Student Name....Answers!........................

Teacher (circle one) DKI JOR VNA

Homegroup ........................................

# MATHEMATICAL METHODS (CAS) UNIT 1

# EXAMINATION 1

## Wednesday November 2nd 2016

Reading Time: 1:00 – 1:15pm (15 minutes)
Writing time: 1:15 – 2:15pm (1 hour)

**Instructions to students**

This exam consists of **17** questions.
All questions should be answered in the spaces provided.
There are **65** marks available in this examination.
A decimal approximation will not be accepted if an exact answer is required.
Where more than one mark is allocated to a question working must be shown.
Students **may not** bring any notes or any calculators into this exam.
Diagrams in this exam are not to scale except where otherwise stated.

## FORMULAS

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A \cap B) = \Pr(A).\Pr(B)$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Newton's Iterative formula for approximating roots of a polynomial:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1) Given $A = \begin{bmatrix} -2 & 4 \\ -6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ -2 & 7 \end{bmatrix}$, calculate the following:

| a) $B - 2A$ | b) $AB$ |
|---|---|
| $= \begin{bmatrix} 5 & 0 \\ -2 & 7 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ -12 & 2 \end{bmatrix}$<br>$= \begin{bmatrix} 9 & -8 \\ 10 & 5 \end{bmatrix}$ | $= \begin{bmatrix} -2 & 4 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -2 & 7 \end{bmatrix}$<br>$= \begin{bmatrix} -18 & 28 \\ -32 & 7 \end{bmatrix}$ |

(2 + 2 = 4 marks)

2) Consider the set of simultaneous equations:

$$5x - 6y = 21$$
$$x - 2y = 5$$

**a** Write the set of equations as a matrix equation.

$$\begin{bmatrix} 5 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 5 \end{bmatrix}$$

**b** Use a matrix method to solve the equations and hence determine the values of $x$ and $y$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 21 \\ 5 \end{bmatrix}$$
$$= \frac{1}{-10+6} \begin{bmatrix} -2 & 6 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 21 \\ 5 \end{bmatrix}$$
$$= -\frac{1}{4} \begin{bmatrix} -12 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

So $x = 3$, $y = 1$

(1 + 3 = 4 marks)

3)

a) Calculate the coordinates of the image of the point $(17,-5)$ under the translation defined by $T=\begin{bmatrix}-8\\9\end{bmatrix}$.

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}17\\-5\end{bmatrix}+\begin{bmatrix}-8\\9\end{bmatrix}$$

$$=\begin{bmatrix}9\\4\end{bmatrix}$$

$\Rightarrow (9,4)$

b) Calculate the coordinates of the image of the point $(6,-13)$ under the linear transformation defined by the matrix $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

$\Rightarrow$ reflection in $x$-axis.

so $(x', y') \overset{?}{=} (6, 13)$

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}6\\-13\end{bmatrix}=\begin{bmatrix}6\\13\end{bmatrix}$$

$\Rightarrow (x', y') = (6, 13)$

c) Describe the transformation defined by the matrix $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$. (See part b) above)

reflection in $x$-axis

(2 + 2 +1 = 5 marks)

4) Find the exact values of

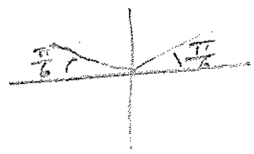
| a) $\sin 60^0$ | b) $\tan\frac{2\pi}{3}$ | c) $\cos\left(-\frac{\pi}{6}\right)$ |
|---|---|---|
| $\frac{\sqrt{3}}{2}$ | $-\sqrt{3}$ | $\frac{\sqrt{3}}{2}$ |

(3 marks)

5) Solve the following equation $2\sin x=\sqrt{3}, -2\pi\le x\le 2\pi$

$\sin x = \frac{\sqrt{3}}{2}$

$x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

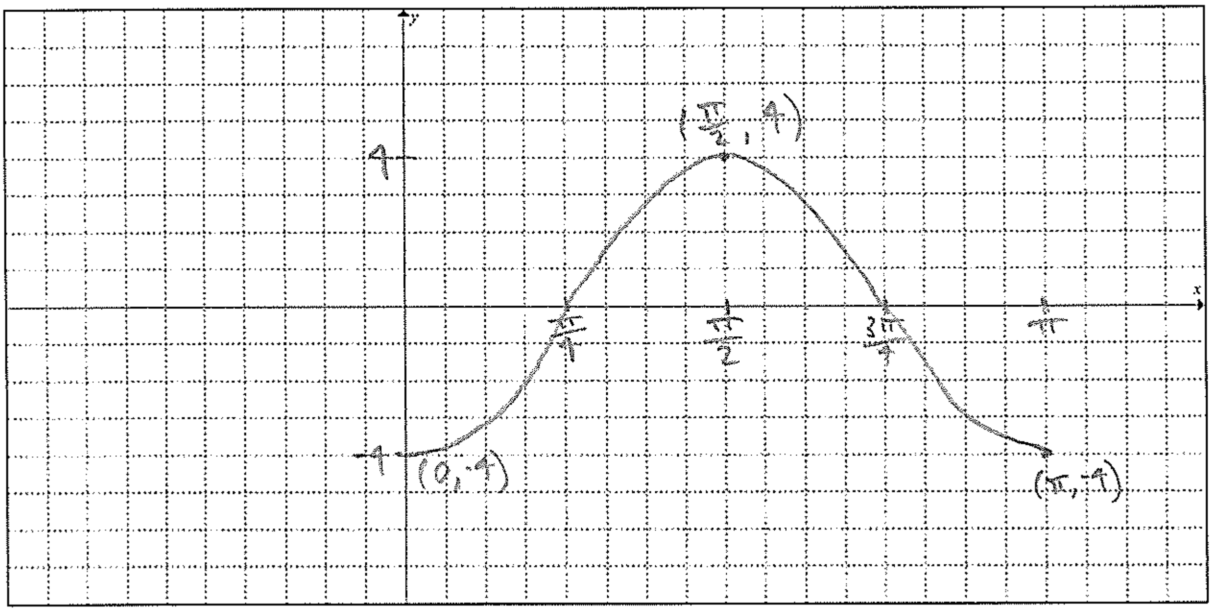


(4 marks)

6) a) What is the period and the amplitude of the graph of $y = -4\cos 2x$?

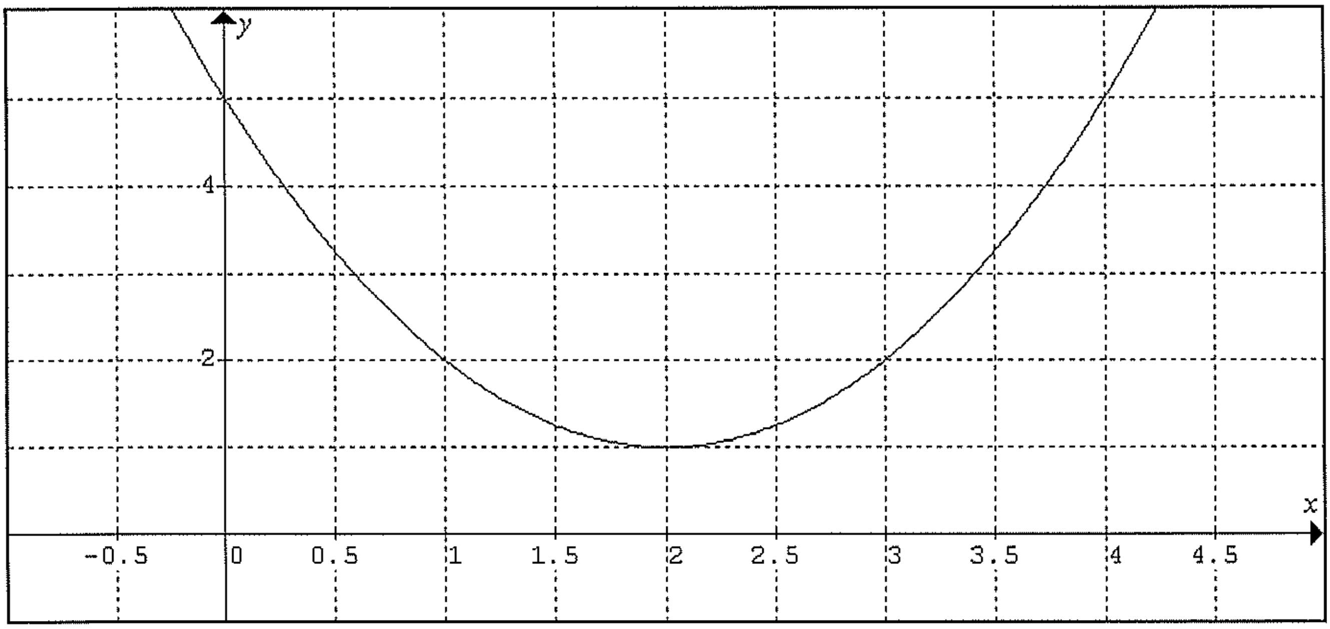
Period = $\pi$ Amplitude = 4

b) Sketch the graph, showing <u>one complete cycle</u>. Clearly label key points.




(2 + 3 = 5 marks)

7) Part of the graph of the function $f : R \to R, f(x) = (x-2)^2 + 1$ is shown below.




a) Find the average rate of change of $y = f(x)$ with respect to $x$, between x = 0 and x = 3.

$$\text{Avg change} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2-5}{3-0}$$

$$= -1$$

b) Find the instantaneous rate of change of $y = f(x)$ with respect to $x$ at the point where $x = 5$.

$f(x) = x^2 - 4x + 4 + 1$

$f'(x) = 2x - 4$

so $f'(5) = 10 - 4$

$= 6$

(2 + 2 = 4 marks)

8) Find, using first principles, the derivative of $y = x^2 + 5x + 1$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 + 5(x+h) + 1 - x^2 - 5x - 1}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 5h}{h}$$

$$= \lim_{h \to 0} 2x + h + 5$$

so $\frac{dy}{dx} = 2x + 5$

(3 marks)

9) If $f(x) = x^2(3x^2 - x) + 7$ , find $f'(-1)$.

$f(x) = 3x^4 - x^3 + 7$

$f'(x) = 12x^3 - 3x^2$

$f'(-1) = 12(-1)^3 - 3(-1)^2$

$= -12 - 3$

$= -15$

(3 marks)

10) Find the derivatives of with positive indices?

a) $y = \dfrac{7x^2 - 2x}{x}$

$= 7x - 2$

$\Rightarrow \dfrac{dy}{dx} = 7$

b) $f(x) = \dfrac{4}{3x^4}$

$= \dfrac{4}{3}x^{-4}$

$\Rightarrow f'(x) = \dfrac{-16}{3}x^{-5}$

$= \dfrac{-16}{3x^5}$

(2 + 2 = 4 marks)

11) Simplify

a) $\int (5x^3 + 3x^2 + 2)dx$

$= \dfrac{5}{4}x^4 + x^3 + 2x + c$

b) $\int \sqrt[3]{x^2}\,dx$

$= \int x^{\frac{2}{3}} dx$

$= \dfrac{3}{5}x^{\frac{5}{3}} + c$

$= \dfrac{3}{5}\sqrt[3]{x^5} + c$

more room?

(2 + 2 = 4 marks)

12) Find $\lim_{x \to 2} \dfrac{x^2 + 3x - 10}{x^2 - 4}$

$= \lim_{x \to 2} \dfrac{(x+5)(x-2)}{(x+2)(x-2)}$

$= \lim_{x \to 2} \dfrac{x+5}{x+2}$

$= \dfrac{7}{4}$

(2 marks)

13) A particle moves in a straight line such that its displacement, $x$ metres, from a fixed origin at time $t$ seconds is modelled by $x = t^2 - 6t + 8, t \geq 0$.

**a** Identify its initial position.

$t = 0, \quad x = 8$

**b** Show, using calculus, that the particle is momentarily at rest at $t = 3$ seconds.

$v = 2t - 6$ at rest $\Rightarrow v = 0$

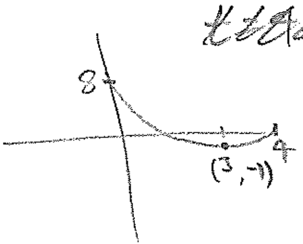
$v = 0 \quad 0 = 2t - 6$

$6 = 2t$

$t = 3$

**c** What is the average speed of the particle over the first 4 seconds?

avg speed $= \dfrac{\text{dist travelled}}{\text{time taken}}$



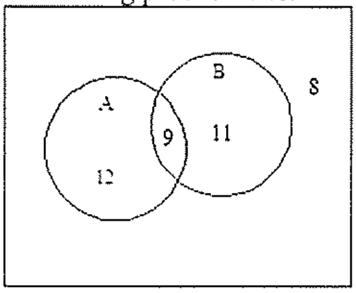

$= \dfrac{9 + 1}{4}$

$= \dfrac{10}{4}$

$= 2.5 \text{ m/s}$

(1 + 2 + 3 = 6 marks)

14) Use this Venn Diagram to find the following probabilities:




| a) $Pr(B' \cap A)$ | b) $Pr(B \mid A)$ | c) $Pr(A' \cup B)$ |
|---|---|---|
| $\frac{12}{40} = \frac{3}{10}$ | $\frac{9}{21}$ | $\frac{11+8+9}{40} = \frac{28}{40} = \frac{7}{10}$ |

(3 marks)

15) If $Pr(B) = 0.42$, $Pr(A' \cap B) = 0.16$ and $Pr(A') = 0.51$,

a) complete this probability table

| | $B$ | $B'$ | |
|---|---|---|---|
| $A$ | 0.26 | 0.23 | 0.49 |
| $A'$ | 0.16 | 0.35 | 0.51 |
| | 0.42 | 0.58 | 1 |

b) Find $Pr(A \cap B')$

0.23

c) Find $Pr(A \cup B)$ $= Pr(A) + Pr(B) - Pr(A \cap B)$

$= 0.49 + 0.42 - 0.26$

$= 0.65$

d) Find $Pr(A' | B)$

$= \frac{Pr(A' \cap B)}{Pr(B)}$

$= \frac{0.16}{0.42} = \frac{8}{21}$

( 2 +1 +1 + 2 = 6 marks)

16) Mr Oates needs two students to take some parents on a school tour. He chooses them randomly from a group of ten that were standing near his office. How many different groups of two could he choose?

$^{10}C_2 = \frac{10!}{2!\,8!}$

$= \frac{10 \times 9}{2 \times 1}$

$= \frac{90}{2}$

$= 45$

2 marks

17)

a) Define an iterative formula using Newton's Method for the function $f(x) = 6x^3 + 4x - 3$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f'(x) = \cancel{\ } 18x^2 + 4$$

so $$x_{n+1} = x_n - \frac{6x_n^3 + 4x_n - 3}{18x_n^2 + 4}$$

b) Use this to calculate the value of $x_1$ when $x_0 = 1$. Give your answer as an exact value.

$x_0 = 1$, then $x_1 = x_0 - \dfrac{6x_0^3 + 4x_0 - 3}{18x_0^2 + 4}$

$$= 1 - \frac{6 + 4 - 3}{18 + 4}$$

$$= 1 - \frac{7}{22}$$

$$= \frac{15}{22}$$

(2 + 1 = 3 marks)

**END OF EXAMNATION**