Student Name: SOLUTIONS

Home Group: ____________

Teacher's name: (please circle): Mrs O'Rielly Ms Webb

Mathematical Methods

Unit 2

November 2015

Part II Total 78 marks

Topics covered:

- Probability
- Circular Functions
- Rates of Change
- Differential Calculus
- Integral Calculus
- Matrices
- Combinatorics

Complete working must be shown and simplified wherever possible in order to gain full marks.

Reading Time: 15 minutes

Writing Time: 90 minutes

Students may bring one bound reference book into the exam.
Students may bring a CAS calculator into the exam.

No paper or electronic dictionaries may be used.

Teacher's name: (please circle): Mrs O'Rielly Ms Webb

SECTION A: MULTIPLE CHOICE SECTION: CHOSE THE CORRECT ANSWER AND RECORD THE LETTER ON THE COLOURED SHEET.

1 A bowl contains 2 white chocolates and 3 dark chocolates. Andrew chooses a chocolate at random and eats it. He then chooses a second chocolate at random from those that remain and eats it. The probability that Andrew has eaten different colour chocolates is:

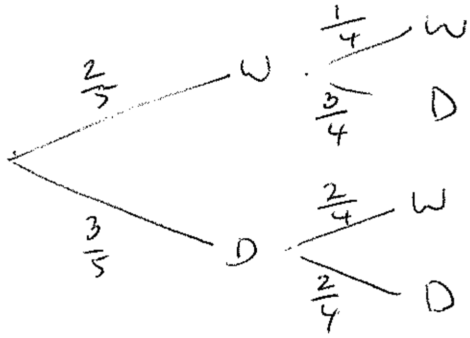
A $\frac{6}{25}$

B $\frac{3}{10}$

C $\frac{2}{5}$

D $\frac{12}{25}$

E $\frac{3}{5}$



$Pr(WD) + Pr(DW)$

$= \frac{\cancel{2}^1}{5} \times \frac{3}{\cancel{4}_2} + \frac{3}{5} \times \frac{\cancel{2}^1}{\cancel{4}_2} = \frac{3}{10} + \frac{3}{10} = \frac{6}{10} = \frac{3}{5}$

2 For two events A and B, $\Pr(A \cup B) = 0.9$, $\Pr(A) = 0.8$ and $\Pr(B) = 0.5$. It can be concluded that:

~~**A**~~ event A is a subset of event B.

~~**B**~~ event B is a subset of event A.

C events A and B are independent. yes $\Rightarrow Pr(A) \times Pr(B) = 0.4 = Pr(A \cap B)$

D events A and B are mutually exclusive.

E events A and B are neither mutually exclusive nor independent.

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$0.9 = 0.8 + .5 - Pr(A \cap B)$

$\Rightarrow Pr(A \cap B) = .4$

3 If $\sin\theta = -\frac{2\sqrt{2}}{3}$, then a possible value of $\cos\theta$ is:

A $-\frac{1}{3}$

B $\frac{2\sqrt{2}}{3}$

C $-\frac{\sqrt{5}}{9}$

D $-\frac{1}{\sqrt{8}}$

E $\frac{1}{9}$

$\cos\theta = \sqrt{1 - \left(\frac{-2\sqrt{2}}{3}\right)^2}$

$= \sqrt{1 - \left(\frac{8}{9}\right)}$

$= \sqrt{\frac{1}{9}}$

$= \pm\frac{1}{3}$

4 Over the interval $[-\pi, \pi]$, the graphs of y= $-\cos\theta$ and y= $\sin\theta$ intersect at:

A $-\frac{\pi}{4}, -\frac{\pi}{4}$

B $\frac{3\pi}{4}, \frac{5\pi}{4}$

C $-\frac{5\pi}{4}, -\frac{13\pi}{4},$

D $\frac{\pi}{4}, \frac{5\pi}{4}$

E $-\frac{\pi}{4}, \frac{3\pi}{4}$

5 The quadrant or quadrants the function $y = \frac{sin\theta}{cos\theta}$ has a positive solution is:

tan θ

A quadrant 1 only

B quadrants 1 and 3

C quadrants 2 and 3

D quadrants 3 and 4

E quadrant 4 only

6 Let $y = x^2 - 7x + 10$. The average rate of change of y with respect to x over the interval $4 \le x \le 6$ is:

$\frac{f(6) - f(4)}{6-4} = \frac{4 - -2}{2}$

$= 3$

A -3

B -2

C 1

D 3

E 4

7 If $A = \begin{bmatrix} 2 & 2 \\ 3 & ^-1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & ^-3 \\ 4 & ^-5 \end{bmatrix}$, then AB is represented by:

A $\begin{bmatrix} 14 & ^-16 \\ 5 & ^-4 \end{bmatrix}$

~~B~~ $\begin{bmatrix} ^-3 & 9 \\ ^-7 & 13 \end{bmatrix}$

~~C~~ $\begin{bmatrix} 5 & ^-1 \\ 7 & ^-6 \end{bmatrix}$

~~D~~ $\begin{bmatrix} ^-1 & 5 \\ ^-1 & 4 \end{bmatrix}$

E $\begin{bmatrix} 14 & 16 \\ 5 & 4 \end{bmatrix}$

8 If $y = 3x(x-6)$, then $\frac{dy}{dx}$ is equal to:

A $6x^2 - 3$

B $6x + 8$

C $6x - 6$

D $6x - 18$

E $6x^2$

$y = 3x^2 - 18x$

$\frac{dy}{dx} = 6x - 18$

9 For the curve with equation $y = x^2 - 8x$, the gradient of the curve at $x = 5$ is:

A -30

B -15

C -8

D 2

E 17

$\frac{dy}{dx} = 2x - 8$

at $x = 5$

$= 10 - 8 = 2$

10 The coordinates of the point on the curve with equation $y = \frac{1}{2}x^2 - 2x + 5$ at which the tangent is parallel to the line $y = -4x + 2$ are:

~~A~~ (−6, 35)

B (−2, 10)

C (−2, 11)

~~D~~ (2, 3)

~~E~~ (2, 6)

$\frac{dy}{dx} = x - 2$

gradient $= -4$

$x - 2 = -4$

$x = -2$

$y = \frac{1}{2}(-2)^2 - 2(-2) + 5$

$= 2 + 4 + 5 = 11$

11 Let $\frac{dy}{dx} = 3 - 4x$ and $y = 1$ when $x = 0$. The value of y when $x = 2$ is:

A -9

B -7

~~C~~ -2

D -1

E 1

$y = \int (3 - 4x)\,dx = 3x - 2x^2 + c$

$1 = 3(0) - 2(0)^2 + c$

$\Rightarrow c = 1$

$y = -2(x)^2 + 3x + 1$

$= -2(2)^2 + 3(2) + 1$

$= -8 + 6 + 1 = -1$

12 $\lim\limits_{h \to 0} \frac{9 - (3+h)^2}{h}$ is equal to:

A -6

B -4

C -2

D 0

E 8

$= \lim\limits_{h \to 0} \frac{(3+3+h)(3-(3+h))}{h}$

$= \lim\limits_{h \to 0} \frac{(6+h)(-h)}{h}$

$= \lim\limits_{h \to 0} -(6+h), \ h \neq 0$

$= -6$

13 A hardware shop stocks five colours of paint. New colours can be made by taking two of the existing colours and mixing them. Assuming each such mixing gives a new colour, the number of new colours possible is:

A 5
B 10
C 20
D 25
E 5! × 4!

${}^5C_2 = 10$

14 A team of four tennis players must be chosen from a squad of nine. The number of different teams that can be chosen if Leisel must be on the team as captain is:

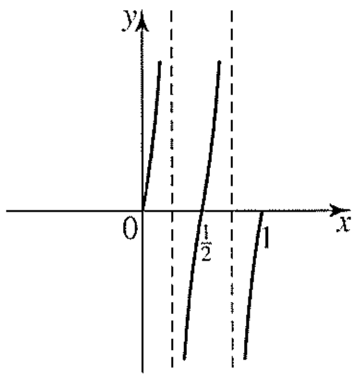
A ${}^8C_3$
B ${}^8C_4$
C ${}^9C_4$
D ${}^8P_3$
E ${}^9P_4$

${}^1C_1 \times {}^8C_3 =$

15 The graph depicted could have the equation:

A $y = 2\tan \pi x$
B $y = \tan \pi x$
C $y = \tan \dfrac{\pi x}{2}$
D $y = \dfrac{1}{2}\tan \pi x$
E $y = \tan 2\pi x$

tan
period $= \frac{1}{2}$
$\frac{\pi}{b} = \frac{1}{2}$
$b = \frac{\pi}{\frac{1}{2}} = 2\pi$



16 If $f(x) = \begin{cases} x-1, & x \in (0,1] \\ x^3 - 1, & x \in [-1, 0) \end{cases}$ then

$\lim\limits_{x \to 0} f(x)$ equals:

$\lim\limits_{x \to 0^-} f(x) = 0 - 1 = -1$

$\lim\limits_{x \to 0^+} f(x) = 0 - 1 = -1$

A 0
B 1
C -1
D does not exist
E *x*

END OF SECTION A

SECTION B- SHORT ANSWER SECTION:

1 If A and B are events such that $\Pr(A) = 0.7$, $\Pr(A \cap B') = 0.3$ and $\Pr(A' \cap B) = 0.2$, find:

	A	A'	
B	0.4	0.2	0.6
B'	0.3	0.1	0.4
	0.7	0.3	1

a $\Pr(A \cap B)$

0.4

b $\Pr(B)$

0.6

c $\Pr(A \cup B)$ $\left(\text{also} = 1 - \Pr(A' \cap B')\right)$

$\Pr(A) + \Pr(B) - \Pr(A \cap B) = .7 + .6 - .4$
$= 0.9$

d $\Pr(A' \mid B')$

$= \dfrac{\Pr(A' \cap B')}{\Pr(B')} = \dfrac{0.1}{0.4} = 0.25$

(1 + 1 + 1 + 2 = 5 marks)

2 If Pr(A) = 0.3, Pr(B) = 0.5 and Pr ($A \cup B$) = 0.8, determine whether A and B are:

$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.3 + 0.5 - 0.8 = 0$

a independent.

if A + B are independent, then $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$0 \neq 0.3 \times 0.5 \Rightarrow$ A + B are not independent

b mutually exclusive.

yes as $\Pr(A \cap B) = 0$

(1 + 1 = 2 marks)

3 From a group of five boys and 4 girls, three school captains must be selected.

a) How many different combinations of three school captains, without any restrictions, are possible?

${}^9C_3 = 84$

b) How many combinations contain a majority of female captains?

2 girls or 3 Girls

${}^4C_2 \times {}^5C_1 + {}^4C_3 = 34$

3 continued:

c) How many different combinations of three captains are possible if both genders must be represented?

2 girls or 2 boys

$= {}^4C_2 \times {}^5C_1 + {}^5C_2 \times {}^4C_1$

$= 70$

(1+2+2 = 5 marks)

4 A nursery worker plants 5 identical pumpkin seedlings, 3 identical tomato seedlings and 4 identical zucchini seedlings in a row.

(a) The nursery worker in question decides to arrange the same seedlings in a circle. How many different ways can this be done?

12 objects in a circle $= \dfrac{11!}{5!3!4!} = 2310$

(b) What is the probability that the pumpkin seedlings remain together?

$= \dfrac{\dfrac{7!.5!}{5!.3!.4!}}{2310} = \dfrac{35}{2310} = \dfrac{1}{66}$ (0.015)

(2+2 = 4 marks)

5 If $\cos\theta = 0.4$, write down the value of:

a $\cos(\pi - \theta)$ $= -\cos\theta = -0.4$

b $\cos(\pi + \theta)$ $= -\cos\theta = -0.4$

c $\cos(2\pi - \theta)$ $= \cos\theta = 0.4$

(3 marks)

6 Solve the equation $1 - 2\cos x = 2$, where $-2\pi \leq x \leq 2\pi$, giving answers as exact values.

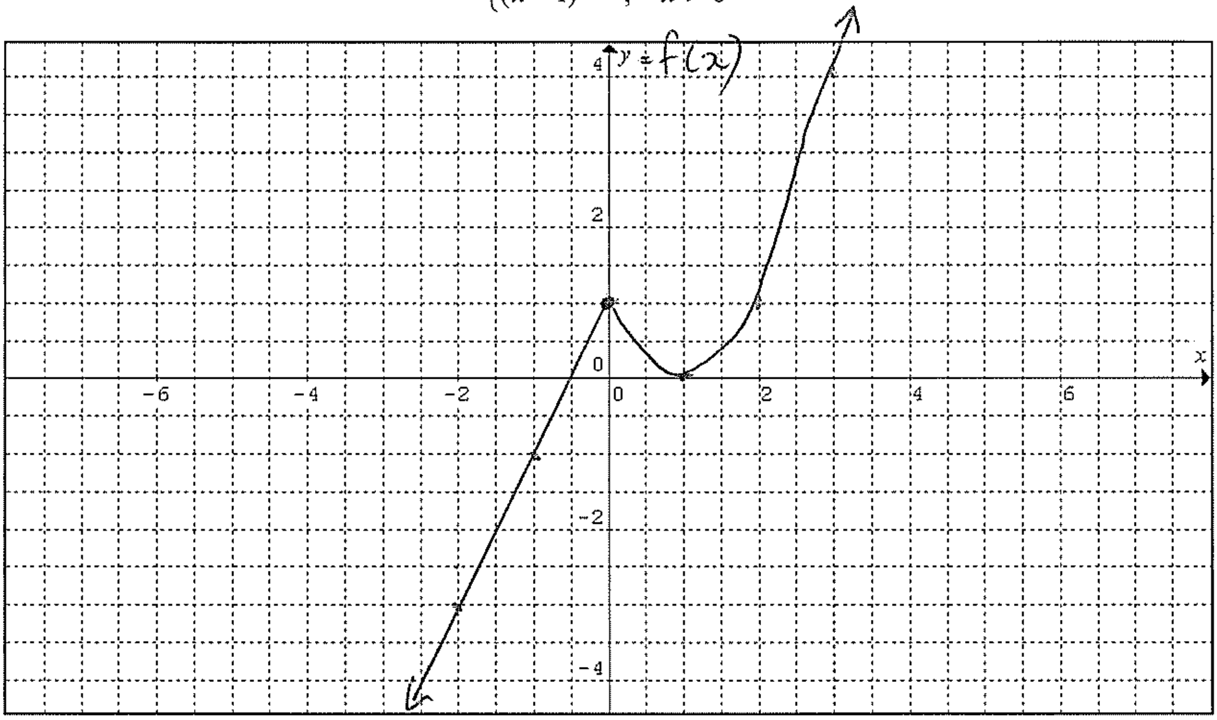
$2\cos x = -1$

$\cos x = -\frac{1}{2}$

$x = \frac{-4\pi}{3}, \frac{-2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

(2 marks)

7 **(a)** Sketch the graph of $f(x) = \begin{cases} 2x+1 & , \quad x \le 0 \\ (x-1)^2 & , \quad x > 0 \end{cases}$



(b) For what values of *x* is the graph a continuous function?

$x \in R$

(c) Evaluate $\lim_{x \to 3} f(x)$ $= (3-1)^2$

$= 4$

(3 + 1+1 = 5 marks)

8 Find the derivatives of:

a) $y = \dfrac{4}{x^2}$ $= 4x^{-2}$ $\dfrac{dy}{dx} = -8x^{-3}$ $= \dfrac{-8}{x^3}$	b) $y = x^{\frac{5}{4}}$ $\dfrac{dy}{dx} = \dfrac{5}{4} x^{\frac{1}{4}}$

(2 marks)

9 Find a function that differentiates to:

(a) $2x^4$	(b) $^{-}x^{-2}$
$f(x) = \int (2x^4)\,dx$ $= \frac{2x^5}{5} + c$	$f(x) = \int (-x^{-2})\,dx$ $= x^{-1}$ $= \frac{1}{x} + c$

(4 marks)

10 If $\frac{dy}{dx} = (x-2)^2$ and the y-intercept is $^{-}3$, find the equation for y.

$\frac{dy}{dx} = x^2 - 4x + 4$

$y = \int (x-2)^2 + c$

$= \frac{1}{3}x^3 - 2x^2 + 4x + c$

y-int $= -3$

$\Rightarrow c = -3$

$\underline{y = \frac{x^3}{3} - 2x^2 + 4x - 3}$

(2 marks)

11 Describe ***geometrically*** the transformation matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Dilates a point by a factor of 2 from both x + y axis.

(1 mark)

12 Tickets for a one-way trip on a Melbourne to Sydney passenger train can be purchased as either adult, child (under 15 years old) or pensioner. The table below shows the number of passengers and the total takings for three trips.

Number of adult passengers	Number of child passengers	Number of pensioner passengers	Total takings ($)
145	103	121	20 260
130	110	90	18 400
142	115	80	19 200

a) Let x equal the cost of an adult's ticket. Let y equal the cost of a child's ticket. Let z equal the cost of a pensioner's ticket. Construct three equations in terms of x, y and z

$$145x + 103y + 121z = 20\,260$$

$$130x + 110y + 90z = 18\,400$$

$$142x + 115y + 80z = 19\,200$$

b) Using matrices, express the equations in the form $AX = B$.

$$\begin{bmatrix} 145 & 103 & 121 \\ 130 & 110 & 90 \\ 142 & 115 & 80 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20\,260 \\ 18\,400 \\ 19\,200 \end{bmatrix}$$

c) Find A^{-1} to 2 decimal places: $\begin{bmatrix} 0.03 & -0.09 & 0.07 \\ -0.04 & 0.09 & -0.04 \\ 0.01 & 0.03 & -0.04 \end{bmatrix}$

d) Determine the costs of a train ticket for an adult, a child and a pensioner.

$AX = B$

$X = A^{-1}B = \begin{bmatrix} 75 \\ 50 \\ 35 \end{bmatrix}$

cost of adult = $75

child = $50

pensioner = $35

(2+1+1+1= 5 marks)

13 For the curve with equation $y = \frac{1}{3}x^3 + x^2 - 3x + 6$

a) Find $\frac{dy}{dx}$ $= x^2 + 2x - 3$

b) What are the co-ordinates of the points when the gradient is zero?

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x + 3 = 0$ or $x - 1 = 0$

$x = -3, 1$

2 points $(-3, 15)$; $(1, \frac{13}{3})$

if $x = -3$,

$y = \frac{1}{3}(-3)^3 + (-3)^2 - 3(-3) + 6$

$= 15$

if $x = 1$

$y = \frac{1}{3}(1)^3 + (1)^2 - 3(1) + 6$

$= \frac{13}{3}$

c) Prove the type (nature) of the turning points:

a) $x = -3$

$f'(-4)$ $f'(-3)$ $f'(0)$

$= +5$ $= 0$ $= -3$

$(-3, 15)$ is a local maximum

b) $x = 1$

$f'(0)$ $f'(1)$ $f'(2)$

$= -3$ $= 0$ $= 5$

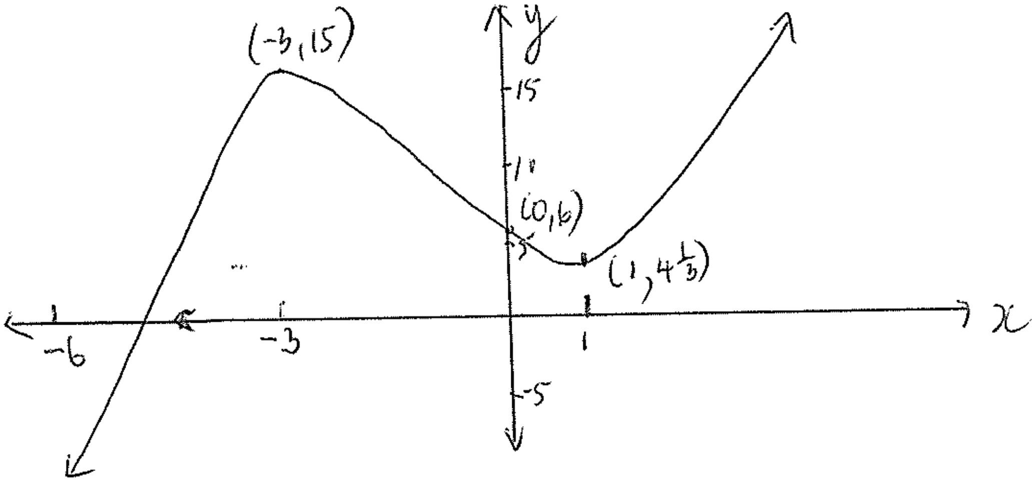
$(1, \frac{13}{3})$ is a local minimum

d) Where does the graph cross the y axis?]

if $x = 0$, $y = 6$

$(0, 6)$

e) Hence sketch the graph of $y = \frac{1}{3}x^3 + x^2 - 3x + 6$ (you do NOT need to find the x intercepts). Mark all relevant points with their co-ordinates.



(1 + 3 + 2 + 1 + 3 = 10 marks)

14 The tide level on a pier pylon can be modeled by the equation $H = 4 + 2\sin\frac{\pi}{6}t$ where H is the height in metres of the tide from the bottom of the pylon and t is time in hours after midnight.

a) What is the period of this tide?

$$\text{period} = \frac{2\pi}{1} \div \frac{\pi}{6} = \frac{2\pi}{1} \times \frac{6}{\pi} = 12 \text{ hrs}$$

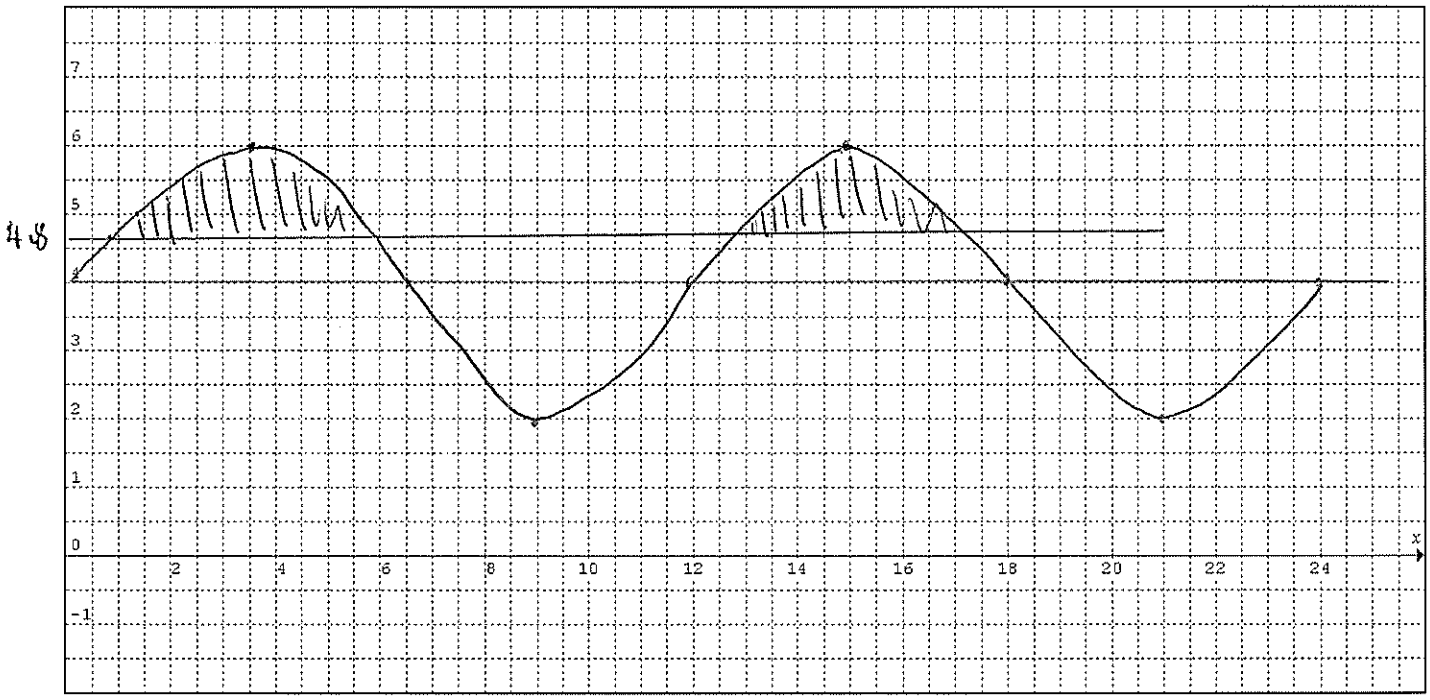
b) What is the maximum and minimum heights of the tide level and when do they occur in the first 24 hours?

max = 2 + 4 = 6 m at 9 am   min = 4 − 2 = 2 m at 9 pm

c) What is the water level on the pier at 4 am and at 4 pm? Give your answers as exact and correct to 2 decimal places.

5.73 m (from CAS)

d) Sketch the graph of the tide function showing the first 24 hours.



e) Show on the graph when the tide level is above 4.8m. (shaded part of graph)
At what times in the first 24 hours is the tide level above 4.8 m on the pylon?

.7859 → 5.2141 , 12.7859 → 17.2141

.7859 hrs × 60 = 47 min

.2141 × 60 hrs = 13 min

∴ Times = 00.47 → 05.13 am
12.47 pm → 5.13 pm

(1 + 3 + 3 + 3 + 2 = 12 marks)

END OF SECTION B