}\,dx$ c*. $\int (6x^2+3x)/(x^3+x^2)\,dx$

Solution

a. $\int -\frac{2}{x}\,dx$

Using the formula after taking the -2out since it is a constant

$$\int -\frac{2}{x}\,dx = -2\int \frac{1}{x}\,dx = -2\log_e|x| + c$$

Solution

b. $\int \frac{3}{2x+5}\,dx$

Using the formula

$$\int \frac{a}{ax+b}\,dx = \log_e|ax+b| + c$$

$$\int \frac{3}{2x+5}\,dx = \int \frac{3\left(\frac{2}{2}\right)}{2x+5}\,dx = \int \left(\frac{3}{2}\right)\frac{2}{2x+5}\,dx = \left(\frac{3}{2}\right)\int \frac{2}{2x+5}\,dx = \frac{3}{2}\log_e|2x+5| + c$$

Solution

c*. $\int (6x^2+3x)/(x^3+x^2)\,dx$

Simplify the equation

$$(6x^2+3x)/(x^3+x^2) = 3(2x+1)/(x^2+x)$$

$$\int \frac{3(2x+1)}{x^2+x}\,dx$$

Check if the satisfy the $\int \frac{g'(x)}{g(x)}\,dx = \log_e|g(x)| + c$

$$g(x) = x^2 + x \text{ then } g'(x) = 2x+1$$

You will need to make the numerator factors g'(x) then you should end up with 2g'(x)/g(x) -x/g(x) where g(x)=x^3+x^2. For the -x/(x^3+x^2) you will need to do partial fraction then integrate using log_e

Afterwards you can show using log identities that the solution is the same as the first one.

Note the numerator is $g'(x)$

So $\int \frac{3(2x+1)}{x^2+x} dx = 3\int \frac{g'(x)}{g(x)} dx = 3\log_e|x^2+x| + c$

You may try to integrate without simplifying.

$\int (6x^2+3x)/(x^3+x^2)\, dx$

But this will require you to understanding some advance technique which is mainly used in Specialist Mathematics.

For those who are interested, your tutor may show you the advance method to solve the question without the first simplification.

Testing Understanding

1. Find:

a. $\int e^x \, dx$

e^x+c

b. $\int 3e^{2x} \, dx$

3/2 e^2x+c

c. $\int -e^{2x-1} \, dx$

-1/2 e^(2x-1)+

d. $\int 1/e^x + x \, dx$

-e^-x+1/2 x^2 +c

e. $\int (e^{2x})^3 + e^{-x} \, dx$

1/6 e^6x -e^-x+c

f. $\int (e^{2x} + e^x)^2 \, dx$

1/4 e^4x+ 2/3e^3x+1/2e^2x

2. Find:

a. $\int \sin 5x \, dx$

-1/5 cos 5x+c

b. $\int -\cos 6x \, dx$

-1/6sin 6x+c

c. $\int 2\sin\frac{x}{3} \, dx$

-6 cos x/3 +c

d. $\int 2\sin(4x+1) \, dx$

-1/2 cos (4x+1) +c

e. $\int -\cos 6(x-1) \, dx$

-1/6 sin (6x-6) +c

f. $\int 2 - 5\cos\left(\frac{x}{4} - 1\right) dx$

2x-20sin(x/4-1)+c

3. Find:

a. $\int \frac{2}{x} dx$

2loge x +c

b. $\int \frac{4}{3x} dx$

4/3 loge x +c

c. $\int -\frac{1}{7x} dx$

-1/7 loge x +c

d. $\int \frac{2}{2x+1} dx$

loge (2x+1) +c

e. $\int -\frac{4}{3x-2} dx$

-4/3 loge (3x-2) +c

f. $\int \frac{3}{4-5x} dx$

-3/5 loge(4-5x) +c

4. Find:

a. $\int (\sin 5x - 2\cos x) dx$

-1/5cos 5x -2sinx +c

b. $\int e^{2x} - \cos 6x \, dx$

1/2 e^2x -1/6 sin 6x +c

c. $\int x^3 + 2\sin\frac{x}{3} dx$

x^4/4-6cosx/3+c

d. $\int \cos x + e^{7x} - x^3 \, dx$

sinx + 1/7e^7x - x^4/4

e. $\int e^{2x-1} + 2\sin(4x-3) \, dx$

1/2 e^2x-1 -1/2 cos (4x-3)+c

f. $\int (e^{2x})^3 - \sin x \cos x \, dx$

1/6e^6x +1/4 cos2x+c

g. $\int \frac{2}{x} + \sin(3x-1) \, dx$

2logex + 1/3 cos (3x-1)+c

h. $\int e^{-3x} - \frac{3}{x} - \frac{\sin x/4}{6}) \, dx$

-1/3 e^-3x -3loge x +2/3cos x/4+c

i. $\int e^{3x-10}/e^{-2x+1} \, dx$

1/5e^(5x-11)+c

j. $\int e^{-x/3} + (3x-2)^3 \, dx$

-3xe^-x/3+(3x-2)^4/12+c

k. $\int e^{-1} + 2\sin(3-x) + \frac{1}{1-x} dx$

xe^-1+2cos(3-x)-loge(1-x)+c

5. Challenge Questions

a. $\int x e^{2x^2} \, dx$

1/4 e^2x^2 +c

b. $\int \sin^2 x \, dx$

-1/4 sin2x+x/2+c

c. $\int -\tan x \quad dx$

-loge(cosx)+c

d. $\int (4x + 2\cos x)/(x^2 + \sin x)\, dx$

2loge(x^2+sin x)+c

e. $\int \sin 2\text{x} \cos 3\text{ x} - \sin 3x \cos 2x\, dx$

-cos(-x)+c

f. $\int \cos 2x\, (\sin 2x)^3\, dx$

1/8 (sin2x)^4+c

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 21

Topic: Integration

Integrating by recognition

Area under the curve

Definite integral and its properties

Integrating by Recognition

By now you should know the differentiation and anti-differentiation are reverse process like how subtraction is the reverse of addition.

We can say

$If\ f'(x) = g(x)\ \ then \int g(x)\ dx = f(x) + C$

Here are some examples that might need integration by recognition

Example 1. Find the derivative of $(4-3x)^3$. Hence solve for $\int (4-3x)^2\, dx$

First, find the derivative of the function

$$f(x) = (4-3x)^3 \text{ and by using chain rule}$$

$f'(x) = -9(4-3x)^2$

Since

$$\int f'(x)dx = f(x) + C$$

Then

$\int -9(4-3x)^2 dx = (4-3x)^3 + C$

Simplify the integral such that it looks like $\int (4-3x)^2\, dx$

Therefore $\int (4-3x)^2 dx = -\frac{1}{9}(4-3x)^3 + C'$

Note that the $C' = \frac{C}{9}$. Generally we don't write $\frac{C}{9}$ since it is already a constant regardless what operation is used on C.

Here is a harder example

Example 2. Derive $x \sin x$. Hence using your answer to find $\int x \cos x \ dx$

Firstly derive $x \sin x$ *by using product rule*

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let $u = x$ *and* $v = \sin x$

$$So \ \frac{du}{dx} = 1 \ and \ \frac{dv}{dx} = \cos x$$

Thus $f'(x) = x\cos x + \sin x$

Using our integration by recognition formula

$$\int f'(x)dx = f(x) + C$$

$$\int (x\cos x \ + \sin x)dx = xsin\ x + C$$

We can separate integrals

$$\int x\cos x \ dx + \int \sin x \, dx = xsin\ x + C$$

Rearrange the equation such that it looks like $\int x \cos x \ dx$

$$\int x\cos x \ dx = xsin\ x - \int \sin x \, dx + C$$

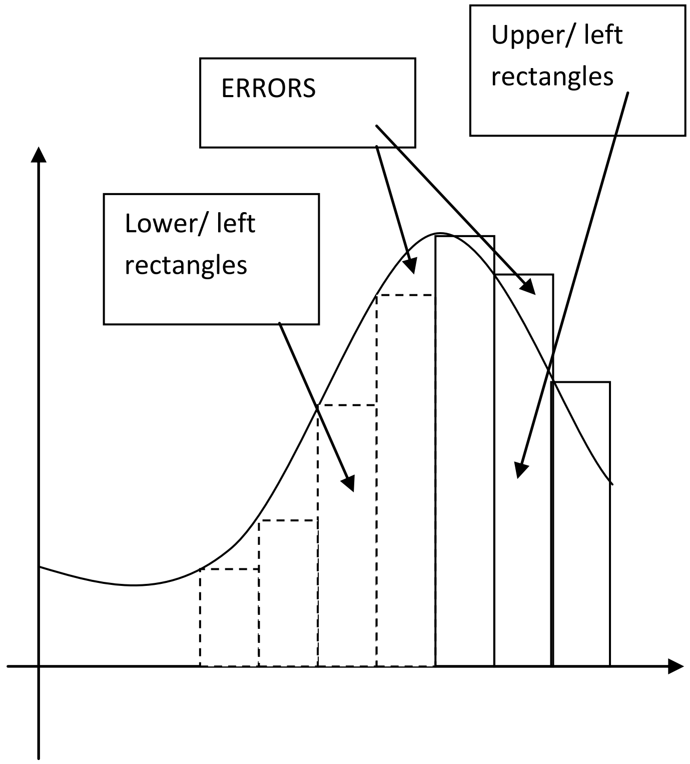
Since we know the integration of trigonometric function our solution is

$$\int x\cos x \ dx = xsin\ x + \cos x + C$$

Note: C is an arbitrary constant can the constant from the integration of sin is absorbed by C

Areas under the curves

In order to solve the area under the curve, we can use various techniques to approximate the solution. A simple technique is using rectangular strip of equal width to find the area enclosed from the limits a to b. However, to minimise the inaccuracy of using this technique is to take very small step sizes. The error is the extra or missing area of the rectangle.



- The **dotted lines** rectangle are the lower rectangles
- The **straight lines** rectangle are the upper rectangles

Either use the upper or lower rectangles to approximate the area of the curve. Most cases you will be taught using the left and right rectangles. Simply, which rectangle edge touches the curve determines the left or right rectangle.

In this case, it is a left rectangle approximation because all the rectangle top left edge is touching the curve!

Note the gap or excess between the curve and the rectangle is the error.

Here is the general equation, given that you know the equation of the curve.

$$\sum_{x=a}^{b} |f(x)|\,\Delta x = \Delta x\,(|f(a)| + |f(a+1)| + \ldots + |f(b-1)| + |f(b)|)$$

$$Where\ \Delta x = \frac{b-a}{n}, a\ and\ b\ are\ limits\ of\ the\ area\ of\ interest\ and\ n\ is\ the\ number\ of\ rectangles$$

Don't be frightened with the equation above! What it means is adding all the rectangles up.

The area of a rectangle is width multiplied by height. Basically adding all the area of the rectangle is

$$Total\ area = Area_1 + Area_2 + \ldots = w_1 h_1 + w_2 h_2 + \ldots$$

Δx is the width of the rectangle, and since the width of all rectangle is the same we can take it out as a common factor as shown in the equation above.

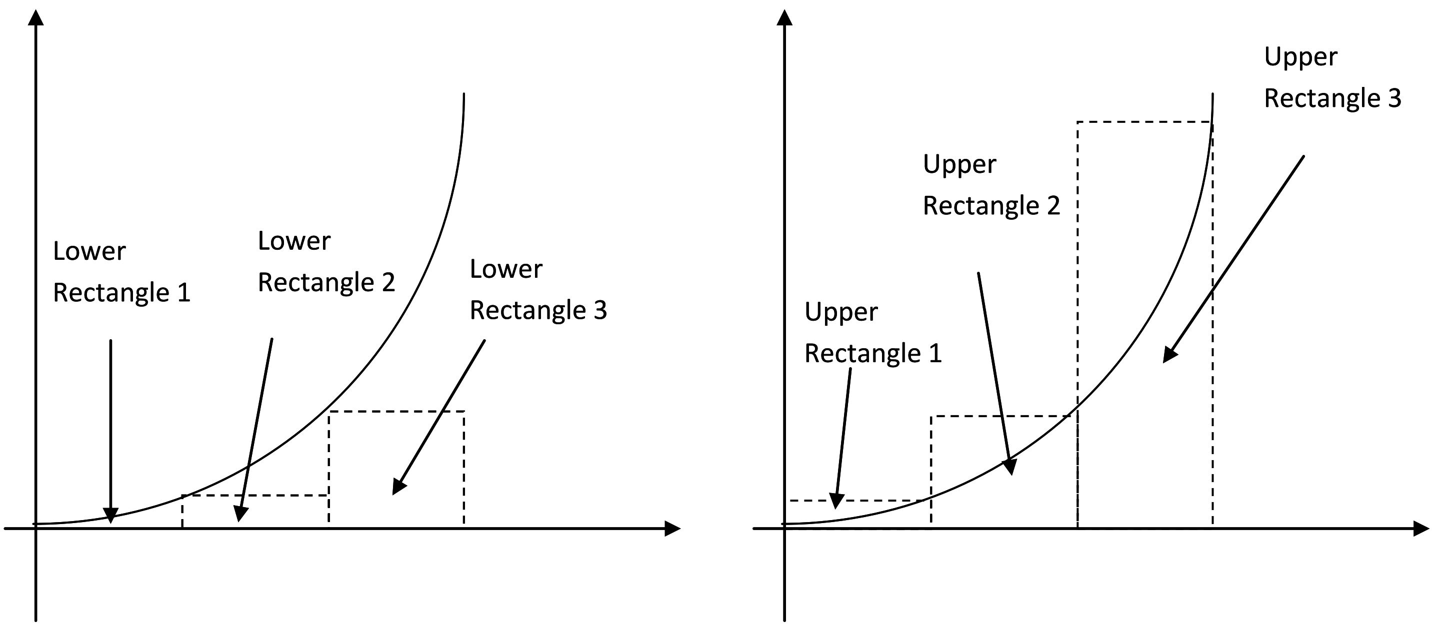
Now the only thing to find now is the height. You will notice that the height is the dependent value of the equation which is simply $f(x)$. The reason why we take the absolute value is to avoid subtracting areas,

since we are finding the **total area**. Once you compute the height and width and sum all the area, the equation will be the same as the one above!
Here is an example

Example 3.
Use 3 upper and lower rectangles to approximate the function $y = x^2$ between 0 and 6

First illustrate the graph.

Lower Rectangle **Upper Rectangle**



Next you will need to label the information given:

- Area of interest to estimate is 0 to 6
- Step size is 2 from $Step\ size = \frac{b-a}{3} = \frac{6-0}{3} = 2$
  - For lower rectangle the height values are taken from
    - $x = \{0,2,4\}$
  - For lower rectangle the height values are taken from
    - $x = \{2,4,6\}$
- Height of the rectangle is given by using the equation $f(x) = y = x^2$
  - For lower rectangle height values are
    - $f(x) = \{0,4,16\}$
  - For upper rectangle height values are
    - $f(x) = \{4,16,36\}$

Now computing the values together

$Lower\ rectangle = 2(0 + 4 + 16) = 40\ unit^2$
$Upper\ rectangle = 2(4 + 16 + 36) = 112\ unit^2$

You will notice there is a big discrepancy between the lower and upper rectangle results. The exact answer is 72 $unit^2$. The lower rectangle is too conservative with the approximation while the upper rectangle over estimated the area. However there are other techniques such as middle point method (Taking the middle of the rectangle as an approximation, trapezoidal method (Trapezium shapes) and Rungka method (High order polynomial i.e. Parabolas and cubics!) to accurately approximate the area under the curve. Luckily you will only need to know rectangular shape approximation.

Testing Understanding

1. Find the derivative of $y = xcos(x)$ and hence solve for $\int xsin\ x\ dx$

dy/dx=d(xcosx)/dx= -xsin x + cos x

so integrate (d(xcosx)/dx)= integrate (-xsin x + cos x)

xcosx - integrate(cos x)= integrate (-xsin x)

integrate (xsin x)= integrate(cos x) - xcosx = sinx -xcosx +C

2. Using integration by recognition method to solve for $\int xe^{x^2}\ dx$. Start with $y = e^{x^2}$

dy/dx=d e^x^2/dx = 2xe^x^2

so integrating [d e^x^2/dx] =integrating[2xe^x^2]
so integrating[xe^x^2]= 1/2 e^x^2 +C

3. Solve for $\int xlog_e x\ dx$

consider y= x^2 loge x

so dy/dx= 2x log e x + x
integrating d[x^2 loge x]/dx= integrating [2x log e x + x]

x^2 loge x - 1/2 x^2= integrating [2x log e x]
integrating [x log e x]=1/2 x^2 loge x - 1/4 x^2 +C

4. Using step size of 0.5, approximate the function and sketch $y = x^2$ between 0 and 3 with upper and lower rectangles method.

5. Use 4, and 8 right and left rectangles , approximate the function and sketch $y = xe^{x^2}$ between 0 and 1. Comment on the results.

Definite Integrals

You may wonder how can we exactly solve the area under the curve? Integration of a function determines the area under the curve. However you will need to know not all functions can be integrated and that is why we need approximating technique such as the lower and upper or the left and right rectangle method or better techniques.

Note how we integrate functions for indefinite integrals, we basically use the same techniques but applying it with lower and upper limits.

Such that

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

This is also known as **fundamental theorem of calculus**, and the expression on the left hand side is the **definite integral.**

This expression is possible if the function is continuous for all $x \in [a, b]$ and that the $F(x)$is an anti-derivative of $f(x)$.

Note that the constant disappears in a definite integral.

Another general rule is that if the upper and lower limits is reversed the sign of the definite integral changes.

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

Here is an example.

Evaluate $\int_1^2 x^2 + 2x + 1\ \ dx$

Since we know that for $f(x) = x^2 + 2x + 1$, the integration is $F(x) = \frac{x^3}{3} + x^2 + x + c$

So $\int_1^2 x^2 + 2x + 1\ \ dx = \left[\frac{x^3}{3} + x^2 + x\right]_1^2 = \left(\frac{2^3}{3} + 2^2 + 2\right) - \left(\frac{1^3}{3} + 1^2 + 1\right) = \left(\frac{26}{3}\right) - \left(\frac{7}{3}\right) = \frac{19}{3}$

Properties of definite integrals

Let $f(x)$ and $g(x)$ are functions which are continuous between $x = a$ and $x = b$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

In a case where b $lies$ $between$ a and c $then$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Example

Given $\int_2^6 f(x)\,dx = 5$ Find

a. $\boldsymbol{\int_2^6 2f(x)\,dx}$

SOLUTION

Since we know the property $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

Then $\int_2^6 2f(x)\,dx = 2\int_2^6 f(x)\,dx = 2(5) = 10$

b. $\boldsymbol{\int_6^2\ [f(x) + x]\,dx}$

SOLUTION

Separate the integral to $\int_6^2\ f(x)dx + \int_6^2 x\ dx$ to integrate by parts.

Using the property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_6^2\ f(x)dx + \int_6^2 x\ dx = -\int_2^6\ f(x)dx + \left[\frac{x^2}{2}\right]_6^2 = -5 + \left(\frac{4}{2}\right) - \left(\frac{36}{2}\right) = -21$

Testing Understanding

6. Evaluate the following

a. $\int_0^{\frac{\pi}{2}} \sin x \ \ dx$

[-cosx]= 1

b. $\int_2^4 x^3 + \sqrt{x} \ \ dx$

[1/4 (4)^4 +2/3 (4)^3/2]- [1/4 (2)^4 +2/3 (2)^3/2]

c. $\int_1^2 (3x+2)^2 dx$

[1/9(8)^3]- [1/9(5)^3]

d. $\int_0^3 \frac{1}{\sqrt{x+2}} dx$

[2(5)^(1/2)]- [2(2)^(1/2)]

e. $\int_1^4 \frac{2}{x} \ \ dx$

2ln3

f. $\int_2^0 x^3 + x \, dx$

-6

g. $\int_0^a \sin x + x^4 \ \ dx$

-cos a +1/5 a^5

h. $\int_0^{4\pi} \cos x + \sin 4\,x \ \ dx$

0

7. Given $\int_0^2 f(x) dx = 2$, find

a. $\int_0^2 2f(x) dx$

4

b. $\int_2^0 [f(x) - 1] dx$

-2 +2 =0

c. $\int_0^2 3f(x) - x^2 dx$

6 -(8/3)

8a. Solve for $\int (x^2 + x + 1) \ dx$

1/3 x^3 +1/2 x^2 +x +C

b. $Find \ \int_a^1 (x^2 + x + 1) \, dx$

1/3 +1/2 +1 -(1/3a^3 +1/2a^2 +a)

c. $If \ \int_a^1 (x^2 + x + 1) \, dx = 4 \ then \ solve \ for \ a$

1/3a^3 +1/2a^2 +a +13/6 =0

a=-1.817 CALC

9. Find the derivative of $2 \log_e sinx$, hence solve for $\int_1^2 \cot x \ \ dx$

dy/dx = 2cosx /sinx =2 cot x

so integrating 2 cot x = 2 loge sinx

so $\int_1^2 \cot x \ \ dx$ = [log e sinx]= logesin1 - logesin 2

NQT EDUCATION

YEAR 12

VCE MATHS METHODS UNIT 4 TERM 3 WORKBOOK

WEEK 22

Topic: Integration

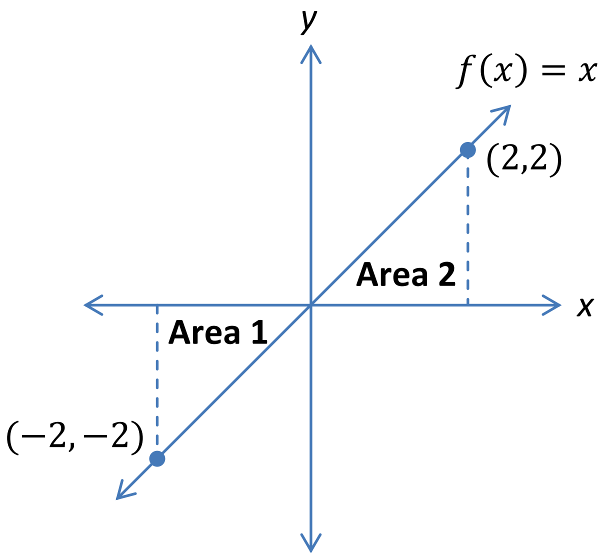
Area enclosed by curve

Area between two curves

Sketching Graphs with given derivatives

Area Enclosed by Curve

In previous topics we have learnt how to find the area of the curve. However sometimes it is required to find the total unsigned area rather than the signed area. Most problem solving questions requires you to find the total or true area, so be aware what the questions are asking you to find!
Here is an example to distinguish the differences.



Total Unsigned Area

The sum of areas enclosed by the curve is the true area.

Area 1 = 2 $units^2$

Area 2 = 2 $units^2$

So the true area is $4\ units^2$

Signed Area

If the signs are considered then areas above the x axis is considered positive and the areas below the x axis is negative.

$$\int_{-2}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-2}^{2}$$

$$= (2) - (2) = 0\ units^2$$

In order to solve for the total unsigned area using integration techniques, we break the absolute region enclosed by the curve depending on the number of times it hits the x axis.

Such that the formula will be for the above example

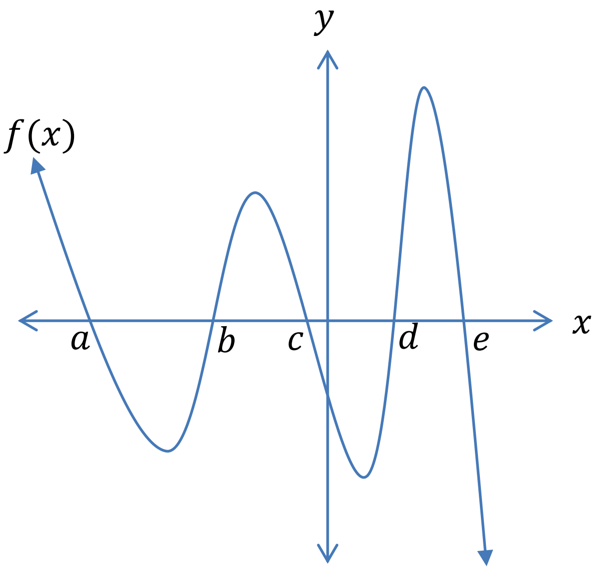
$Area = |Area\ 1| + |Area\ 2|$

Since the curve passes through one x-axis then we split the integral into two; relative to the x intercept.

$= \left|\int_{-2}^{0} x\,dx\right| + \left|\int_{0}^{2} x\,dx\right|$ Note that Area 1 is positive if the integral is multiplied by a negative.

$= -\int_{-2}^{0} x\,dx + \int_{0}^{2} x\,dx = -[0-2] + [2-0] = 4\ units^2$

In a general case



While the signed area is given by

$$\int_a^d f(x)dx$$

The true area enclosed by the curve $f(x)$

$$Area = |Area_a^b| + |Area_b^c| + |Area_c^d| + |Area_d^e|$$

The following equations can be used to solve for the enclosed area.

Equation 1

$$\left|\int_a^b f(x)\,dx\right| + \left|\int_b^c f(x)\,dx\right| + \left|\int_c^d f(x)\,dx\right| + \left|\int_d^e f(x)\,dx\right|$$

Equation 2

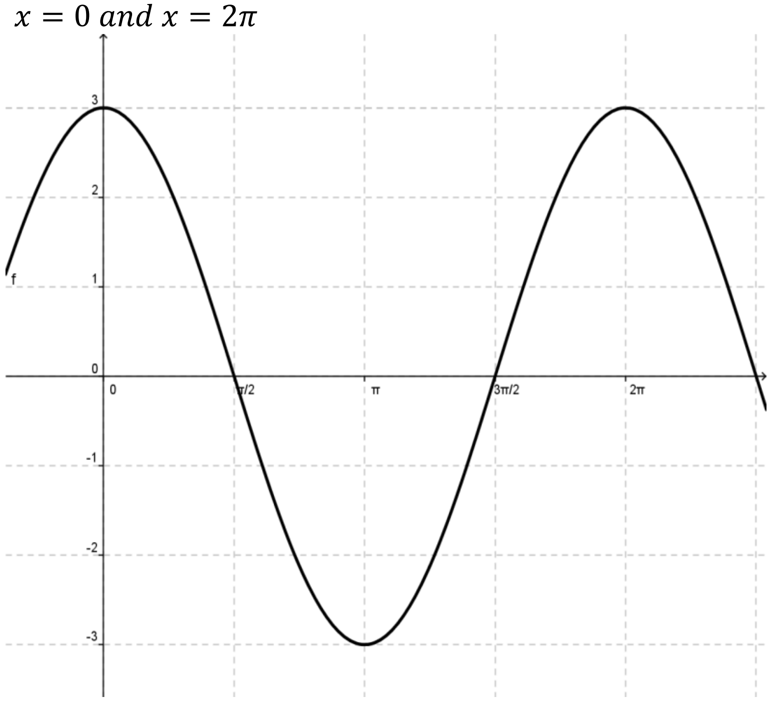
$$-\int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \int_c^d f(x)\,dx + \int_d^e f(x)\,dx$$

Equation 3

$$\int_b^a f(x)\,dx + \int_b^c f(x)\,dx + \int_d^c f(x)\,dx + \int_d^e f(x)\,dx$$

Example

Solve for the area of the region enclosed by the curve $f(x) = 3\cos(x)$, the x-axis, and the lines $x = 0 \ and \ x = 2\pi$



Usually you would like sketch the graph and find the x-intercepts between the points of interest.

The x intercepts are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

Next we write the integrals for the enclosed area. Using Equation 1 as an example

$$\left|\int_0^{\frac{\pi}{2}} f(x)\,dx\right| + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\,dx\right| + \left|\int_{\frac{3\pi}{2}}^{2\pi} f(x)\,dx\right|$$

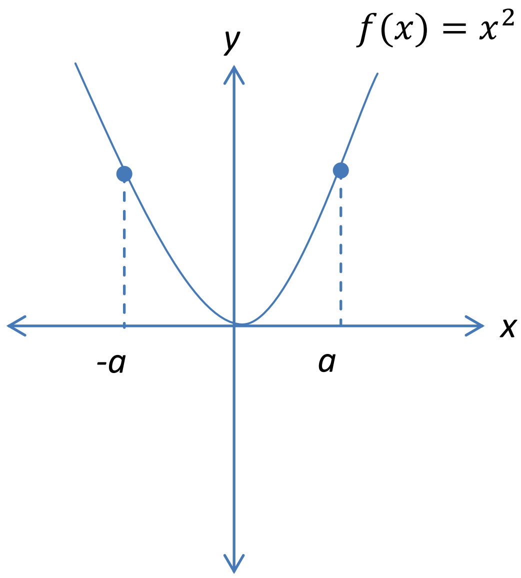
$$= \left|3\sin\frac{\pi}{2} - 3\sin 0\right| + \left|3\sin\frac{3\pi}{2} - 3\sin\frac{\pi}{2}\right| + \left|3\sin 2\pi - 3\sin\frac{3\pi}{2}\right|$$

$$= |3 - 0| + |-3 - 1| + |0 - -1| = 3 + 4 + 1$$
$$= 8\ units^2$$

For symmetry properties, the following expression can be used to solve for the area.
In a simply case where the axis of symmetry is at $x = 0$

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)dx$$

For example consider the graph below:



As long there is a symmetry line we can equate the function into one integral even if the function is reflected. i.e $f(x) = x^3$

$$\left|\int_{-a}^{a} f(x)\, dx\right| = 2\int_{0}^{a} f(x)dx$$

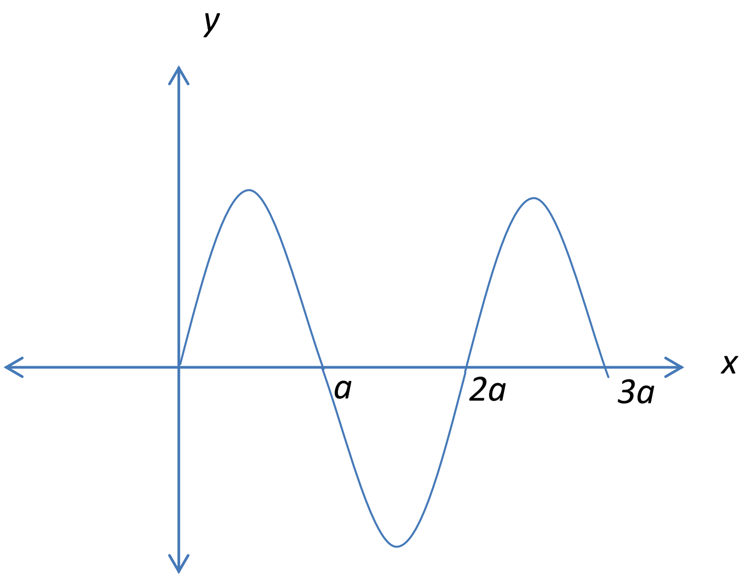
Still solves for the enclosed area, however if it was signs included then

$$\int_{-a}^{a} f(x)\, dx = 0$$

In general, especially in trigonometric functions, the following can be expressed as

$Area = N\int_{a}^{b} f(x)dx$

For example consider the graph below:



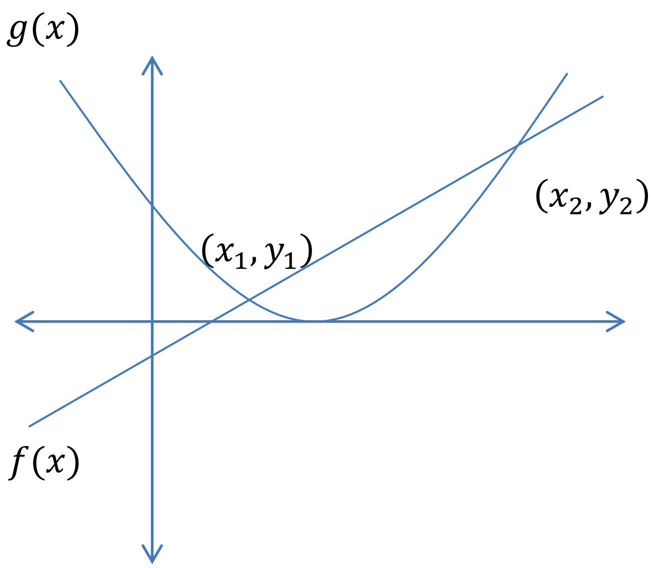
Let the function on the graph be a periodic function. Since the function has 3 same areas, we can equate it as one formula.

Such that the $\left|\int_{0}^{3a} f(x)dx\right|$

$$= 3\int_{0}^{a} f(x)\, dx$$

Area between two curves

Consider the graphs below



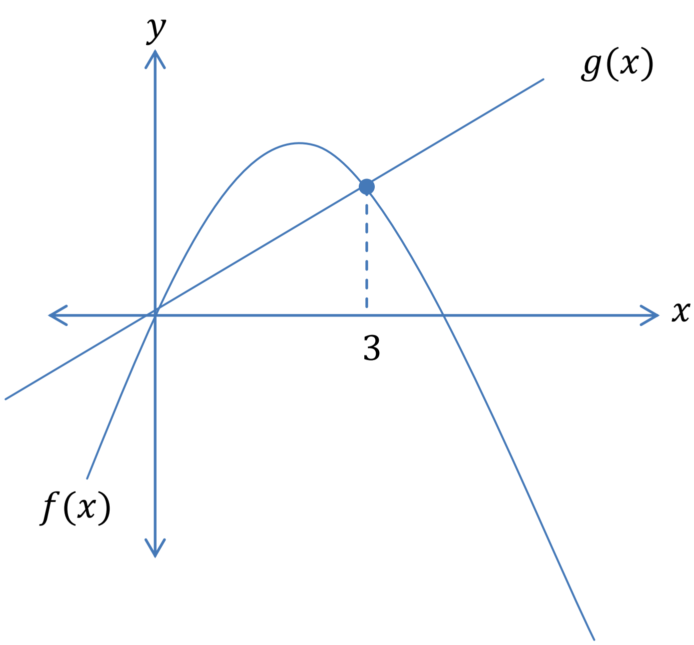
The area bounded by the two curves and line of intersection is the difference between the upper curve and the area below the lower curve.

$$Area = \int [upper\ curve - lower\ curve]\ dx$$
$$= \int_{x_1}^{x_2} f(x)dx - \int_{x_1}^{x_2} g(x)dx$$
$$= \int_{x_1}^{x_2} [f(x) - g(x)]dx$$

Example

Find the area enclosed between graphs of $f(x) = x(4 - x) and\ g(x) = x$

Firstly you would sketch the graphs of $f(x) and\ g(x)$



Afterwards, solve for the intersections of $f(x) and\ g(x)$.

Let $f(x) = g(x)$

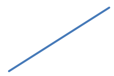
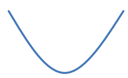
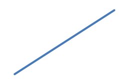
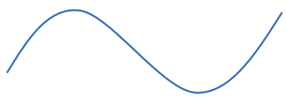
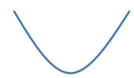
$$x(4 - x) = x$$
$$4x - x^2 - x = 0$$
$$3x - x^2 = x(3 - x) = 0$$

Therefor the x coordinates of the points of intersection are $x = 0\ and\ x = 3$

$$Upper\ curve = f(x) = x(4 - x)$$
$$Lower\ curve = g(x) = x$$
$$f(x) - g(x) = 3x - x^2$$
$$\int_0^3 3x - x^2 dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \left(\frac{27}{2} - \frac{27}{3}\right) - 0$$
$$= 4.5\ units^2$$

Sketching Graphs with given derivatives

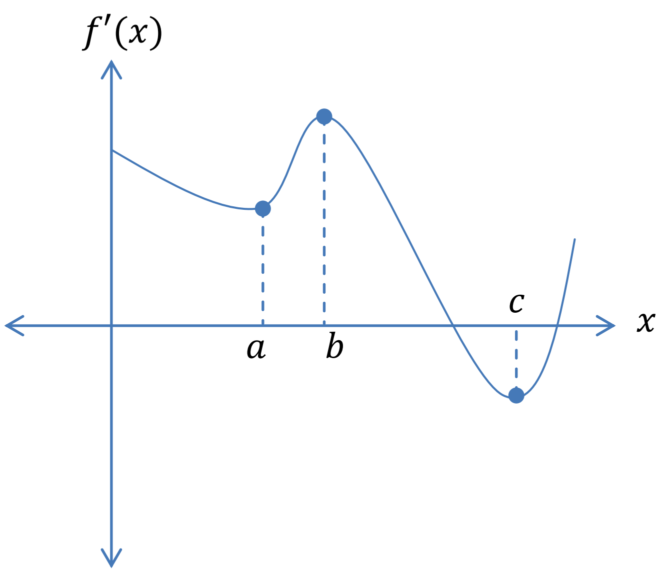
The derivative function $f'(x)$ of a polynomial function can be represented of the graph function $f(x)$ by increasing the degree by one.

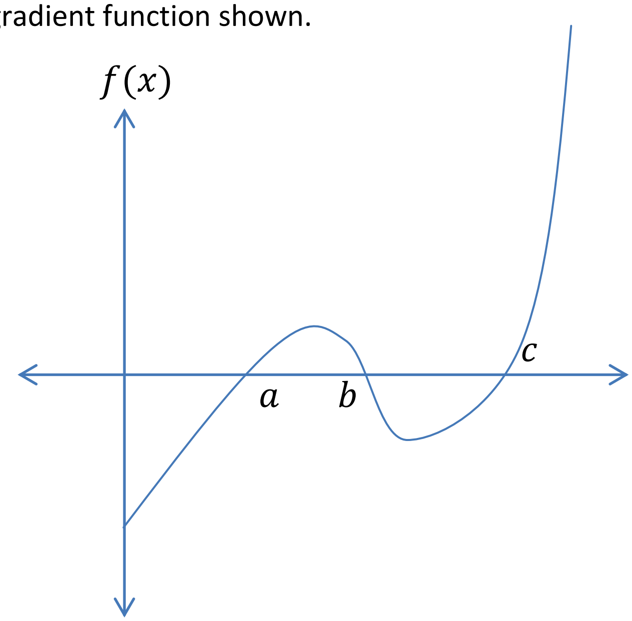
Original function $f(x)$		***Derivative function*** $f'(x)$	
Linear $i.e\ y = x$		**Constant** $i.e\ y = c$	
Quadratic $i.e\ y = x^2$		**Linear** $i.e\ y = x$	
Cubic $i.e\ y = x^3$		**Quadratic** $i.e\ y = x^2$	

- If gradient of $f(x)$is positive we draw the graph $f'(x)$ *above the x-axis.*
- If gradient of $f(x)$is negative we draw the graph $f'(x)$ *below the x-axis.*
- The turnings on the graph of $f(x)$ are intercepts on the $f'(x)$

Example

Sketch the general shape of the graph $f(x)$ given the gradient function shown.





$\{x: 0 < x < a\}$: *the gradient is negative.*
$\{x: x = a\}$: *the stationary point, thus x intercept.*
$\{x: a < x < b\}$: *the gradient is positive; increases then decreases in magnitude*
$\{x: x = b\}$: *the stationary point, thus x intercept.*
$\{x: b < x < c\}$: *the gradient is negative; increases then decreases in magnitude*
$\{x: x = c\}$: *the stationary point, thus x intercept.*
$\{x: x > c\}$: *the gradient is increasingly positive as it has a higher magnitude (steeper).*

Testing Understanding

1. **Find the area of then region enclosed by the curve $f(x) = \sin\left(x - \frac{\pi}{2}\right)$, the x-axis and the line $x = 0$ and $x = \frac{3\pi}{2}$.**

Abs(integration between pi/2 and 0) f(x)+ Abs(integration between 3pi/2 and pi/2) f(x)=4

2. **Find the area of the shaded region, bounded by the x axis $f(x) = 3x^4 - 9x^3 + 3x^2 + 9x - 6$**

Find intercepts at -1,1 and 2

Abs(integration between -1 and 1) f(x)+ Abs(integration between 1 and 2) f(x)=8.8 +6.5=9.45

3. **Find the total area between $0 \leq x \leq 6$ of function $f(x) = x^2 - 4x$**

Abs(integration between 0 and 4) f(x)+ Abs(integration between 4 and 6) f(x)=21.33

4. **Find the total area between $0 \leq x \leq 5\pi$ of function $f(x) = \cos\frac{x}{2}$**
   Abs(integration between 0 and pi) f(x)+ Abs(integration between pi and 3pi) f(x) + Abs(integration between 3pi and 5pi) f(x)=10

5. **Find the value of the x coordinate for which $\int_0^1 e^x\,dx = \int_1^a e^x\,dx$**

E^a –E^1=E^1-1
E^a=2E^1-1
A=ln(2e^1-1)

6. **Find the area of the region enclosed by the graph $f(x) = x^2 and\ g(x) = \sqrt{x}$**

Abs(integration between 0 and 1) [g(x)-f(x)]=0.333

7. **Find the area between the $x\ axis\ \ and\ y = 2$ of function $f(x) = 4 - x^2$**

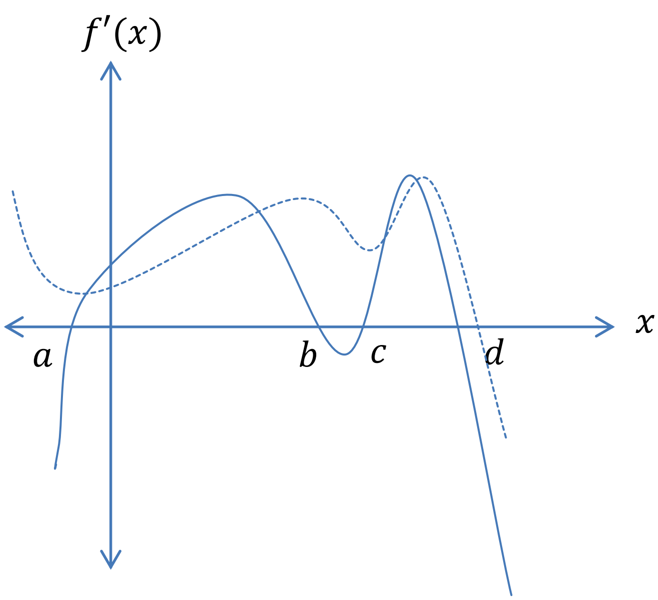
Find the area of Abs(integration between -2 and 2) [f(x)]=10.66
Find the area of Abs(integration between -root2 and root2) [2-x^2] =0.377

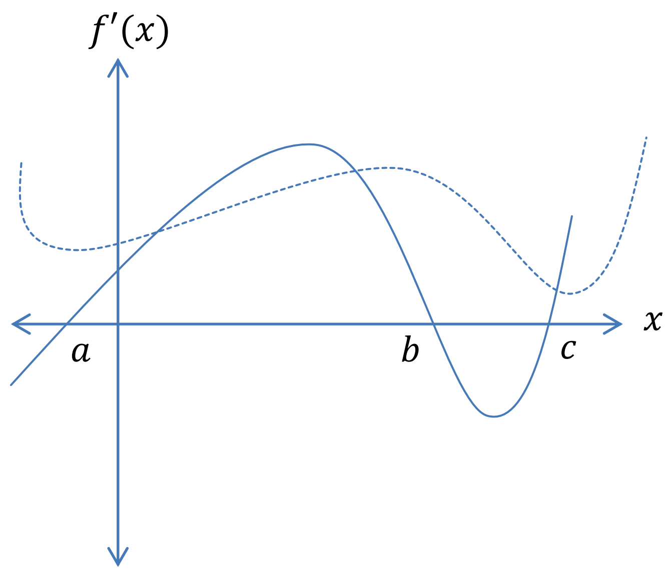
Thus area is 10.66-0.377=6.89

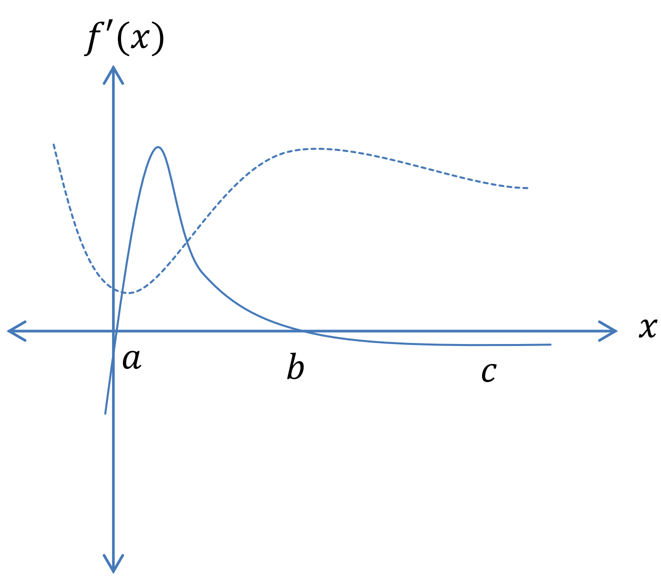
8. Find the exact value for the enclosed area between $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ of function $\boldsymbol{f(x) = 3\cos(x)}$ *and* $\boldsymbol{g(x) = sin(2x)}$

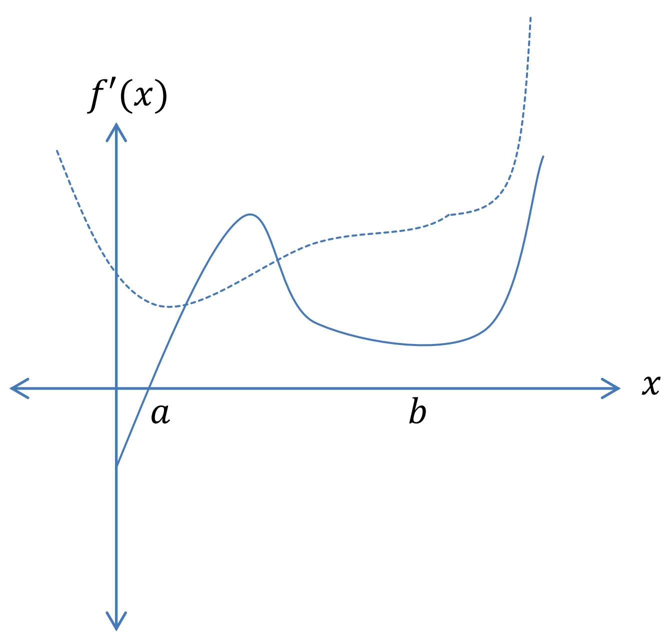
Find the area of Abs(integration between pi/2and 3pi/2) [f(x)-g(x)] =6

9. Sketch the general shape of the graph $f(x)$on the same graph given the gradient function and state when the function is increasing, decreasing or at stationary point.









NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 23

Topic: Probability Discrete Random Variables

Probability Revision
Random Variables
Discrete probability distributions
Expected value

Probability Revision

Before we start learning discrete probability random variables, we need to do some preparation and revision to understand the concepts in this topic.

$$Probability\ of\ an\ event = \frac{Number\ of\ occured\ events}{total\ number\ of\ events}$$

$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ $\qquad$ $i.e\ \ 4! = 4 \times 3 \times 2 \times 1 = 24$

Condition probability $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$ $\qquad$ *Probability of A occurring given B occurred.*

Number of permutations (arrangements): ${}^{n}P_{r} = \frac{n!}{(n-r)!}$ $\qquad$ $i.e\ {}^{5}P_{3} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$

Number of combinations (selections): ${}^{n}C = \binom{n}{r} = \frac{n!}{(n-r)!r!}$ $\qquad$ $i.e\ {}^{5}C_{3} = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = 10$

Revision

1. **A coin is tossed three times.**
   a. **Draw a tree diagram**

		HHH
	HH	HHT
H-		HTH
	HT	HTT
	TH	THH
T-		THT
	TT	TTH
		TTT

   b. **Find the probability that exactly two tails are obtained.**

   3/8

   c. **Find the probability that exactly two tails are obtained, given that the first toss resulted in a head?**

   1/4

2. **Calculate**

a. $^{7}C_{4}$ $\qquad$ **b.** $^{6}P_{2}$

7!/4!3!=35 $\qquad$ 6!/4!=30

3. **How many different ways can seven dots be arranged if there are three whites, one blue, one black one purple, and one greens and all dot must be displayed?**

210

4. **In a multiple-choice test consist of 5 questions, each of which has 5 options. What is the probability that you can guess all the correct answers?**

0.00032

5. **A card is drawn at random from a normal pack of playing cards. Find the probability that the card is either a red, or a three, or both.**

0.538

Random Variables

The outcome of which the possible outcome is countable is considered to be a discrete random variable. Generally, discrete random variables are usually associated with numbers or size. For example; shoe sizes, number of cans bought, etc.

However, where the listed possible outcomes are not countable, we have a continuous random variable. Generally, continuous random variables are usually associated with height, mass and time. For example, measurement of height or age; the results can range between certain values and depends on the accuracy of the measuring instrument.

Sometimes variables are assigned to a particular outcome from experimentation.

Try listing some measurements and determine whether it's a discrete random variable or a continuous random variable.

Discrete probability distributions

At first we look at the discrete probability distribution. Consider the table below

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This is a *probability distribution* where it gives a list of x values possible and their associated probability. The probability distribution can also be presented vertically.

Here are some properties for the probability distribution

- $The\ probability\ must\ be\ 0 \leq \Pr(X = x) \leq 1$ for all values of x
- $\sum \Pr(X = x) = 1$

These are the conditions needed to have a discrete probability distribution

If we look at the discrete probability distribution before the $\Pr(X = x)$ satisfy the first condition and the sum of the probability, $\sum \Pr(X = x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$, is also satisfied.

Expected value

When it comes to probability, we generally calculated the mean of a set of values. Another way to describe it is to call it the expected value.

The formula to find the expected value is

$$\mu = \sum_{i=1}^{n} x_i \Pr(X = x_i)$$

The expected value of discrete random variable is the sum of each possible outcome multiplied by its probability.

For example

Find the expectation of the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Since we know

$$\mu = \sum_{i=1}^{n} x_i \Pr(X = x_i) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8} = \frac{3}{2}$$

Therefore, the expected value is $\frac{3}{2}$. On average you are expected to get 1.5 as time becomes infinite. If the value of the expected value matches the experimental mean then the game is considered fair or unbiased and if not it is considered biased.

Expectation of a discrete probability is linear

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

Given that $a, b\ and\ c$ are constants and X and Y are random variables

Testing Understanding

1. **Find the value for k which makes the table represent a discrete probability distribution and find the expected value.**

x	0	1	2	3
$\Pr(X = x)$	k	$1/2$	$1/4$	$\frac{1}{8}$

K=0.125

E=1.375

x	0	1	2	3
$\Pr(X = x)$	k	$2k$	k	$1/2$

K=0.125

E=2

x	-1	0	1	2
$\Pr(X = x)$	$2k$	k	$3k$	k

K=1/7

E=0.428

2. **In a box which contains 5 white marbles, three red marbles and two blue marbles. You draw two marbles, with replacement, from the jar. Let X be the number of reds marbles drawn**
   a. **What values can X take**

2,1,0

b. Draw a table to show the probability distribution of X.

x	0	1	2
prX=x	0.49	0.42	0.09

c. Find the expected value of the probability distribution

E=0.6

d. Calculate the probability of drawing exactly one red marble from the box.

0.42

e. Calculate the probability of drawing at least one red marble from the box.

0.51

3. Two fair dice are rolled. Let X be the sum of two dice.

a. **Draw a table to show the probability distribution of X and find the expected value**.

x	2	3	4	5	6	7	8	9	10	11	12
Prx=x	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

b. Find Pr $(X = 7)$

1/6

c. Find Pr $(X > 10)$

1/12

d. Find Pr $(4 \leq X \leq 10)$

5/6

4. Two fair coins are tossed. T stands for the number of tails obtained. Draw a table to show the probability distribution of T and find the expected value.

T	0	1	2
PrT=x	1/4	1/2	1/4

EX =1 = one tail.

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 24

Topic: Probability Discrete Random Variables

Variance and standard deviation
Binomial probability distribution
Binomial probability graph
Expected value, variance and standard deviation of binomial probability

Variance and standard deviation

In statistics we have been interested in the measure of central tendency (average, mean, median, etc) and measure of spread (range, variance, standard deviation, etc).

Variance is the spread of the random variable which is denoted by σ^2 $(Sigma\ squared)$ or $Var(X)$. The larger the value for variance, the more spread out the data.

The variance is the summation of the square difference between the variable and the mean.

$$Var(X) = \sigma^2 = E(X-\mu)^2 = \sum (x_i - \mu)^2 \times \Pr(X = x_i) = E(X^2) - [E(X)]^2$$

However we use standard deviation to measure the spread of the data.

Standard deviation of X is written as $SD(X)$

$$SD(X) = \sqrt{Var(X)} = \sigma$$

For example

1. **Find the expectation, variance and standard deviation of the following distribution.**

x	-1	0	1	2
$\Pr(X = x)$	0.5	0.2	0.2	0.1

$$E(X) = -1 \times 0.5 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.1 = 0.9 Var(X) = E(X^2) - [E(X)]^2$$
$$= 1 \times 0.5 + 0 \times 0.2 + 1 \times 0.2 + 4 \times 0.1 - (0.9)^2 = 1.1 - 0.81 = 0.29$$
$$SD(X) = \sigma = \sqrt{0.29} = 0.539$$

Checking your understanding

1. **Find the values of a and b in the following probability distribution given that the $E(X) = 3.5$**

x	2	3	4	5	6
$Pr(X = x)$	0.3	0.1	a	0.1	b

0.3+0.1+a+0.1+b=1 a+b=0.5 2+2b=2.1

0.6+0.3+4a+0.5+6b=3.5 4a+6b=2.1 b=0.05, a=0.45

a. **Solve for the variances and standard deviation.**

V=1.35 S=1.1

2. **Find the values of a and b in the following probability distribution given that the $E(X) = 1$**

x	-2	0	2	4	6
$Pr(X = x)$	0.3	0.3	a	0.05	b

a+b=0.35

-0.6+2a+0.2+6b=1

2a+6b=1.4

0.7+4b=1.4

b=0.175

a=0.175

b. **Solve for the variances and standard deviation.**

Variance =8 S=2.83

c. **Find the probability when $\Pr(X \geq 0)$**

0.7

d. **Find the probability when $\Pr(X = 2|X \geq 0)$**

0.175/0.7=0.25

Binomial probability distribution

In this type of probability there are only two possible outcomes. Generally we refer the outcomes as success and failure i.e. winning or losing, heads or tails.

The variables in this probability distribution are

- n (number of identical independent trials)
- p (probability of success)
- $q \text{ which can be refer to } 1 - p$ (probability of failure)

Binomial random variable is sometimes referred to as Bernoulli trials, which simply means there are only two possible outcomes; success or failure.

Binomial probability distribution helps us find the probability of multiple combinations i.e. exactly two spades when obtaining three cards with replacements.

For shortcuts we can write probability variables as X. In this case we can write the binomial distribution of n trials with probability p as

$X \sim Bi(n, p) or B(n, p)$

The sample space of picking up three cards and that exactly two are spades is:

$$SS\boldsymbol{S'} \quad S\boldsymbol{S'}S \quad \boldsymbol{S'}SS$$

Generally we work this out as $\Pr(exactly\ two\ spades) = 3 \times \left(\frac{1}{4}\right) \times \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right) = \frac{9}{64}$

In a situation where you are required to find 5 or more combinations this technique is not going to work so well.

However by using binomial probability distribution we can solve these types of probability situations.

$$Binomial\ probability: \quad \Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$Given\ p + q = 1\ the\ probability\ can\ be\ written\ as \quad \Pr(X = x) = \binom{n}{x} p^x q^{n-x}$$

For example:

What is the probability to draw 11 Clubs from 40 draws?

Using binomial probability formula and $n = 40,\ x = 11, p = \frac{1}{4}\ and\ q = \frac{3}{4}$

$$\Pr(11\ Clubs\ from\ 40\ draws) = \binom{40}{11} \frac{1}{4}^{11} \frac{3}{4}^{29} \cong 0.131$$

Example 2

Sam the Shooter has 80% chance to successfully shoot a bulls-eye. Find the probability that if Sam tries shooting five times, he will be successful:

a. Exactly twice **b. More than three times** **c. on the first and third attempt**

Solution A

Find $n, x, p\ and\ q$

$n = 5 \quad x = 2 \quad p = 0.8 \quad q = 0.2$

Using binomial probability formula

$$\Pr(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$\Pr(X = 2) = \binom{5}{2} 0.8^2 0.2^3 = 0.0512$$

Solution B

Find $n, x, p\ and\ q$

$n = 5 \quad x > 3 \quad p = 0.8 \quad q = 0.2$

Using binomial probability formula

$$\Pr(X > 3) = \Pr(X = 4) + \Pr(X = 5)$$
$$= \binom{5}{4} 0.8^4 0.2^1 + \binom{5}{5} 0.8^5 0.2^0$$
$$= 0.4096 + 0.32768$$
$$= 0.73728$$

Solution C

The condition is specified, so no need to use binomial distribution

$Sucessful\ on\ 1^{st} and\ 3^{rd}:\ SFSFF$

$$\Pr(Sucessful\ on\ 1^{st} and\ 3^{rd}) = 0.8 \times 0.2 \times 0.8 \times 0.2 \times 0.2 = 0.00512$$

Remember some properties of probability

$$\Pr(X = a) = 1 - \Pr\,(X = a')$$

$$\Pr(X \geq a) = 1 - \Pr\,(X < a)$$

Note that signs are interchangeable such that following are true as well.

$$\Pr(X < a) = 1 - \Pr\,(X \geq a)$$

$$\Pr(X \leq a) = 1 - \Pr\,(X > a)$$

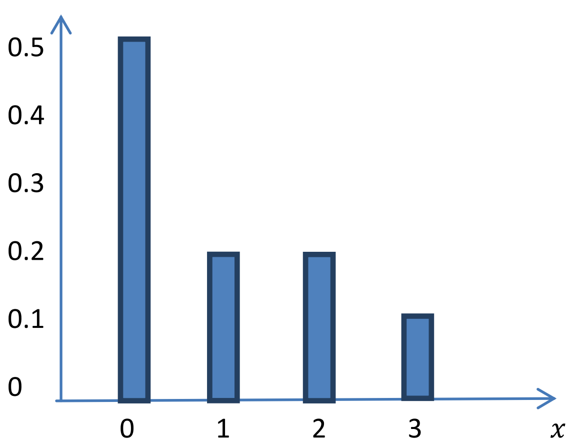
Binomial probability graph

When graphing binomial distribution the independent value is the random variable and the dependent value is the probability of the random variable. *Note the graphs can be drawn as a dot graph.*

Consider the following distribution

x	0	1	2	3
$\Pr\,(X = x)$	0.5	0.2	0.2	0.1

The binomial distribution graph is



From the information given

We have $n = 3\ and\ p = 0.3 < 0.5$

Thus this graph is skewed to the **right or positively skewed**

Now consider the graph below. We see that if $p = 0.5$ we have a symmetrical graph. If $p < 0.5$, the graph is **skewed to the right or positively skewed** (Where the tail is stretch to the right side) while $p > 0.5$ the graph is **skewed to the left or negatively skewed** (Where the tails is stretch to the left side.

$\Pr\,(X = x)$ $\Pr\,(X = x)$ $\Pr\,(X = x)$

0.4 0.4 0.4

In your calculator you can solve for the binomial distribution $\Pr(X = x)$by using the function $binompdf(n, p, x)$

And for cumulative probability $\Pr(X \leq a)$ we use the function $binomcdf(n, p, a)$

For example

Using your calculator solve for $\Pr(X = 0)$, $\Pr(X = 1)$ and $\Pr(X > 1)$ given $X \sim Bi(10, 0.25)$

a. $\Pr(X = 0) : binompdf(10, 0.25, 0) = 0.0563$
b. $\Pr(X = 1) : binompdf(10, 0.25, 1) = 0.1877$
c. $\Pr(X > 1) = 1 - \Pr(X \leq 1)$ *or* $1 - (\Pr(X = 0) + \Pr(X = 1))$
   $: 1 - binomcdf(10, 0.25, 1)$
   $: 1 - (binompdf(10, 0.25, 0) + binompdf(10, 0.25, 1))$

$$: 1 - 0.244 = 0.756$$

Expected value, variance and standard deviation of binomial probability

In binomial distribution, we can solve for the expectation, variance and standard deviation value with ease given that we know the n and p.

In general where $X \sim Binomial\ distribution = Bi(n, p)$ where n represent the number of trials and p is a probability constant of success, we have:

$$E(X) = np$$
$$Var(X) = np(1 - p)$$
$$SD(X) = \sqrt{np(1 - p)}$$

For example

Given $X \sim Bi(10, 0.25)$, find the expected value, variance and standard deviation.

Solution

By following to formula

$E(X) = np = 10 \times 0.25 = 2.5$
$Var(X) = np(1 - p) = 10 \times 0.25(1 - 0.25) = 1.875$
$SD(X) = \sqrt{Var(X)} = 1.37$

Testing Understanding

1. **A drug has been tested 95% chance to cure a person who has the winter flu. If 10 people are tested, find the probability that**

   a. **None are cured**
   9.76*10-14

   b. **At least one is not cured**
   1-Pr(S'=0)=0.4012

   c. **Only the first and fifth person is cured.**
   3.52*10^-11

   d. **Find the expected value, variance and standard deviation of this distribution.**

   E=9.5 V=0.475 S=0.6892

   e. **State whether the graph is positively skewed, negatively skewed or symmetrical graph and why.**

   Negatively skewed because p=0.95>0.5

   f. **Sketch a graph that may represent this probability distribution.**

   Students' own work

2. **If 100 cards are drawn, with replacement, find the probability that**

   a. **Exactly 25 are red**

   1.913*10^-7

   b. **Exactly 30 are spades**

   0.0458

   c. **Exactly 10 are Jacks**

   0.093

   d. **None are threes**

   3.34 *10^-4

3. **In a plant experiment, 50 oak trees were planted to test for surviving to maturity. It is known that the probability of the oak tree survival is 0.4, find the number of oak trees expected to survive. Within what range of number of trees that 95% of the trees would survive.**
   Given that $\Pr(\mu - 2\sigma \leq x \leq \mu + 2\sigma) = 0.95$

E=20

SD=root(varX)=3.46

$\mu - 2\sigma = 13.08$

$\mu + 2\sigma = 26.92$

So $13.08 \leq x \leq 26.92$

4. A scientist with past experience knows that only 80% of the results are correct. If he produced 30 results, what is the probability the he gets

a. Exactly 10 correctly

3.38*10^-8

b. more than 10 correctly

1-3.83^10-8=0.99

c. Find the expected value, variance and standard deviation of this distribution.

E=24 V=4.8 SD=2.19

5. From 100 data, in average only 80 are readable

a. Find the constant probability

0.8

b. Find the variance

16

6. A normal die is rolled fifty times. Find the probability that 20 even numbers are rolled, the expected value for getting a prime number and standard deviation.

0.0419

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 25

Topic: Probability Discrete Random Variables

Markov Sequences

Transition Matrices

Markov Sequences

Markov sequence or Markov Chain is a special type of stochastic process, a mathematical model that changes over time which is based on probability associated with the event in question. In this model, the next condition of the system depends only on the current condition. Generally we use tree diagram to find us find the probability associated with Markov sequences. Mathematically, for a given probability it is represent as

$$\Pr(B) = \Pr(B|A) \times \Pr(A) + \Pr\,(B|A') \times \Pr\,(A')$$

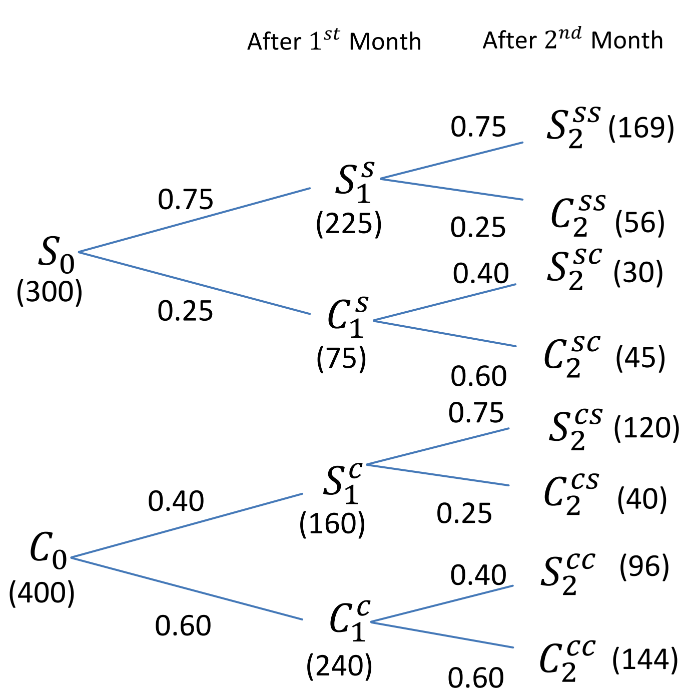
Remember the conditional probability as $\Pr(B|A) = \frac{\Pr(B \cap A)}{\Pr\,(A)}$

Therefore $\Pr(B) = \Pr(B \cap A) + \Pr(B \cap A')$

Here is an example of Markov Sequences

Consumers can shop at Safeway or Coles. Market research shows that in 75% of cases a previous Safeway's shopper will shop from them as well. The research also shows that 60% of Coles' shoppers remained with them. If Safeway has 300 customers and Coles has 400 customers in a particular month, find the number of customers for each in the following month.

Draw a tree diagram to represent the situation.



Using the aid of a tree diagram

The number of consumers in the first month:

Safeway

$$\Pr(S_1) = \Pr(S_1^s|S_0) \times \Pr(S_0) + \Pr(S_1^c|C_0) \times \Pr(C_0)$$
$$\Pr(S_1) = 0.75 \times 300 + 0.40 \times 400 = 385$$

Coles

$$\Pr(C_1) = \Pr(C_1^s|S_0) \times \Pr(S_0) + \Pr(C_1^c|C_0) \times \Pr(C_0)$$
$$\Pr(SC_1) = 0.25 \times 300 + 0.60 \times 400 = 315$$

Using the aid of tree diagram

The number of consumer in the second month:
Safeway

$$\Pr(S_2) = \Pr(S_2^{ss}|S_1^s) \times \Pr(S_1^s) + \Pr(S_2^{sc}|C_1^s) \times \Pr(C_1^s) + \Pr(S_2^{cs}|S_1^c) \times \Pr(S_1^c) + \Pr(S_2^{cc}|C_1^c) \times \Pr(C_1^c)$$
$$\Pr(S_2) = 169 + 30 + 120 + 96 = 415$$

Coles *(Try it yourself. Note these two methods)*

$\Pr(S_2) = 285$

In general $\Pr(A_n)$ is the sum of the conditional probability multiplied with the previous situation that affects the values with A_n

In order to tackle Markov sequence, the best thing to do is **draw the tree diagram** and **label the probabilities**. By doing so, this will help you to produce the *probability formula* to solve for the probability of an event in any given time.

Transition Matrices

An easy and efficient way to solve Markov sequences is using transition matrices. In this case, matrices allows us to obtain information of different variables and at the same time solves for the sequence. The operator we are interested in is the matrix multiplication.

Matrix multiplication can be used only if the number of columns of the first matrix is the same as the number of rows of the second matrices.

Let $Matrix\ A$ of order $m \times n$ and $Matrix\ B$of order $n \times p$

So $A \times B\ exist\ and\ if\ order\ m \times p$ since: $(m \times \boxed{n}) \times (\boxed{n} \times p\ )$

However $B \times A\ does\ not\ exist$!

From the previous example we can set up a matrix to show the situation

	Given from S the previous month	**Given from C the previous month**
From S this month	0.75	0.40
From C this month	0.25	0.60

So the transition matrix is $T = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}$.

Before we solve for Markov sequence, we need a second matrix called **initial state matrix**.

Initial state matrix $S = \begin{bmatrix} 300 \\ 400 \end{bmatrix}$

To find the state after a period of time we multiply the transition matrix with the previous state

$S_0 = \begin{bmatrix} 300 \\ 400 \end{bmatrix}$ Original State

$S_1 = T \times S_0 = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix} \begin{bmatrix} 300 \\ 400 \end{bmatrix} = \begin{bmatrix} 385 \\ 315 \end{bmatrix}$

$S_2 = T \times S_1 = T \times (T \times S_0) = T^2 \times S_0 = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}^2 \begin{bmatrix} 300 \\ 400 \end{bmatrix} = \begin{bmatrix} 415 \\ 285 \end{bmatrix}$

$S_3 = T \times S_2 = T^3 \times S_0$

In general, the state at any given time, n, can be found using

$$S_n = T^n \times S_0$$

As time increases, you will see that matrix converges to a same result. This is known as the **steady state matrix** and it represents the values of the matrix converging as time reaches infinity.

Using the example before

$S_5 = T^5 \times S_0 = \begin{bmatrix} 430.0824 \ldots \\ 269.9175 \ldots \end{bmatrix}$ $S_{10} = T^{10} \times S_0 = \begin{bmatrix} 430.7656 \ldots \\ 269.2344 \ldots \end{bmatrix}$ $S_{20} = T^{20} \times S_0 = \begin{bmatrix} 430.7692 \ldots \\ 269.2307 \ldots \end{bmatrix}$

The steady state matrix is

$$T \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Using the previous example

$$\begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

We can break the matrix into linear combination:

$0.75x + 0.4y = x$ $-0.25x + 0.4y = 0$
$0.25x + 0.6y = y$ $0.25x - 0.4y = 0$

However we know that $x + y = 700$ (Number of customers)

Then we have:

$$0.25x = 0.4y = 0.4(700 - x)$$
$$0.65x = 280$$
$$x = 430.77 \therefore y = 269.23$$

Hence when time reaches to infinity the matrix or solution reaches to the values $\begin{bmatrix} 430.77 \\ 269.23 \end{bmatrix}$

Transition matrix can be used to any number of events. Although the conditions for a transition matrix is it will always be square and the sum of each column must be 1.

Testing Understanding

1. **The weather is recorded and it indicated the probability of raining day following a raining day in winter is 0.55, whereas the probability of a sunny day following a sunny day in that season is only 0.4. Given the first day of winter has the same chance of raining and sunny day.**

   a. **What is the probability of a raining day and sunny day on the first day?**

      Since same chance then 50% raining and 50% sunny

   b. **Draw a tree diagram representing this Markov sequence**

0.55 **0.275 ->** 0.55/0.45 -> **0.15125/0.12375**

0.5 0.45 **0.225->** 0.6/0.4 ->**0.135/0.1**

0.5 0.6 **0.30>** 0.55/0.45->**0.165/0.135**

0.4 **0.20->** 0.6/0.4-> **0.12/0.08**

   c. **Determine the probability of a sunny day on the second day. Write explicitly using the probability formula**

P(S)=P(S|R)*P(R)+ P(S|S)*P(S)=**0.275+0.30=0.575**

   d. **Determine the transition matrix and using it to clarify your result for c .**

0.55	0.6
0.45	0.4

   e. **Using the transition matrix to find the probability of raining and sunny day on the third day**

T^3*S= 0.5714

0.4285

f. **Approximate what is the probability of the raining and sunny day converges to? State how you approximate it**

0.571428 By letting n= very large i.e. n=10

0.4285714

g. **Find the exact value for the steady state values.**

0.6y=0.45(1-y) y=3/7 x=4/7

2. **Sam and Tim often play tennis. They start with even chance of winning but if Tim wins than his confidence increases and he is 80% likely to win their next game. However if Tim loses then he has only 30% chance of winning the next game.**

a. Draw the tree diagram representing this sequence up to 3rd game

0.80 **0.40 ->** 0.80/0.20 -> **0.32/0.08**
T 0.5 0.20 **0.10->** 0.30/0.70 ->**0.03/0.07**

S 0.5 0.30 **0.30>** 0.80/0.20->**0.24/0.06**
0.70 **0.20->** 0.30/0.70-> **0.06/0.14**

b. **Using the information given in the tree diagram, what is the probability of Tim and Sam draws up to the 2nd game?**

PR(DRAW)=PR(TS or ST at second)=PR(S|T)*PR(T)+PR(T|S)*PR(S)=0.1+0.3=0.4

c. **Find the transition matrix and solve for the probability of Tim and Sam winning the second and third game.**

0.8 0.3 second game of winning T=0.575 and S=0.425

0.2 0.7 Third game of winning T=0.5875 S=0.4125

d. **If the match end when the first player wins two games. What is the chance of Sam winning the match?**

MATCH for Sam to win is S1S2 or S1T2S3 or T1S2S3

S1S2=0.5*0.425=0.2125 So addition of all of them =0.418756
S1T2S3=0.5*0.575*0.4125=0.1186

T1S2S3=0.5*0.425*0.4125=0.087656

e. **In the long run, who will likely to win most games?**

Let n=large such T^n *S= 0.6 Tim is likely to win the most!
0.4

3. Past records indicated if a train is late on one day there is 60% probability that the same train will be late the next day. If the train is on time there is 10% chance it will be late the next day.

a. Find the probability the train is on time on Wednesday given it was late on Monday.

0.7

b. Find the probability the train is late on Friday given it was on time on Monday.

0.19375

c. Years later, what can you comment about the train record?

Regardless the initial value: The train is most likely to be on time

4. 15 players can choose to be in Red, Blue, or Green. At first players divide themselves equally into coloured teams. After each game, players can switch in to different teams or stay so that each team has a fair chance of winning. It is informed that

- **50% of previous Red remains in Red**
- **The remaining previous Red equally divided between the other two teams**
- **20% of previous Blue remains with Blue**
- **30% of previous Blue switches to Red**
- **50% of previous Blue switches to Green**
- **50% of previous Green switches to Red**
- **20% of previous Green switches to Blue**
- **30% of previous Green remains with Green**

a. What is the transition matrix for this team in the order stated?

R	0.5	0.3	0.5	or likewise
B	0.25	0.2	0.2	
G	0.25	0.5	0.3	

b. What is the initial state matrix?

5

5

5

c. How many players remained in Red on the second and third game?

6.5 6.85 or round off to the nearest whole number

d. How many players are there in the Blue team after the second and fourth game?

2nd game 3.25 3rd game 3.34 general solution should be 3 for both

e. Overall, what can you comment about the teams distribution when the fair?

Blue has the fewest members while Red has the most members. This state that Blue has experienced or better players!

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 26

Topic: Probability Discrete Random Variables

Revision and Preparation
Continuous Probability distribution
Expectation and variance of continuous probability distribution
Median and mode of continuous probability distribution

Revision and Preparation for Continuous Probability

Before we start learning continuous probability we need to revise on some previous topics that relates to it. Make sure you know your anti-differentiation and its techniques as it is used most frequently in this topic.

Here are some key concepts that you need to know

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq 1$$

$$|x| takes\ the\ positive\ value\ regardless\ to\ its\ sign$$

$$Condition\ Probability = \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr\,(B)}$$

$$Binomial\ Probability: X{\sim}Bi(n,P) \quad then\ E(X) = np\ \ and\ Var(X) = np(1-p)$$

Revision

1. **Integrate the following**

a. $\int_2^5 5x^3\,dx$ **b.** $\int_0^4 x^{\frac{5}{2}}\,dx$ **c.** $\int_1^2 -\frac{1}{x^2}\,dx$

5x^4/4 ->461.250

4/7 x^(7/4)-> 9.1428

(1/2) (1/x^3)-> -0.4375

2. **Find the area enclosed by the equation $y = x^2 - 4x$ and the x axis.**

32/3

3. **Solve for the values of x given $|-x+4| < 3$**

$$|-x+4| => \ -x+4 < 3\ if\ x < 4 \quad : \ x > 1$$

$$x - 4 < 3\ if\ x > 4 \quad : x < 7$$

$$1 < x < 7$$

4. **Given $A = \{2,3,4,5,7,8,9,10,12,15\}$. A number is drawn from set A**

a. $\Pr(x < 5|\ 3 < x \leq 10)$

1/6

b. $\Pr\,(\text{even number}|\text{divisble by } 3)$

1/4

5. **Consider $X \sim Bi(5, 0.3)$**

   a. **Find the expectation and standard deviation**

E=1.5 sd=1.02

   b. **Find $\Pr(X = 1)$**

0.36015

   c. **Find $\Pr(X < 2)$**

0.52822

   d. **Find $\Pr(X \geq 2)$**

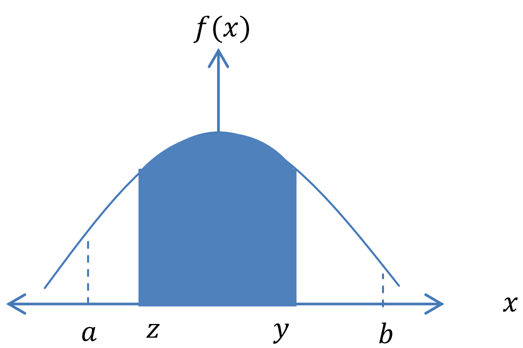
0.47178

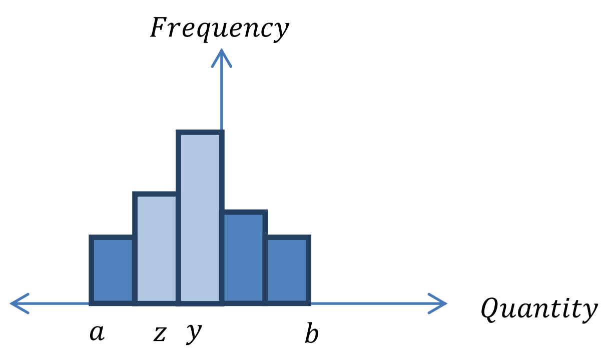
   e. **Find $\Pr(1 \leq X < 3)$**

0.66885

Continuous Probability distribution

Continuous random variable is when a variable can take any value over an interval. Quantities such as height, weight, and time can be modeled using continuous probability distribution.





We model the quantity against its frequency and the area under the graph is effectively the probability. Sometimes we use formulas to approximate the area under the graph such as the upper rectangular rules. If possible if we have the function that describes the distribution we can integrate it and solve the probability over the interval.

In continuous random variable, the probability of an exact value is zero.

$$\Pr(X = x) = 0 \; for \; all \; possible \; value \; of \; x$$

However over an certain interval the probability exist

$$\Pr(z \leq x \leq y) = \int_z^y f(x)\, dx \neq 0$$

The above is define as $probability\ distribution\ function\ (pdf)\ of\ X$

Another key feature is that the graph cannot go below the horizontal axis since the probability must be positive.

And the probabilities must lie between $0\ and\ 1$ so that the graph area $\int_a^b f(x)\,dx = 1$

Refer to the two graphs for visual interpretation of the continuous probability conditions.

In general for the function $\boldsymbol{to\ be\ probability\ density\ function}$:

- $f(x) \geq 0 for\ all\ values\ for\ x \quad i.e\ must\ be\ positive$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad i.e\ the\ area\ enclosed\ by\ the\ function\ and\ the\ x\ axis\ must\ be\ 1$

Since $\Pr(X = x) = 0$ in continuous probability, the following expression have the same values!

$$\mathbf{Pr}(\boldsymbol{a} < X < b) = \mathbf{Pr}(\boldsymbol{a} \leq \boldsymbol{X} < b) = \mathbf{Pr}(\boldsymbol{a} < X \leq b) = \boldsymbol{Pr}\,(\boldsymbol{a} \leq \boldsymbol{X} \leq \boldsymbol{b})$$

Example

Show that $f(x) = \begin{cases} \frac{2}{3}(x+1) & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$ is a probability distribution function and find $\Pr(x < 0.7)$

Solution to show it is a probability distribution function

Firstly, to satisfy as a probability density function then $\int_{-\infty}^{\infty} f(x)\,dx = 1$

So $\frac{2}{3}\int_0^1 x + 1\,dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^1 = \frac{2}{3} \times \frac{3}{2} = 1$ Thus is a probability distribution function!

Solution to find $\boldsymbol{Pr}\ (\boldsymbol{x} < 0.7)$

Secondly, for $\Pr(x < 0.7)$ and since the lower limit is restricted from $0 \leq x \leq 1$ then the probability is the same as $\Pr(x < 0.7) = \Pr(0 < x < 0.7)$. By using $\Pr(z \leq x \leq y) = \int_z^y f(x)\,dx$ and the conditions of probability i.e $\Pr(X = x) = 0$

$$\Pr(x < 0.7) = \frac{2}{3}\int_0^{0.7} x + 1\,dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^{0.7} = 0.63$$

General note, if the integral evaluated to be more than 1 then you have done something wrong because the entire area of the function as stated of one of the condition is only 1. The solution of the integral must lie between $0 \leq \int_z^y f(x)\,dx \leq 1$

Expectation, variance, median and mode of continuous probability distribution

The *expectation* of a continuous probability distribution is found by

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

The *variance* of the continuous probability distribution is found by

$$Var(x) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} xf(x)\,dx\right]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

The *standard deviation* is the square root of the variance.

You will realise that the mean/expected value and the variance of the continuous and discrete are very similar.

The *median* of the value is such that $\Pr(X < m) = 0.5$ so

$$\int_{-\infty}^{m} f(x)\,dx = 0.5$$

The *mode* is simply the x-value of the maximum value of the function

$$x_{mode}\ where\ f(x_{mode}) = \max(|f(x)|)$$

Example

Given that $f(x) = \begin{cases} \frac{2}{3}(x+1) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$ is a probability distribution function, find the mean, the standard deviation, the median and the mode.

Solution (Mean)

1. Find $xf(x)$

   $xf(x) = \frac{2}{3}(x^2 + x)$
2. Solve for the integral $\int_{-\infty}^{\infty} xf(x)\,dx$ (Note we are finding the integral where its non-zero)

$$\int_{-\infty}^{\infty} xf(x)\,dx = \frac{2}{3}\int_0^1 (x^2 + x)\ dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{9}$$

$$Therefore\ E(X) = \frac{5}{9}$$

Solution (SD)

1. Find $x^2 f(x)$
   $x^2 f(x) = \frac{2}{3}(x^3 + x^2)$
2. Solve for the integral $\int_{-\infty}^{\infty} x^2 f(x)\,dx$ (Note we are finding the integral where its non-zero)
   $$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{2}{3}\int_0^1 (x^3 + x^2)\;dx = \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = 7/18$$
3. Evaluate $Var(x)$
   $$Var(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = 13/162$$
4. $SD(x) = \sqrt{Var(x)} \cong 0.2833$

Solution (Median)

Equate $\int_{-\infty}^{m} f(x)\,dx = 0.5$

$$\int_{-\infty}^{m} f(x)\,dx = \int_0^m \frac{2}{3}(x+1)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^m = \frac{2}{3}\left(\frac{m^2}{2} + m\right)$$

So $\frac{2}{3}\left(\frac{m^2}{2} + m\right) = 0.5$ and using quadratic techniques $2m^2 + 4m - 3 = 0$

$$Using\ quadratic\ method\ m = -2.59\ or\ 0.59$$

Since $m\ is\ between\ 0 \leq x \leq 1$ then $m = 0.59$

The Median is $x_{median} = 0.59$

Solution (Mode)

Graph or observe $f(x) = \begin{cases} \frac{2}{3}(x+1) & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$

You will realise the maximum value is when $x = 1$ so therefore Mode is $x_{mode} = \frac{4}{3}$

Testing Understanding (Show full working out)

1. **Find a if the following function is a probability density function**

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le a \\ 0 & otherwise \end{cases}$$

Integrate 3x^2 between 0 and a =1
A^3=1 so A=1

b. **Find the probability $\Pr(X < 0.2)$ and $\Pr(X > 0.3)$**

0.008 and 0.973

c. **Find the mean and standard deviation of the probability density function**

Mean =0.75 sd=0.19365

d. **median and mode of the probability density function**

median =0.7937 mode =x=1

2. **Show that $f(x) = \begin{cases} \frac{1}{2}\sin x & 0 \le x \le \pi \\ 0 & otherwise \end{cases}$ is a probability density function**

=0.5cosx|=> 1

a. **Find the mean in two decimals, median and the mode in exact values.**

Mean=1.57

Median= pi/2

Mode = pi/2

b. **Find the probability $\Pr\left(\frac{\pi}{4} < x < \frac{\pi}{2}\right)$**

0.35355

3. **Consider $f(x) = \begin{cases} A|x-1| & 0 \le x \le 2 \\ 0 & otherwise \end{cases}$ is a probability density function**

a. **Find the value for A**

A=1

b. Find the mean, standard deviation, median and mode

Mean =0.1666 sd=0.2357 median = 1 mode = 0 and 2

c. i. Find $\Pr(x < 0.5)$

0.375

i. Find $\Pr(x < 1.5)$

0.625

d. Find $\Pr(0.5 < x < 1.5)$

0.25

Extended Responses

1. Let number of years t and an age of wild animal represented by a continuous random variable with probability density function $f(x) = \begin{cases} Ae^{-x} & 0 \leq x \leq \infty \\ 0 & otherwise \end{cases}$

a. Find the value for A

A=1

b. i. Find $\Pr(X < 2)$

0.86

ii. Find $\Pr(X > 5)$

0.0067

iii. Find $\Pr(3 < x < 6)$

0.0473

c. Find $E(X)$ and $Var(X)$

E(X) =1 Var(X)=1

d. Find the median and comment the difference between the median and $\Pr(X < E(X))$

X median=0.6931

probability of x<mean =0.6321

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 4 TERM 3 WORKBOOK

WEEK 27

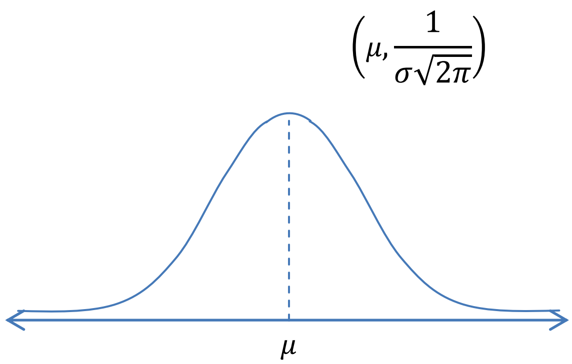
Topic: Probability Discrete Random Variables

Normal distribution

Standard normal distribution

Normal Distribution

vThe normal distribution graph is characterized by a symmetrical bell curve which means the mean, median and mode all coincide for the probability distribution. The graph extends infinitely left and right; as it further approaches infinite of both sides it converges to zero. The bell curve is always located above the x axis.



Normal Distribution

- Symmetric bell shape
- $\text{Mode} = \text{Median} = \text{Mean}$
- Maximum value at the axis of symmetry.
- Reaches to zero when approaches to negative or positive infinity.

This makes it a very useful probability distribution for continuous random variables. Normal distributions are often used to graph variables such as concentration of population, heat distribution and even VCE scores!

The equation is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$

Where μ *is the mean*, σ *is the standard deviation*

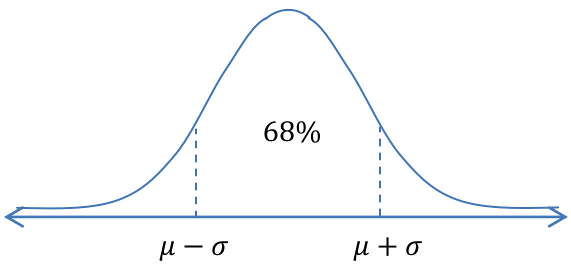
Deriving the solution is very difficult and too advanced for this course, so you don't need to worry about it!

By observing the graph we realise the maximum value occurs when $x = \mu$. And when the x value substitute back into the function we obtain $f(x = \mu) = \frac{1}{\sigma\sqrt{2\pi}}$

The normal distribution satisfy the conditions of probability density function hence, given $f(x) \geq 0$,

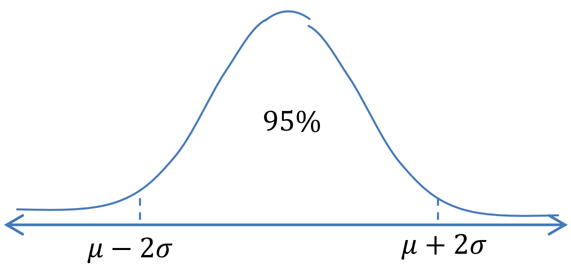
$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} dx = 1$$

One of the key features of normal distribution which is mostly used is the probabilities, the total area, over the interval of standard deviations.



$$\Pr(\mu - \sigma < x < \mu + \sigma) \approx 0.68$$

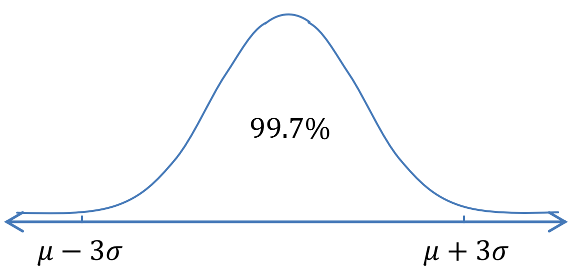
Where the interval lies within one stand deviation of the mean



$$\Pr(\mu - 2\sigma < x < \mu + 2\sigma) \approx 0.95$$

Where the interval lies within two stand deviation of the mean

It is said that the observation is **very likely** to occur within this range



$$\Pr(\mu - 3\sigma < x < \mu + 3\sigma) \approx 0.997$$

Where the interval lies within three stand deviation of the mean

It is said that the observation is **certain** to occur within this range

Since it's symmetrical, by halving the shapes you'll get half of the area/ percentage!

For example

If we knew that the VCE Unit 3 and 4 Score results of graduating students was normally distributed with a mean 30 and a standard deviation of 7, we would expect:

$For\ \mu \pm \sigma\ range - 68\%\ which\ gives\ the\ range\ 23\ to\ 37$

$For\ \mu \pm 2\sigma\ range - 95\%\ which\ gives\ the\ range\ 16\ to\ 44$

$For\ \mu \pm 3\sigma\ range - 99.7\%\ which\ gives\ the\ range\ 9\ to\ 51$

You might want to draw the shape when it's in half so its 34%, 47.5% etc. Since it symmetrical!

With further analysis

47 *Top 0.75%* **44** *Top 2.5%* **40** *Top 8%* **37** *Top 16%* **35** *Top 25%* **30** *Top 50%* **25** *Top 75%*

Using CAS

Firstly we define $n(m, s, x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-m}{s}\right]^2}$ where m is the mean and s is the standard deviation.

Then type $\int_{-\infty}^{\infty} n(30,7,x)dx$. You answer should be 1.

Older calculators use $normalcdf(lower\ limit, upper\ limit, mean, standard\ deviation)$!

In this case $normalcdf(0, x, 30,7)$

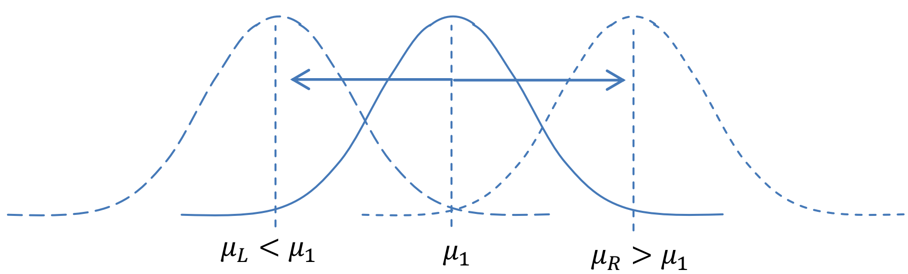
Play around with the upper limits to check the previous solutions!

Characteristic of the Normal distribution

Tutors should go through the characteristic of the Normal distribution. Ask the students to enter different value for mean and SD on their calculator so they can visualize the shape

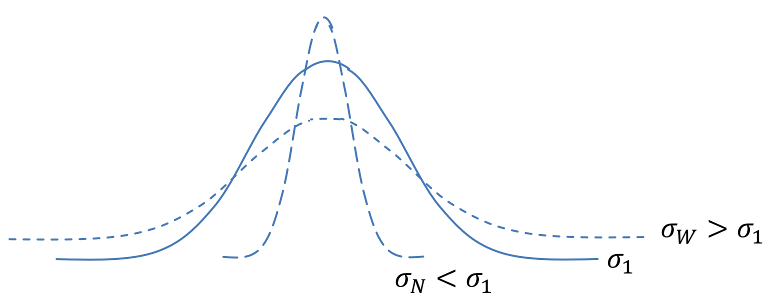
Changes of μ

- Increasing the value μ shifts the curve to the right
- Decreasing the value μ shifts the curve to the left



Changes of σ

- Increasing σ makes the curve flatter and wider
- Decreasing σ makes the curve tall and narrower



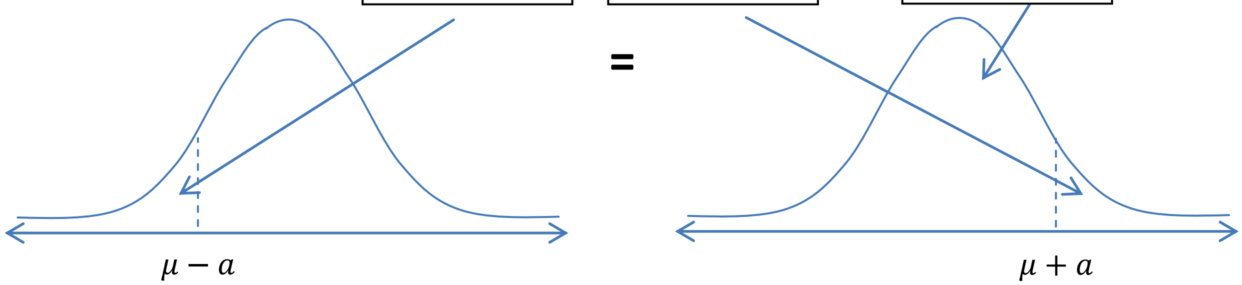
Regardless what has changed in $\mu\ and/or\ \sigma$ the area under the graph is one!

Also, if a continuous random variable X has a normal distribution with expected value μ and variance σ^2 , it can be denoted as $X \sim N(\mu, \sigma^2)$

Here are some properties of normal distribution!

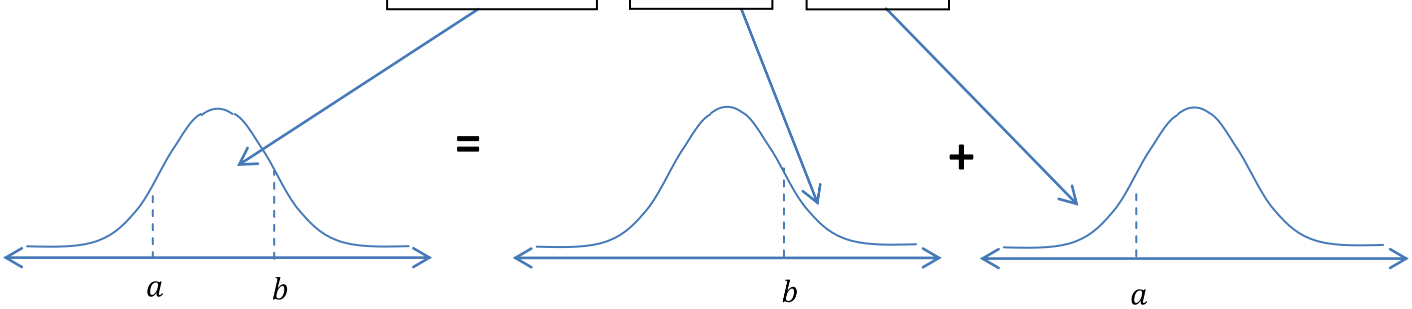
Where a is the distance from the mean and

$$Pr(X < \mu - a) = Pr(X > \mu + a) = 1 - \Pr(X < \mu + a)$$



And

$$\Pr(a < X < b) = \Pr(X < b) - \Pr(X < a)$$



Try solving these by yourself and using the above techniques!

If $X \sim N(50,100)$, *solve for*

a. $\Pr(X > 65)$

b. $\Pr(35 < X < 65)$

c. $\Pr(55 < X < 70)$

0.0668

0.86639

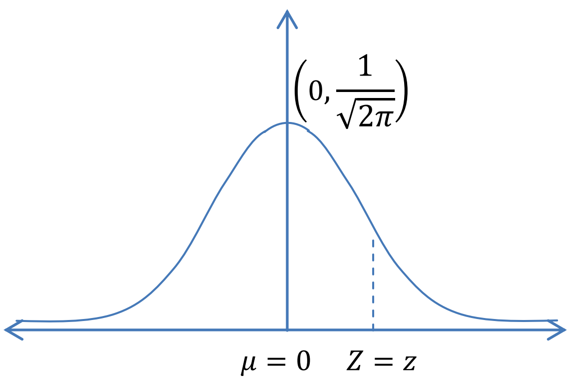
0.28579

Standard normal distribution

Solving for probability using the normal distribution is quite difficult unless we only wanted to find the 68%, 95% and 99.97% ranges for the variable. Although, there are tables of values produced for the standard normal distribution. Standard normal distribution, denoted as the letter Z, has a mean of 0 and a variance 1 and usually it is written as $Z \sim N(0,1)$.

If the distribution is not standard normal distribution we can use our CAS to complete the calculation. However you will need to know how to read off the table and transposing normal distribution to standard normal distribution.

$$If\ X \sim N(\mu, \sigma^2)\ then\ Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$



For standard normal distribution

$$\Pr(Z < z)\ or\ \Pr(Z \le z)$$

As represent on the graph on the left.

Note $\Pr(Z < 0) = 0.5$ due to the symmetry.

If you look at the percentage range

$For\ \mu \pm \sigma\ range - 68\%\ which\ gives\ the\ range - 1\ to\ 1 \qquad so\ \ Pr(-1 < Z < 1) = 0.68$

$For\ \mu \pm 2\sigma\ range - 95\%\ which\ gives\ the\ range - 2\ to\ 2 \qquad so\ \ Pr(-2 < Z < 2) = 0.95$

$For\ \mu \pm 3\sigma\ range - 99.7\%\ which\ gives\ the\ range - 3\ to\ 3 \qquad so\ \ Pr(-3 < Z < 3) = 0.997$

Without a CAS, you are required to read off the table to find the value of $\Pr\ (Z < z)$ and it is not that hard to do.

Let's say you got given $\Pr\ (X < x)$ but it is not a tech active question!

Firstly you need to convert the distribution to standard normal so

$$\Pr(X < x) = \Pr\left(\frac{X-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) = \Pr\left(Z < \frac{x-\mu}{\sigma}\right) = \Pr\ (Z < z)$$

Afterwards use the table to find $z = \frac{x-\mu}{\sigma}$ so that you can solve for $\Pr\ (Z < z)$

How to read from the table

Part 2: Then read the **'hundredth'** on first row and obtain the value.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	1	2	3	4	5	6	7	8	9
…																			
1.0	0.8413	0.8438	0.8461	0.8485	0.8505	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	11	14	16	18	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	19
…																			

Part 1: Read the '**unit and tenth digit'** on first column

Part 3: If there is a **thousandth** digit, read the **right row** and match on the same unit and tenth digit row. Add the value $\times 10^{-4}$ to **Part 2**

Using table to find $\mathbf{Pr}\ (\boldsymbol{Z} < 1.0)$

First, we look at the 1.0 *row* in the table and secondly, look at the value 0.00 *column.*

So the value $\Pr(Z < 1.0) = 0.8413$

Using table to find $\mathbf{Pr}(\boldsymbol{Z} < 1.05)$

First, we look at the 1.0 *row* in the table and secondly, look at the value 0.05 *column.*

So the value $\Pr(Z < 1.05) = 0.8531$

Using table to find $\mathbf{Pr}(\boldsymbol{Z} < 1.051)$

First, we look at the 1.0 *row* in the table and secondly, look at the value 0.05 *column*.

Since the thousandth is 1, look at the right 1 column and you obtain a number 2 which is 0.0002. The value 0.0002 is added onto the $\Pr\ (Z < 1.05)$ value

So the value $\Pr(Z < 1.0) = 0.8415$

Using table to find $\mathbf{Pr}(\boldsymbol{Z} > 1.155)$

$$\mathbf{Pr}(\boldsymbol{Z} > 1.155) = \mathbf{1} - \mathbf{Pr}(\boldsymbol{Z} < 1.155)$$

First, we look at the 1.1 *row* in the table and secondly, look at the value 0.05 *column*.

So $\Pr(Z < 1.15) = 0.8749$

Afterwards look at the right 5 column, because of the thousandth digit is 5, and you obtain a number 10 which is 0.0010. The value 0.0010 is added into $\Pr(Z < 1.15)$ value.

Therefore $\Pr(Z < 1.155) = 0.8759$

And $\Pr(Z > 1.155) = 1 - 0.8759 = 0.1241$

Test the students by giving them a question to do i.e pr(Z<1.029) on the whiteboard

Example

Let X be a random variable $X \sim N(8,2)$, a normal distribution with mean 8 and variance 2.

a. Find the Z value which would be used to represent a x value of 11.
b. Find $\Pr\,(X < 11)$.
c. Find the $X \; and \; Z$ ranges when the area is 68%
d. Find $\Pr(6 < X < 10)$

Solution to a:

First Step: Identify the mean and standard deviation

$\mu = 8$
$\sigma = \sqrt{2}$

Second Step: Formulate $Z = \frac{x-\mu}{\sigma}$
$Z = \frac{11-8}{\sqrt{2}} \cong 2.121$

Solution to b:
Which means $\Pr(X < 11) = \Pr(Z < 2.121) \approx 0.983 \quad (Using\ calculator)$

Solution to c:
We know $\Pr\,(\mu - \sigma < x < \mu + \sigma) \approx 0.68$

For $X \sim N(8,2)$ and $Z \sim N(0,1)$

Then
$\Pr\left(8 - \sqrt{2} < X < 8 + \sqrt{2}\right) \approx 0.68$

$\Pr\,(-1 < Z < 1) \approx 0.68$

Solution to d:
Find $\Pr(6 < X < 10)$

Use calculator: define$n(m,s,x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-m}{s}\right]^2}$ then equate $\int_6^{10} n\left(8, \sqrt{2}, x\right)dx = 0.842$

If Calculator-Free

Change into standard normal distribution $\Pr(6 < X < 10) = \Pr\,(-1.414 < Z < 1.414)$
$$\Pr(-1.414 < Z < 1.414) = \Pr(Z < 1.414) - \Pr(Z < -1.414) = 0.9213 - 0.0786 = 0.842$$

Testing Understanding

1. **$X \sim N(5, 9)$. Find the range of the x values in which you would expect to find:**

a. **68%** b. 99.7% c. 16%

$2<x<8$ $-4<x<14$ $x<2$

2. **CAS-FREE: Let X (hours, taking 12am as reference) be the time taken for humans to wake up on a Monday morning. It was found to follow a normal distribution where $X \sim N(7, 4)$.**
   a. **Find the time range in which you expect to find the middle 68%**
   $5<x<9$
   b. **Find the percentage of humans you would expect to wake up before 9am**

50+34=84

   c. **Find the percentage of humans you would expect to wake up after 11am**

½(1-95)=2.5%

   d. **Find the percentage of humans you would expect to wake up from $5\text{am to }1\text{pm}$**

34+1/2(99.7)=83.85%

3. **A company collects apples such that the weight of the apple is distributed as $W(grams) \sim N(60, 9)$. Each apple is weighted to test for quality.**
   a. If the company rejects apples weighted less than 54g, what is the probability of apples rejected?

1-0.5*0.95=0.525

   b. **If there were 50,000 apples weighted, how many apples were kept?**

(1-0.525)*50000=23750

The company realised that they should categorise the weight of the apple into medium, large or grand.

   c. **Using standard normal distribution table find the probability of an apple being medium given weight is between $63.063 < W < 63.306$.**
      i) **Find the range in standard normal form, Z.**

$1.021<Z<1.102$

      ii) **Solve for the probability of the standard normal range. Show each step and working out.**

0.0184

d. **Using your CAS, find the probability of an apple being large, given weight is between $63.307 < W < 66$**

0.1124

e. **Using your CAS, find the probability of an apple being grand, given weight is between $W > 66$**

0.0228

4. **CAS-FREE: A particular day temperature in Australian Summer is graphed as a normal distribution with mean 33 and variance 9.**

a. **Find the probability of a particular day when the temperature is less than 30**

0.16

b. **Find the probability of a particular day when the temperature is more than 30**

0.74

c. **Find the probability of a particular day when the temperature is more than 36**

0.025

d. **Given that forecast predicted the temperature is less than 36, what is the probability of the particular day is less than 30.**

0.74/0.975=0.75899

e. **Given that forecast predicted the temperature is more than 36, what is the probability of the particular day is less than 39.**

0.025/0.9985=0.02503

5. **In a bowling match, James scored 90 while Bob scored 95. If James' scores follow a normal distribution $J \sim N(95, 4)$ and Bob's scores follow a normal distribution $B \sim N(100, 9)$, find:**

a. **Who has done better on the day compared to their usual performance?**

J~Pr(Z<-2.5) B~Pr(Z<-1.67)

Since Pr of B is higher in standard normal distribution then Bob performed better

b. **The score James would need to score to have a comparative performance equal to Bob's**

-1.67=(x-95)/2
James need 91.67 or 92