MATHEMATICAL METHODS (CAS) UNITS 3 & 4

2014 Trial **EXAMINATION 2**

July 2014

Solutions

Section A: CAS and reference book permitted

There is a total of 47 marks available for this section.

Reading Time: 15 minutes
Writing time: 1 hour

Instructions to students

Section A has 2 parts. Part 1consists of 12 multiple-choice questions, which should be answered on the answer sheet supplied. Part 2 has 3 extended-answer questions.
All questions should be answered in the spaces provided.
The marks allocated to each of the questions are indicated throughout.
An exact answer is required for all questions unless specified otherwise.
Where more than one mark is allocated to a question, appropriate working must be shown.
Diagrams in this trial exam are not drawn to scale.

Part 1: Multiple Choice. Choose the best answer and circle it on the answer sheet supplied.

Question 1

The maximal domain of the function $y = \log_e(2x+1)$ is

A. $x \in R$

B. $x \in R \setminus \left\{-\frac{1}{2}\right\}$

C. $x \in \left[-\frac{1}{2}, \infty\right)$

D. $x \in \left(-\frac{1}{2}, \infty\right)$

E. $x \in \left[\frac{1}{2}, \infty\right)$

Question 2

The simultaneous linear equations

$$2x + (k-2)y = 2$$
$$(k+1)\,x + 2y = -1$$

have no solutions for

A. $k = -2$

B. $k = 3$

C. $k \in \{-2, 3\}$

D. $k \in R$

E. $k \in R \setminus \{-2, 3\}$

Question 3

The rate of change of the function $y = \dfrac{e^x - e^{3x}}{x^2}$ with respect to *x*, at the point where $x = 2$ is

A. e^{-6}

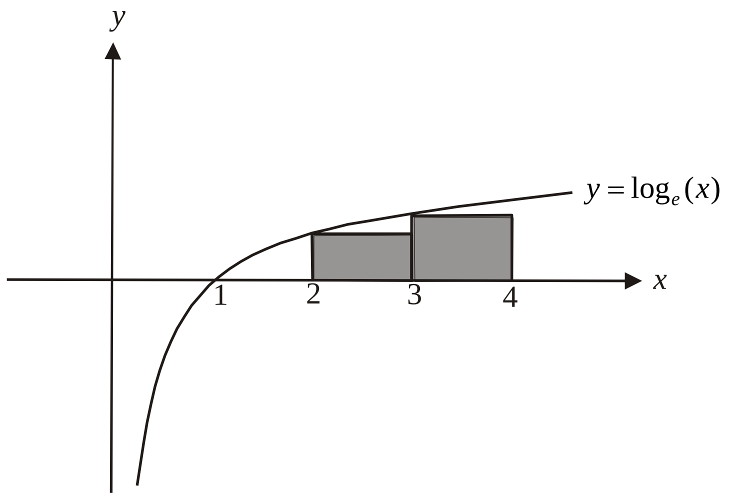
B. $\dfrac{-e^6}{2}$

C. $e^2 + e^{-4}$

D. $e^2 - \dfrac{e^6}{2}$

E. $\dfrac{-e^2(e^4-1)}{4}$

Question 4



The area under the curve $y = \log_e(x)$ between $x = 1$ and $x = 4$ is approximated by the two shaded rectangles shown above. This approximate area in square units is

A. $\log_e(1.5)$
B. $\log_e(5)$
C. $\log_e(6)$
D. $2\log_e(2)$
E. $3\log_e(2)$

Question 5

The graph of $y = x - 3$ and $y = x^2 + kx - 1$ do **not** intersect for

A. $k = 1 \pm 2\sqrt{2}$
B. $-1 - 2\sqrt{2} < k < -1 + 2\sqrt{2}$
C. $1 - 2\sqrt{2} < k < 1 + 2\sqrt{2}$
D. $k < -1 - 2\sqrt{2}$ and $k > -1 + 2\sqrt{2}$
E. $k < 1 - 2\sqrt{2}$ and $k > 1 + 2\sqrt{2}$

Question 6

The average rate of change of the function $y = |(x-1)(x-5)|$ between $x = 2$ and $x = 3$ is

A. -1
B. 0
C. $\frac{1}{6}$
D. 1
E. 7

Question 7

Let $f:[2,\infty)\to R,\ f(x)=\sqrt{x-2}$. The inverse function f^{-1} is given by

A. $f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=x^2+2$

B. $f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=(x+2)^2$

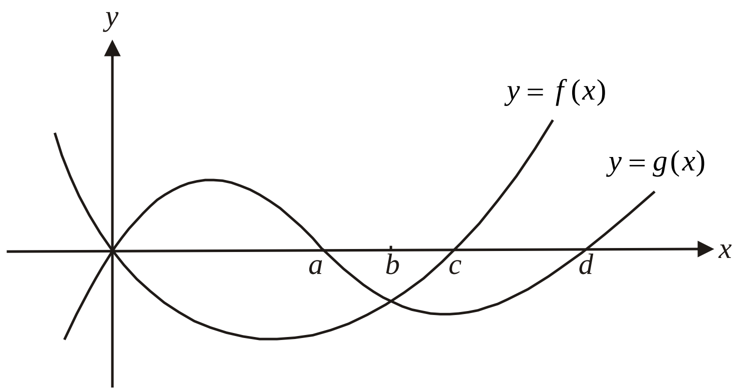
C. $f^{-1}:[2,\infty)\to R,\ f^{-1}(x)=x^2+4$

D. $f^{-1}:[2,\infty)\to R,\ f^{-1}(x)=x^2+2$

E. $f^{-1}:[0,\ 2]\to R,\ f^{-1}(x)=(x+2)^2$

Question 8

The graphs of $y=f(x)$ and $y=g(x)$ are shown below.



The area enclosed by these two graphs is given by

A. $\int_0^b (g(x)-f(x))dx$

B. $\int_0^b (g(x)+f(x))dx$

C. $\int_0^a (g(x))dx-\int_0^b f(x)dx$

D. $\int_0^a (g(x))dx-\int_0^c f(x)dx$

E. $\int_0^d (g(x))dx-\int_0^c f(x)dx$

Question 9

The equation of the tangent to the curve with equation $y = x^{\frac{1}{3}}$ at the point $(8, 2)$ is given by

A. $y = \frac{x}{12} + \frac{4}{3}$

B. $y = \frac{x}{12} + \frac{47}{6}$

C. $y = \frac{x}{6} + \frac{2}{3}$

D. $y = \frac{x}{6} + \frac{23}{3}$

E. $y = \frac{4x}{3} - \frac{26}{3}$

Question 10

A transformation $T: R^2 \to R^2$, which maps the curve with equation $y = e^x$ onto the curve with equation $y = 4e^{x+1}$ could be given by

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Question 11

The function g is continuous and differentiable for $x \in R$. The function g satisfies the following conditions.

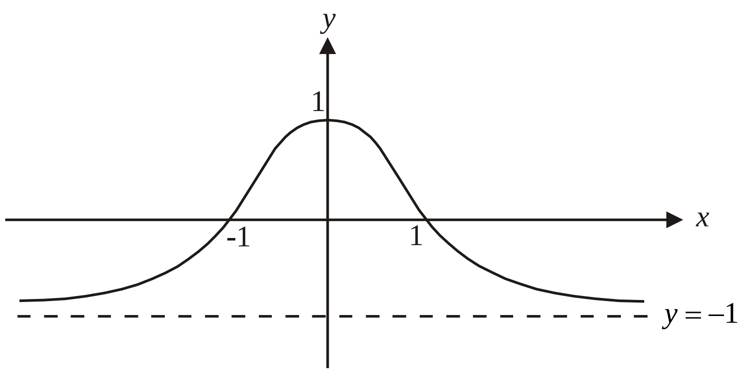
- $f'(x) = 0$ where $x = -1,\ 0$ and 4
- $f'(x) > 0$ where $x < -1$ and $0 < x < 4$
- $f'(x) < 0$ where $-1 < x < 0$ and $x > 4$

It is true to say that the graph of $y = g(x)$ has

A. a stationary point of inflection where $x = 0$
B. a point of inflection where $x = 0$
C. a local maximum where $x = 0$
D. a local minimum where $x = 0$
E. two local minimum

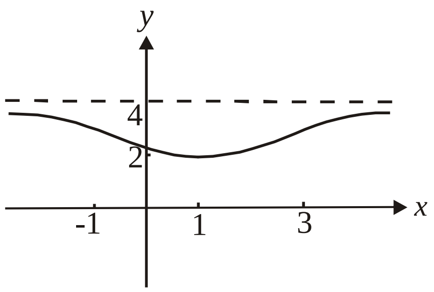
Question 12

The graph of $y = f(x)$ is shown below.

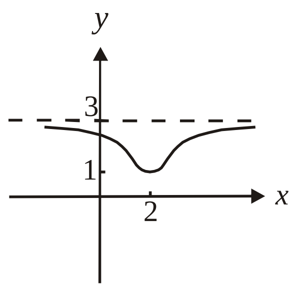


The graph of $y = 3 - f(2x-2)$ is given by

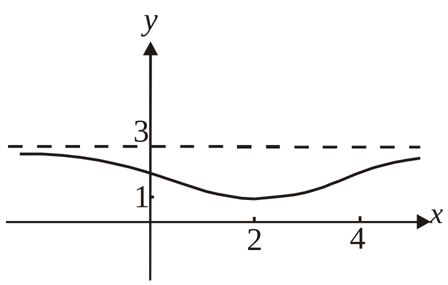
A.



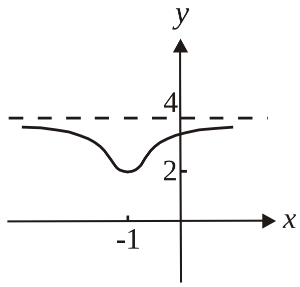
B.



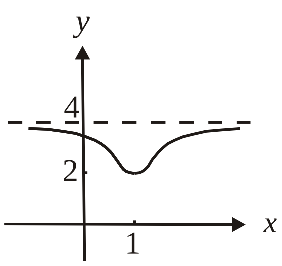
C.



D.



E.



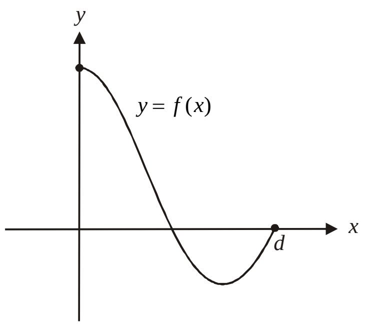
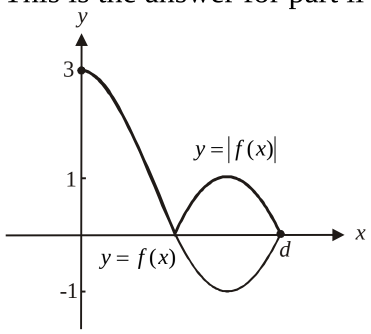
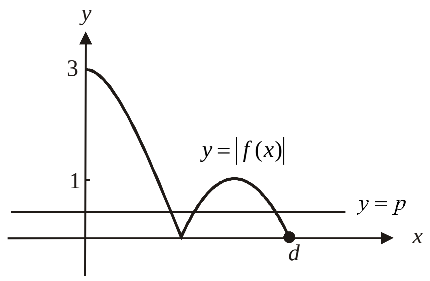
E is the correct answer

Part 2

Question 13

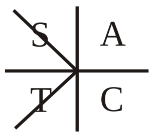
a. Let $f:[0,\ d]\rightarrow R,\ f(x)=2\cos\left(\frac{x}{4}\right)+1$ where d is a constant.

The graph of $y=f(x)$ is shown below.

<table>
<tr>
<td>This is the original graph

</td>
<td>This is the answer for part ii

</td>
<td>This is the graph used for part iv.

</td>
</tr>
</table>

i. Find the value of *d*.

Solve $2\cos\left(\frac{x}{4}\right)+1=0$ for x **(1 mark)**

<table>
<tr>
<td><u>Method 1</u> – using CAS
$x=\ldots-\frac{-8\pi}{3},\frac{8\pi}{3},\frac{16\pi}{3},\ldots$

</td>
<td><u>Method 2</u> – by hand
$2\cos\left(\frac{x}{4}\right)=-1$
$\cos\left(\frac{x}{4}\right)=-\frac{1}{2}$
$\frac{x}{4}=\ldots-\frac{2\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\ldots$
$x=\ldots-\frac{8\pi}{3},\frac{8\pi}{3},\frac{16\pi}{3},\ldots$</td>
</tr>
</table>

From the graph, we are looking for the second smallest positive value of *x*; that is, not the first positive *x*-intercept.So $d=\frac{16\pi}{3}$. *As no decimal places are specified, exact answers are needed.*

(1 mark)

ii. Using a different colour, sketch the graph of $y=|f(x)|$ on the set of axes shown above.

iii. State the maximum value of $|f(x)|$.

From the graph, the maximum value of $|f(x)|$ is 3.

iv. Find the value(s) of *p* for which the equation $|f(x)|=p$ has three solutions.

Again from the graph, if $p<1$ and $p>0$ then there are three solutions to the equation $|f(x)|=p$. *This means that horizontal lines must cross the graph in 3 places.*

If $p=1$, there are two solutions and if $p=0$ there are 2 solutions.
So $p\in(0,\ 1)$ or $\{p: 0<p<1\}$.

(1 mark) – correct values **(1 mark)** – endpoints excluded

$2+1+1+2=6$ marks

b. Let $g:[0,\ q]\rightarrow R,\ g(x)=2\cos(nx)+1,$ where n is a real constant and q is the largest possible value so that the inverse function g^{-1} exists.

i. Find q in terms of n

$\text{period}=\dfrac{2\pi}{n}$

For g^{-1} to exist, g must be 1:1

This means that the domain for $g^{-1}=\left[0,\dfrac{period}{2}\right]=\left[0,\dfrac{\pi}{n}\right]$

so $q=\dfrac{2\pi}{n}\div 2$

$=\dfrac{\pi}{n}$

(1 mark)

The function g passes through the point $T\left(t,\ 1-\sqrt{3}\right)$

ii. Find the value of n in terms of t.

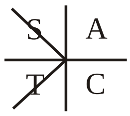
$g(x)=2\cos(nx)+1$

Since $T\left(t,\ 1-\sqrt{3}\right)$ lies on g,

$1-\sqrt{3}=2\cos(nt)+1$ **(1 mark)**

$-\sqrt{3}=2\cos(nt)$

$-\dfrac{\sqrt{3}}{2}=\cos(nt)$



$nt=\dfrac{5\pi}{6}$

$n=\dfrac{5\pi}{6t}$

(1 mark)

$2+2=4$ marks

Total Question 13 = 10 marks

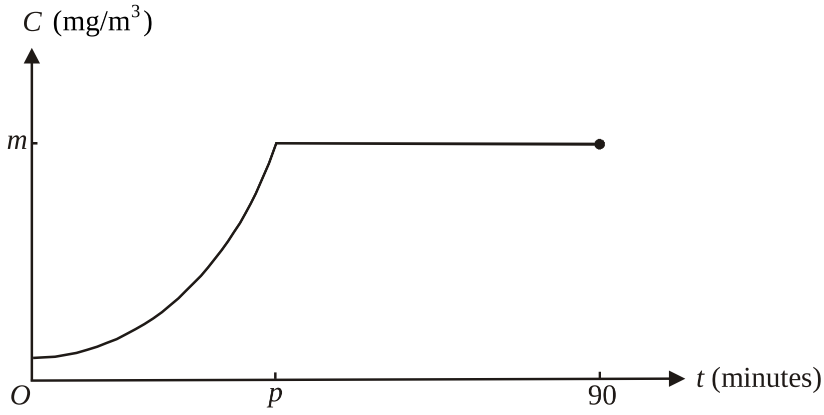
Question 14

Victoria James is a spy.
She is trapped in a space where poisonous gas is leaking.
The concentration *C*, in mg/m^3, of the gas *t* minutes after Victoria became trapped is given by the continuous function

$$C(t)=\begin{cases}\dfrac{500}{100-t}, & 0\le t\le p\\ m, & p<t\le 90\end{cases}$$

where *m* and *p* are constants.

A graph of the function is shown below.



a. What is the initial concentration of the gas in mg/m^3?

$$C(0)=\frac{500}{100-0}$$
$$=5\text{mg/m}^3$$

(1 mark)

b. Find an expression for *m* in terms of *p*.

Since the function is continuous,
$C(p)=\dfrac{500}{100-p}$ and $C(p)=m$.

So $m=\dfrac{500}{100-p}$

(1 mark)

c. Find the minimum and maximum values of *m*.

From the graph and part **a.**, the minimum value of *m* is 5 which occurs when $p=0$.

(1 mark)

From the graph and part **b.**, the maximum value of *m* must occur when $p=90$ (since the function is continuous).

So $m=\dfrac{500}{100-90}$
$=50$

(1 mark)

d. Find the function $C'(t)$ for $0 < t < p$.

Method 1 – using CAS	Method 2 – by hand
$C(t) = \frac{500}{100-t}$ for $0<t<p$ $C'(t) = \frac{500}{(t-100)^2}$ Note $\frac{500}{(100-t)^2} = \frac{500}{(-1(t-100))^2} = \frac{500}{(t-100)^2}$ confirming the CAS answer above.	Let $y = \frac{500}{100-t}$ $= \frac{500}{u}$ where $u = 100-t$ $= 500u^{-1}$ $\quad \frac{du}{dt} = -1$ $\frac{dy}{du} = -500u^{-2}$ $= \frac{-500}{u^2}$ $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$ (chain rule) $= \frac{-500}{u^2} \times -1$ So $\frac{dy}{dt} = \frac{500}{(100-t)^2}$ So $C'(t) = \frac{500}{(100-t)^2}$

1 mark

e. If the rate at which the concentration of the gas is increasing was 1 mg/m^3 per minute, find the value of t. Express your answer in minutes correct to 2 decimal places.

c. Solve $C'(t) = 1$

i.e. $\frac{500}{(t-100)^2} = 1$ for t.

(1 mark)

$t = 77.6393\ldots$ or $t = 122.361\ldots$

but $t < 90$

So $t = 77.64$ minutes (correct to 2 decimal places)

(1 mark)

f. If $p = 10$, find the average concentration of the gas between $t = 0$ and $t = p$, correct to 2 decimal places.

$$\text{average concentration} = \frac{1}{10-0}\int_0^{10} \frac{500}{100-t}\,dt$$ **(1 mark)**

$$= 5.26803\ldots$$

$$= 5.27 \text{ mg/m}^3 \text{ (correct to 2 decimal places)}$$

(1 mark)

If the concentration of the gas reaches 6 mg/m^3 then a human cannot survive.

g. Given that Victoria is trapped for 90 minutes in this space, find the possible values of p in order for her to survive.

For $0 \le t \le 90$ the maximum concentration is m mg/m^3.

From part **b.**, $m = \frac{500}{100-p}$.

For Victoria to survive, we require that

$m < 6$,

Solve $\dfrac{500}{100-p} < 6$ for p. **(1 mark)**

$500 < 6(100-p)$

$500 < 600 - 6p$

$-100 < -6p$

$50 > 3p$

$p < \dfrac{50}{3}$ (since $0 \le p < 90$, reject $p > 100$)

So in order for Victoria to survive, $p \in \left[0, \dfrac{50}{3}\right)$.

(1 mark)

Total Question 14 = 11 marks

Question 15

a. Consider the functions

$$f : R \to R, \quad f(x) = e^{-x} \quad \text{and} \quad g : [0, \infty) \to R, \quad g(x) = 2(x+1)$$

i. Write down the rule for $f(g(x))$.

$f(g(x)) = f(2(x+1)) = e^{-2(x+1)}$

(1 mark)

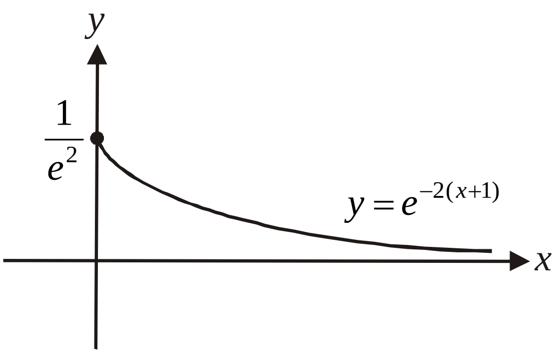
ii. What is the domain of $f(g(x))$?

$d_{f \circ g} = d_g = [0, \infty)$

(1 mark)

iii. Sketch the graph of $f(g(x))$. Hence find range of $f(g(x))$.

To find $r_{f \circ g}$ sketch the graph of $y = e^{-2(x+1)}$ and restrict the domain to $x \in [0, \infty)$.



y-intercept occurs when $x = 0$

$$\begin{aligned} y &= e^{-2(0+1)} \\ &= e^{-2} \\ &= \frac{1}{e^2} \end{aligned}$$

So $r_{f \circ g} = \left(0, \frac{1}{e^2}\right]$

(1 mark) – correct graph **(1 mark)** – correct left endpoint and bracket **1 mark)** – correct right endpoint and bracket

iv. Find the area enclosed by the function $y = f(g(x))$, the x and y axes and the line with equation $x = 1$. Express your answer as an exact value.

$$\text{Area} = \int_0^1 e^{-2(x+1)} \, dx$$ **(1 mark)**

$$= \frac{e^{-4}(e^2 - 1)}{2} = \frac{e^{-2} - e^{-4}}{2} \text{ square units}$$

(1 mark)

v. Describe the two transformations that the graph of $y = f(x)$ undergoes to become the graph of $y = f(g(x))$.

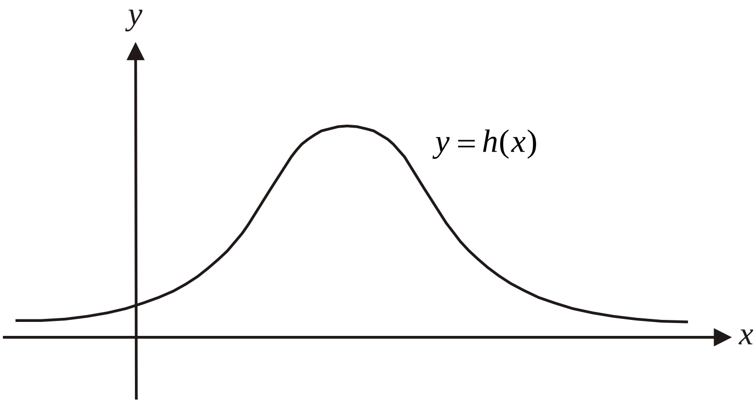
To become the graph of $y = e^{-2(x+1)}$, the graph of $y = e^{-x}$ is

- dilated by a factor of $\frac{1}{2}$ from the y-axis OR factor of 2 towards the y-axis and
- translated 1 unit left

(1 mark) – description of dilation **(1 mark)** – description of translation

$1 + 1 + 3 + 2 = 7$ marks

b. The graph of the function $h: R \to R,\ h(x) = e^{-(x-a)^2}$ where a is a constant and $a > 0$ is shown below.



i. Show that the stationary point of this graph occurs at $(a, 1)$.

Stationary point occurs when $h'(x) = 0$. **(1 mark)**

**Use CAS, Solve $h'(x) = \left(2a e^{-a^2} - 2x e^{-a^2}\right) e^{2ax - x^2} = 0$ for x. so $x = a$	**Use chain rule, $h(x) = e^{-(x-a)^2}$ $h'(x) = e^{-(x-a)^2} \times -2(x-a) \times 1 = -2(x-a)e^{-(x-a)^2} = 0$ $\to x - a = 0 \quad OR \quad e^{-(x-a)^2} = 0(impossible)$, so $x = a$ Since $h(a) = e^{-(a-a)^2} = e^0 = 1$, stationary point occurs at $(a, 1)$.

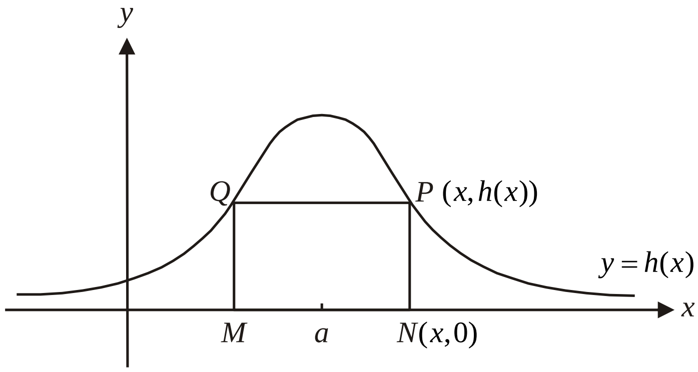
(1 mark)

ii. Find the values of x for which $h(x)$ is strictly increasing.

$h(x)$ is strictly increasing for $x \in (-\infty, a]$.

(1 mark)

The diagram below shows the graph of $y = h(x)$ where $h(x) = e^{-(x-a)^2}$ and the rectangle *MNPQ*.



The side length *MN* lies on the *x*-axis, points *P* and *Q* lie on the graph of $y = h(x)$ and the line $x = a$ bisects the rectangle.

The coordinates of points *N* and *P* are $(x, 0)$ and $(x,\ h(x))$ respectively.

iii. Show that the area *A*, in square units, of the rectangle is given by

$$A = 2(x-a) \times e^{-(x-a)^2}.$$

$A = \text{length} \times \text{width}$
$= MN \times NP$
$= 2(x-a) \times h(x)$
$= 2(x-a) \times e^{-(x-a)^2}$

(1 mark)

iv. Find the maximum area of rectangle *MNPQ*. You do not need to prove this is a maximum.

Maximum occurs when $\frac{dA}{dx} = 0$. **(1 mark)**

**Using CAS, solve $\frac{dA}{dx} = 0$ for x.	**Using product rule,
$x = \frac{2a-\sqrt{2}}{2}$ or $x = \frac{2a+\sqrt{2}}{2}$	$A' = e^{-(x-a)^2} \times 2 + 2(x-a) \times e^{-(x-a)^2} \times -2(x-a) \times 1$
$x = a - \frac{\sqrt{2}}{2}$ or $x = a + \frac{\sqrt{2}}{2}$	$A' = 2e^{-(x-a)^2}\left(1 - 2(x-a)^2\right) = 0$
	$(x-a)^2 = \frac{1}{2}$
	$x = a \pm \frac{1}{\sqrt{2}} = a \pm \frac{\sqrt{2}}{2}$

Since $x > a$, (from the diagram) $x = a + \frac{\sqrt{2}}{2}$ **(1 mark)**

Substitute $x = a + \frac{\sqrt{2}}{2}$ into $A = 2(x-a)e^{-(x-a)^2} = \sqrt{\frac{2}{e}}$

Maximum area is $\sqrt{\frac{2}{e}}$ square units. **(1 mark)**

$2+1+1+3 = 7$ marks

Total Question 15 = 14 marks

End of Section A

Mathematical Methods Multiple Choice Answer Sheet

Choose and circle the best answer for each question

1	A	B	C	**D**	E
2	A	**B**	C	D	E
3	A	**B**	C	D	E
4	A	B	**C**	D	E
5	A	B	**C**	D	E
6	A	B	C	**D**	E
7	**A**	B	C	D	E
8	**A**	B	C	D	E
9	**A**	B	C	D	E
10	A	**B**	C	D	E
11	A	B	C	**D**	E
12	A	B	C	D	**E**

12 x 1 = 12 marks

Name: ________________solutions___________________________ Home Group: __________

Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\,(x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\,dx$