BILLANOOK COLLEGE

NAME: Answers .	Student Number:

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

Practice July Exam

Exam 1 TECHNOLOGY FREE

Friday 21st July, 2017
Reading time: 15 minutes 11:15am- 11:30am
Writing time: 1 hour 11:30am – 12:30pm

QUESTION AND ANSWER BOOKLET

Structure of Booklet

Number of Questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the test room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the test room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book with a detachable sheet of miscellaneous formulas.

Instructions

- Write your **name** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

Instructions

Answer **all** questions in the space provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.

Question 1 (5 marks)

a. If $y = x^2 \sin(x)$, find $\dfrac{dy}{dx}$. 2 marks

$$\frac{dy}{dx} = x^2 \cos(x) + 2x \sin(x)$$

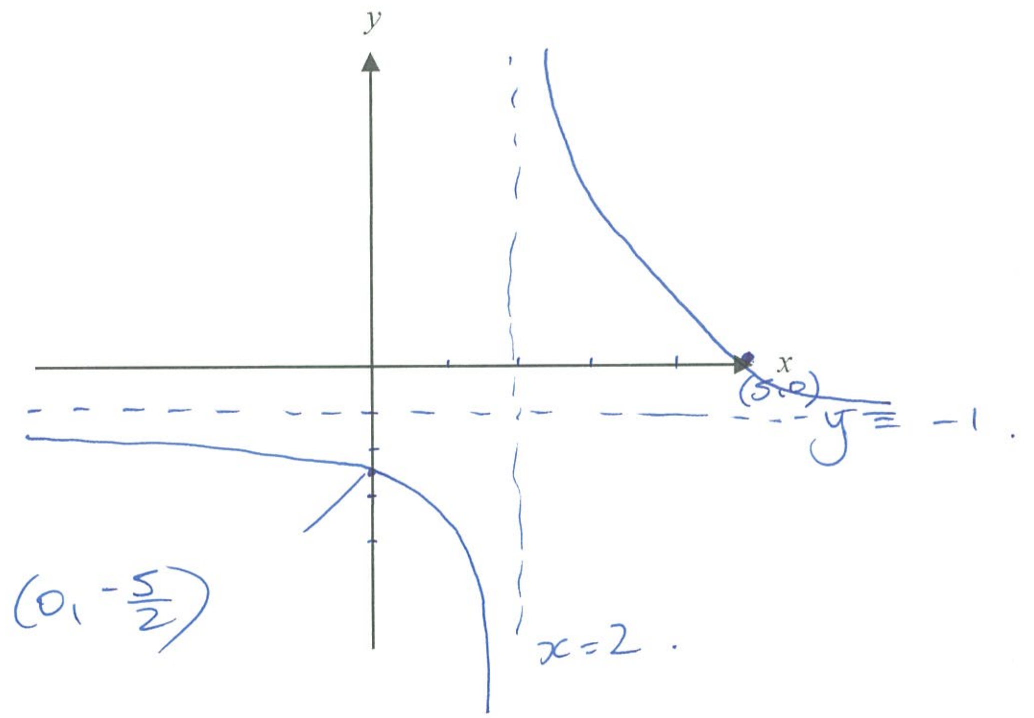
b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$. 3 marks

$$f'(x) = \frac{1}{2\sqrt{x^2+3}} \times 2x$$

$$f'(x) = \frac{x}{\sqrt{x^2+3}}$$

Question 2 (3 marks)

Sketch the graph of $f: R\setminus\{2\} \to R,\ f(x) = -1 + \frac{3}{x-2}$ on the set of axes below. Label axes intercepts with their coordinates. Label asymptotes with their equations



X int ($y=0$) $\quad 0 = -1 + \frac{3}{x-2}$

$1 = \frac{3}{x-2}$

$x - 2 = 3$

$x = 5. \qquad (5,0)$

Y int $(x) = 0 \quad y = -1 + \frac{3}{-2}$

$y = \frac{-5}{2} \qquad (0, \frac{-5}{2})$

Question 3 (2 marks)

Find $\int_1^3 \left(\frac{2}{x}+1\right)dx$.

$\left[2\log_e(x)+x\right]_1^3$

$= \left(2\log_e(3)+3\right) - \left(2\log_e(1)+1\right)$ $\quad \log_e 1 = 0$.

$= 2\log_e(3)+2$.

Question 4 (3 marks)

Let $f(x)=\frac{1}{\sqrt{3}}\cos(x)$ and $g(x)=\sin(x)$.

a. Solve the equation $f(x)=g(x)$ for $x\in[0,2\pi]$. 2 marks

$\frac{1}{\sqrt{3}}\cos(x) = \sin(x)$

$\frac{1}{\sqrt{3}} = \tan(x) \qquad x \in [0, 2\pi]$.

$x = \pi/6,\ 7\pi/6$.

b. Evaluate $f(g(0))$. 1 mark

$f(g(0)) = \frac{1}{\sqrt{3}}\cos(\sin(0))$

$= \frac{1}{\sqrt{3}}\cos(0)$

$= \frac{1}{\sqrt{3}}$.

Question 5 (2 marks)

Solve the following equation:

$$\frac{4000}{2+7^{3x}} = 5$$

$4000 = 5(2+7^{3x})$

$800 = 2+7^{3x}$

$798 = 7^{3x}$

$\log_7(798) = \log_7(7)^{3x}$

$\log_7(798) = 3x$

$x = \frac{1}{3}\log_7(798)$

Question 6 (2 marks)

The tangent to the curve $y = \frac{3}{x} - 2$ at the point $x = a$, where $a > 0$, has a gradient of -9.

Find the value of a.

$\frac{dy}{dx} = \frac{-3}{x^2}$ grad of tangent $= \frac{-3}{a^2}$

$-\frac{3}{a^2} = -9$ $a^2 = \frac{-3}{-9}$ $a = \pm\sqrt{\frac{1}{3}}$

but as $a > 0$ $a = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$

or $\frac{\sqrt{3}}{3}$.

Question 7 (3 marks)

Solve the equation $\log_e(x)+\log_e(3x+2)=2\log_e(x+1)$ for x, where $x>0$.

$\log_e(x(3x+2)) = \log_e(x+1)^2.$

So $3x^2+2x = x^2+2x+1.$

$2x^2 = 1$

$x^2 = \frac{1}{2}.$

$x = \pm\sqrt{\frac{1}{2}}$ but as $x>0$. reject $x=-\sqrt{\frac{1}{2}}$

$\therefore \; x = \sqrt{\frac{1}{2}}.$

Question 8 (9 marks)

Consider the function with the rule $f(x)=\frac{x-2}{x+2}$

a. Find the rule, f^{-1}, for the inverse of f. 3 marks

$y = \frac{x-2}{x+2}$ swap x & y to find inverse

$x = \frac{y-2}{y+2} \Rightarrow yx+2x = y-2$

$yx - y = -2-2x$

$y(x-1) = -2-2x$

$y = \frac{-2-2x}{x-1}$

or

$y = \frac{2x+2}{1-x}.$

b. Find the domain and range of the inverse of f. 2 marks

Domain : $\mathbb{R} \setminus \{1\}$

Range : $\mathbb{R} \setminus \{-2\}$

c. Show that $f^{-1}(x)$ can be written in the form of $a + \frac{b}{1-x}$ and hence find $\int_0^{\frac{1}{2}} f^{-1}(x)\, dx$

4 marks

Use long division.

$\cancel{1-x \,\big)\, 2+2x \quad 2 \quad -(2-2x) \quad 4x}$

$$\begin{array}{r} -2 \\ -x+1 \,\big)\, 2x+2 \\ -(2x-2) \\ \hline 4 \end{array}$$

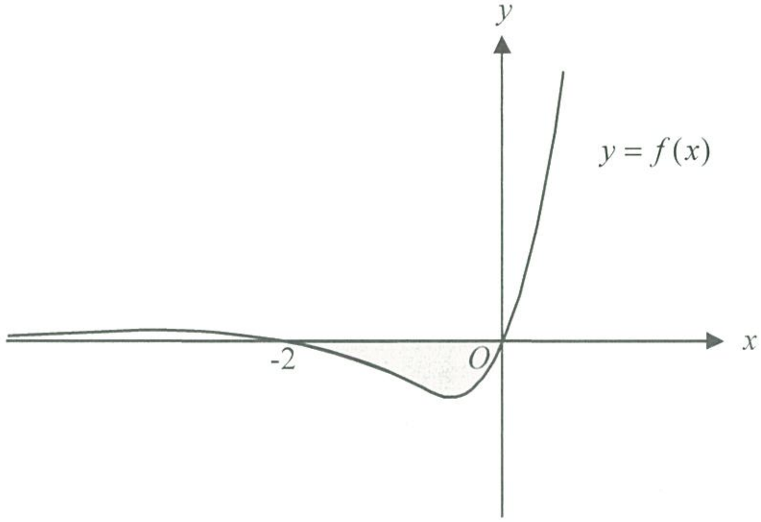
$\Rightarrow \dfrac{2+2x}{1-x} = -2 + \dfrac{4}{1-x}$

$$\int_0^{1/2} \left(-2 + \frac{4}{1-x}\right) dx = \Big[-2x - 4\log_e(1-x)\Big]_0^{1/2}$$

$$= -1 - 4\log_e\left(\tfrac{1}{2}\right) \Rightarrow \boxed{-1 + 4\log_e(2)}$$

Question 9 (4 marks)

The graph of $f: R \rightarrow R,\ f(x) = (x^2 + 2x)e^x$ is shown below.



The region enclosed by the graph of *f* and the *x*-axis is shaded.

a. Find the derivative of $(3 - x^2)e^x$. Give your answer in the form $ae^x - f(x)$, where *a* is a positive constant. 1 mark

$\frac{d}{dx}(3-x^2)e^x\,dx \Rightarrow (3-x^2)e^x + {}^-2xe^x$

$\Rightarrow 3e^x - x^2e^x - 2xe^x \Rightarrow 3e^x - (x^2+2x)e^x$

$\Rightarrow 3e^x - f(x)$

b. Use your answer to part **a.** to find the area of the shaded region. 3 marks

$-\int_{-2}^{0} f(x)\,dx \qquad \frac{d}{dx}(3-x^2)e^x\,dx = 3e^x - f(x)$

$f(x) = 3e^x - \frac{d}{dx}(3-x^2)e^x\,dx$

So. $\int_{-2}^{0} f(x)\,dx = \int_{-2}^{0} 3e^x\,dx - \int_{-2}^{0} \frac{d}{dx}(3-x^2)e^x\,dx$.

$= \left[3e^x\right]_{-2}^{0} - \left[(3-x^2)e^x\right]_{-2}^{0}$

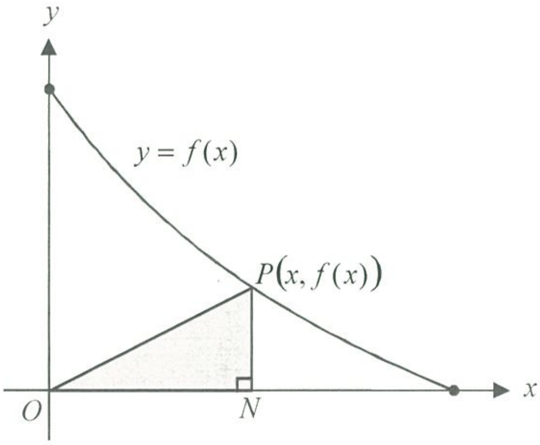
$= (3 - 3e^{-2}) - (3 - e^{-2})$

$= {}^-4e^{-2}$

$\therefore \quad -\int_{-2}^{0} f(x)\,dx = 4e^{-2}$.

Question 10 (7 marks)

Let $f:[0,1]\to R,\ f(x)=1-x^{\frac{2}{3}}$. The graph of f is shown below.



The right-angled triangle *NOP* has vertex *N* on the *x*-axis, and vertex *O* at the origin. The vertex *P* lies on the graph of *f* and has coordinates $(x, f(x))$ as shown.

a. Find the area *A*, of triangle *NOP* in terms of *x*. 1 mark

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times x \times f(x)$

$= \frac{1}{2} \times x \times (1 - x^{2/3})$

$= \frac{x}{2}(1 - x^{2/3})$

b. Find

i. the value of x for which A is a maximum. 2 marks

$$\frac{dA}{dx} = \left(\frac{x}{2} \times -\frac{2}{3}x^{-1/2}\right) + \left(\frac{1}{2}(1-x^{2/3})\right)$$

$$= \frac{-x^{2/3}}{3} + \frac{1}{2} - \frac{x^{2/3}}{2}$$

$$= \frac{-2x^{2/3} - 3x^{2/3}}{6} + \frac{1}{2} \qquad \frac{dA}{dx} = 0$$

$$0 = -\frac{5x^{2/3}}{6} + \frac{1}{2}$$

$$-\frac{1}{2} = -\frac{5x^{2/3}}{6} \Rightarrow 3 = 5x^{2/3} \Rightarrow x^{2/3} = \frac{3}{5}$$

$$x = \left(\frac{3}{5}\right)^{3/2}$$

ii. the maximum area of triangle *NOP*. Give your answer in the form $\frac{a\sqrt{b}}{c}$ where a, b and c are positive integers. 1 mark

$$A = \frac{1}{2}\left(\frac{3}{5}\right)^{3/2}\left(1 - \left(\left(\frac{3}{5}\right)^{3/2}\right)^{2/3}\right)$$

$$= \frac{1}{2}\left(\frac{3}{5}\right)^{3/2}\left(1 - \frac{3}{5}\right)$$

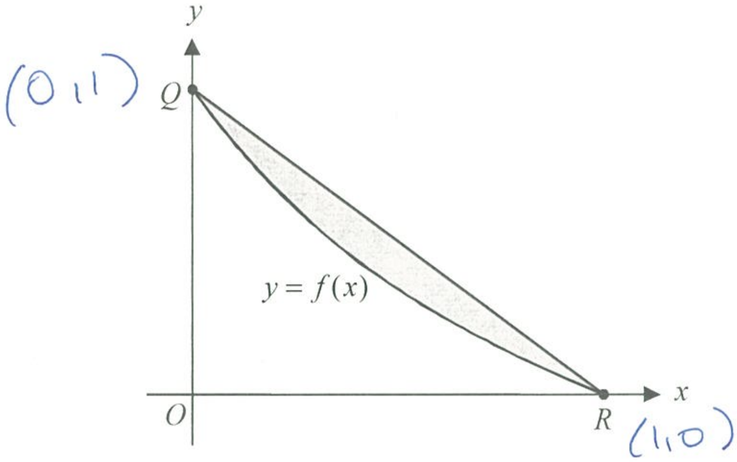
$$= \frac{1}{2}\left(\frac{3}{5}\right)^{3/2}\left(\frac{2}{5}\right)$$

$$= \frac{1}{5}\left(\frac{3}{5}\right)^{3/2}$$

$$= \frac{1}{5} \times \frac{\sqrt{27}}{\sqrt{125}}$$

$$= \frac{3\sqrt{3}}{125\sqrt{5}} \quad \text{OR} \quad \frac{3\sqrt{15}}{125}$$

c. The point Q lies on the graph of f and on the y-axis. The point R lies on the graph of f and on the x-axis.



Find the area enclosed by the line segment QR and the graph of f. 3 marks

Find Q (Y int of $f(x)$.

$f(x) = 1 - x^{2/3}$ y int ($x = 0$) $\therefore$ y int $(0,1)$

Find R (x int of $f(x)$.

$f(x) = 1 - x^{2/3}$.

$x^{2/3} = 1$ so $x = 1$ $(1,0)$.

$y - 1 = -1(x - 0)$.

$y = -x + 1 \Rightarrow y = 1 - x$.

Area $\int_0^1 (1 - x)\,dx - \int_0^1 (1 - x^{2/3})\,dx$.

$= \int_0^1 (\cancel{1} - x - \cancel{1} + x^{2/3})\,dx \Rightarrow \int_0^1 (x^{2/3} - x)\,dx$.

$= \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{x^2}{2}\right]_0^1 \Rightarrow \left(\frac{3}{5} - \frac{1}{2}\right) - 0$.

END OF QUESTION BOOKLET

$= \frac{6-5}{10}$

$= \frac{1}{10}$ sq units.