BILLANOOK COLLEGE

NAME: Answers . Student Number:

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

Practice July Exam

Test B TECHNOLOGY ACTIVE

Friday 21st July, 2017
Reading time: 15 minutes 1:00pm – 1:15pm
Writing time: 2 hour 1:15pm – 3:15pm

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of Questions*	*Number of questions to be answered*	*Number of marks*
1	20	20	20
2	5	5	60

Students are permitted to bring into the SAC room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality maybe used.

Students are NOT permitted to bring into the SAC blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book, Multiple Choice answer sheet, Formula sheet

Instructions

- Write your **name** in the space provided above on this page.
- All written responses must be in English.

09999 : VICTORIAN SECONDARY COLLEGE

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

VICTORIA State Government

VCE MATHEMATICAL METHODS
Written Examination 2
ANSWER SHEET – 2017

STUDENT NAME:

STUDENT NUMBER

INSTRUCTIONS:

USE PENCIL ONLY

SIGN HERE IF YOUR NAME AND NUMBER ARE PRINTED CORRECTLY.

SIGNATURE: ANSWERS

If your name or number on this sheet is incorrect, notify the Supervisor.
Use a **PENCIL** for **ALL** entries. For each question, shade the box which indicates your answer.
All answers must be completed like **THIS** example: A B C D E
Marks will **NOT** be deducted for incorrect answers.
NO MARK will be given if more than **ONE** answer is completed for any question.
If you make a mistake, **ERASE** the incorrect answer – **DO NOT** cross it out.

SUPERVISOR USE ONLY

USE PENCIL ONLY

Shade the "**ABSENT**" box if the student was absent from the examination.

ABSENT ☐

SUPERVISOR'S INITIALS

OFFICE USE ONLY

☐ ☐

ONE ANSWER PER LINE | **ONE ANSWER PER LINE**

1	A	B	C	D	E	11	A	B	C	D	E
2	A	B	C	D	E	12	A	B	C	D	E
3	A	B	C	D	E	13	A	B	C	D	E
4	A	B	C	D	E	14	A	B	C	D	E
5	A	B	C	D	E	15	A	B	C	D	E
6	A	B	C	D	E	16	A	B	C	D	E
7	A	B	C	D	E	17	A	B	C	D	E
8	A	B	C	D	E	18	A	B	C	D	E
9	A	B	C	D	E	19	A	B	C	D	E
10	A	B	C	D	E	20	A	B	C	D	E

Please DO NOT fold, bend or staple this form.

SECTION A – Multiple-choice questions

Question 1

The period of the function $y = 2\tan\left(\frac{3\pi x}{4}\right)$ is

A. $\frac{3}{4}$

B. $\frac{3\pi}{4}$

C. $\frac{4}{3}$

D. $\frac{3\pi}{2}$

E. $\frac{8}{3}$

Question 2

The function with rule $f(x) = (x-1)^2$ has a range of $[1, \infty)$.
The domain of f could be

A. $x \in (-\infty, 0]$

B. $x \in (-\infty, 0)$

C. $x \in (2, \infty)$

D. $x \in [0, 2]$

E. $x \in (-\infty, 0) \cup (2, \infty)$

Question 3

The graph shown has the equation

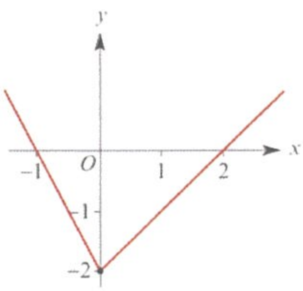
A. $y = \begin{cases} x - 2, x > 0 \\ -2x - 2, x \le 0 \end{cases}$

B. $y = \begin{cases} 2x - 2, x \ge 0 \\ -2x - 2, x < 0 \end{cases}$

C. $y = \begin{cases} x - 2, x > 0 \\ -2x - 1, x \le 0 \end{cases}$

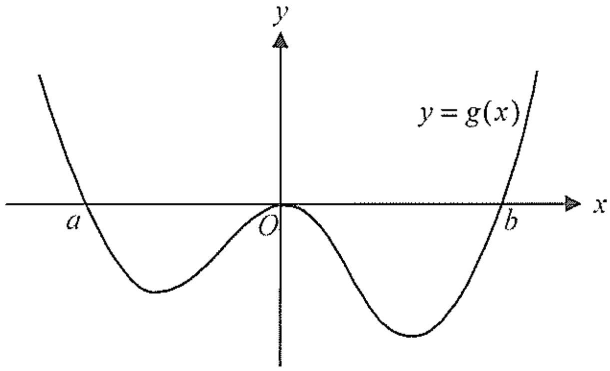
D. $y = \begin{cases} x + 2, x > 0 \\ -2x - 2, x \le 0 \end{cases}$

E. $y = \begin{cases} x - 2, x > 0 \\ -x - 2, x \le 0 \end{cases}$



Question 4

The graph of the function g is shown below.



The rule for g could be

A. $g(x) = -x(x+a)(x+b)$

B. $g(x) = -x^2(x-a)(x-b)$

C. $g(x) = x(x-a)(x-b)$

D. $g(x) = x^2(x-a)(x-b)$

E. $g(x) = x^2(x+a)(x-b)$

Question 5

If $\frac{2(x-1)}{3} - \frac{x+4}{2} = \frac{5}{6}$, then x equals

A. 5

B. $\dfrac{7}{5}$

C. $\dfrac{21}{5}$

D. 21

E. 3

Question 6

Let $f:[0,a]\to R,\ f(x)=\cos\left(2\left(x+\frac{\pi}{3}\right)\right)$.

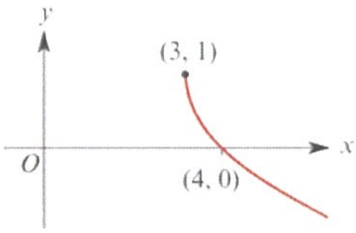
If the inverse function f^{-1} exists, then the maximum possible value of a is

A. $\frac{\pi}{6}$
B. $\frac{\pi}{4}$
C. $\frac{\pi}{3}$
D. $\frac{\pi}{2}$
E. $\frac{2\pi}{3}$

Question 7

A possible equations for the graph shown is:

A. $y = 2\sqrt{x-3}+1$
B. $y = -2\sqrt{x-3}+1$
C. $y = \sqrt{x-3}+1$
D. $y = -\sqrt{x-3}+1$
E. $y = -2\sqrt{x-3}+2$



Question 8

The tangent to the graph of $y=\log_e(ax),\ a>0$ at the point where $x=\frac{1}{a}$, has a y-intercept of

A. -1
B. $-a$
C. 0
D. 1
E. a

Question 9

If x^3-5x^2+x+k is divisible by $x+1$, then k equals

A. -7
B. -5
C. -2
D. 5
E. 7

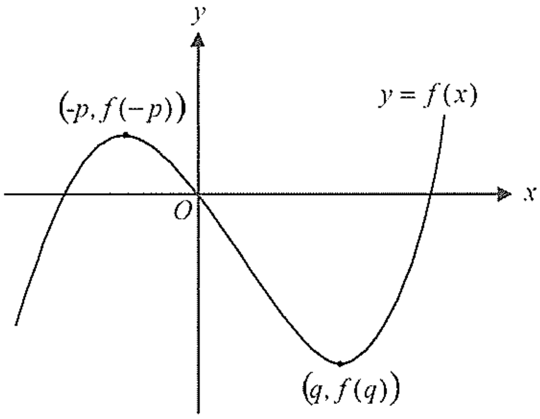
Question 10

The gradient of the curve with equation $y = \sin(2x) + 1$ at $(0,1)$ is:

A. 1
B. -1
C. 0
D. 2
E. -2

Question 11

The graph of $y = f(x)$ is shown below.



The graph has stationary points at the points where $x = -p$ and $x = q$.
The largest interval for which the function f is strictly decreasing is

A. $x = (-\infty, -p)$
B. $x \in (-p, q)$
C. $x \in [-p, q]$
D. $x \in (0, q)$
E. $x \in [0, q]$

Question 12

The inverse function of $g : [1, \infty) \to R,\ g(x) = \sqrt{x-1} + 2$ is

A. $g^{-1} : [-2, 1] \to R,\ g^{-1}(x) = (x+2)^2 + 1$
B. $g^{-1} : [1, \infty) \to R,\ g^{-1}(x) = (x+2)^2 - 1$
C. $g^{-1} : [2, \infty) \to R,\ g^{-1}(x) = (x+2)^2 - 1$
D. $g^{-1} : [1, \infty) \to R,\ g^{-1}(x) = (x-2)^2 + 1$
E. $g^{-1} : [2, \infty) \to R,\ g^{-1}(x) = (x-2)^2 + 1$

Question 13

If $f:[0,2\pi]\to R$ where $f(x)=\sin(2x)$ and $g:[0,2\pi]$ where $g(x)=2\sin(x)$, then the value of $(f+g)(\frac{3\pi}{2})$ is:

A. 2
B. 0
C. -1
D. 1
E. -2

Question 14

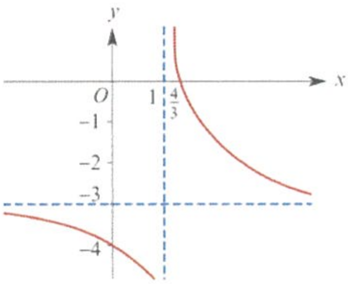
For $f(x)=e^x-2x$, the average rate of change with respect to x over the interval $[0,1]$ is

A. $e-1$
B. $e-2$
C. $e-3$
D. $\frac{e-1}{2}$
E. $\frac{1}{e-2}$

Question 15

A possible equation for the graph shown is:

A. $y-3=\frac{1}{x-1}$
B. $y+3=\frac{1}{x+1}$
C. $y-3=\frac{1}{x+1}$
D. $y-4=\frac{1}{x+1}$
E. $y=\frac{1}{x-1}-3$



Question 16

If $\int_1^4 f(x)dx=6$ find $\int_1^4 (5-2f(x))dx$ is equal to

A. 3
B. 4
C. 5
D. 6
E. 16

Question 17

The simultaneous linear equations $ax+3y=a-3$ and $2x+(a+1)y=-1$ have no solution for

A. $a=2$
B. $a=-3$
C. $a \in R\setminus\{-3,2\}$
D. $a=2$ and $a=-3$
E. $a \in R\setminus\{2\}$

Question 18

If $g(x+2)=x^2+3x+7$, then $g(x)$ is equal to

A. x^2-2x
B. x^2+x
C. x^2-x-3
D. x^2-x+5
E. x^2+x-1

Question 19

Consider the cubic function $g: R \to R,\ g(x)=ax^3+2bx^2+x+5$, where a and b are positive constants. The graph of g has more than one stationary point when

A. $a<\dfrac{3b^2}{4}$
B. $a>\dfrac{3b^2}{4}$
C. $a<\dfrac{4b^2}{3}$
D. $a>\dfrac{4b^2}{3}$
E. $a>\dfrac{2\sqrt{3}b^2}{3}$

Question 20

The transformation $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, maps the graph of the function *f* to the graph of the function $y = \sqrt{x}$.

The rule of *f* is

A. $f(x) = -\sqrt{x} - 1$
B. $f(x) = \sqrt{-x} + 1$
C. $f(x) = \sqrt{x-1} - 1$
D. $f(x) = -\sqrt{x} + 1$
E. $f(x) = -\sqrt{x-1}$

SECTION B

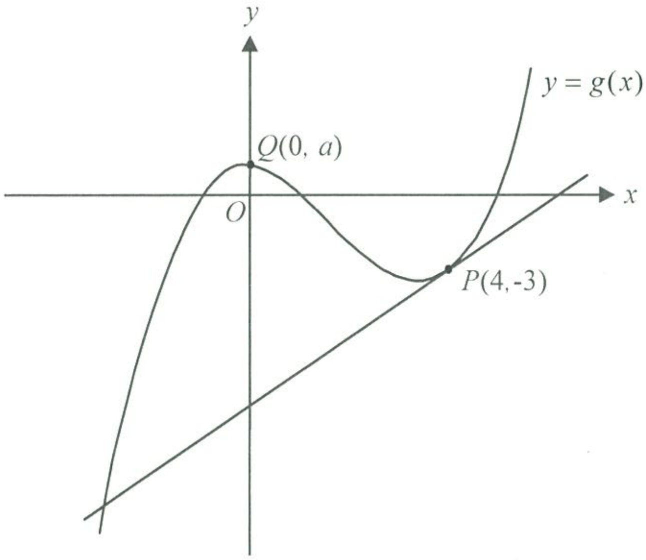
Answer all questions in this section.

Question 1 (8 marks)

Let $g : R \to R,\ g(x) = \frac{1}{5}(x^2 - 1)(x - 5)$.

The points $P(4,-3)$ and $Q(0,a)$ lie on the graph of g where a is a positive constant.

The graph of g and the tangent to the graph of g at the point $P(4,-3)$ are shown below.



a. Find the value of a. 1 mark

$g(0) = \frac{1}{5}(0^2-1)(0-5) \Rightarrow 1$

$\therefore a = 1$

b. For the tangent to the graph of g at the point $P(4,-3)$, find

i. its gradient 1 mark

$g'(4) = \frac{7}{5}$

ii. its equation. 1 mark

$y - -3 = \frac{7}{5}(x-4)$

$y + 3 = \frac{7}{5}(x-4$

$y = \frac{7x}{5} - \frac{43}{5}$

c. Find the distance PQ. Give your answer in the form $b\sqrt{c}$, where b and c are positive integers.

2 marks

Using $(0,1)$ $(4,-3)$

$d = \sqrt{(4-0)^2+(-3-1)^2}$

$= \sqrt{32}$

$= 4\sqrt{2}$

d. Find the area enclosed by the graph of g and the tangent to the graph of g at the point $P(4,-3)$.

3 marks

$\int_{-3}^{4} \left(g(x) - \left(\frac{7x}{5} - \frac{43}{5}\right)\right) dx$

$= \frac{2401}{60}$ square units.

These values from

$\frac{1}{5}(x^2-1)(x-5) = \frac{7x}{5} - \frac{43}{5} \Rightarrow x = -3$

$x = 4$

Question 2 (12 marks)

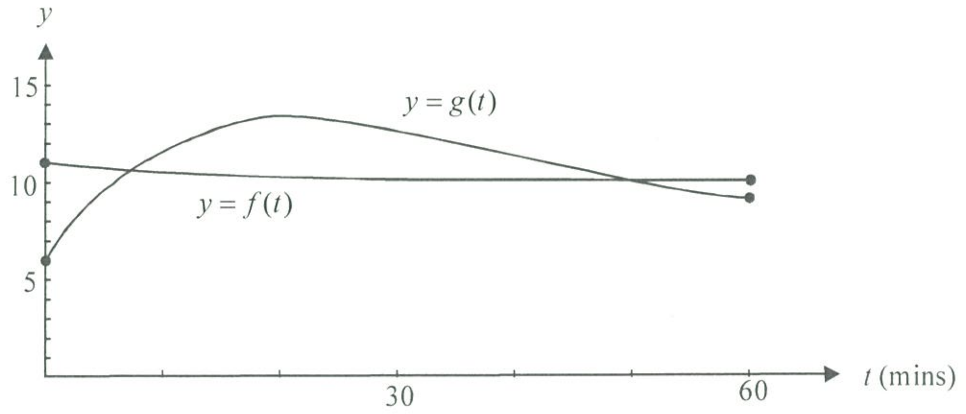
The stock market value of two stocks, Foolsgold and Gold Inc., are modelled respectively by the functions

$$f:[0,60] \to R, f(t) = e^{-\frac{t}{20}} + 10$$

$$\text{and } g:[0,60] \to R, g(t) = te^{-\frac{t}{20}} + 6$$

where f and g represent the value of the respective stocks, in dollars, t minutes after the opening of trade on a particular day.

The graphs of the functions are shown below.



a. Find the values of t when the values of the two stocks were equal. Give your answers correct to three decimal places. 2 marks

$f(t) = g(t)$ solve for t

$t = 6.550$ $t = 50.186$

b. Find the maximum value of the Gold Inc. stock during the first hour of trade. Give your answer to the nearest cent. 2 marks

$g'(t) = 0$ $t = 20$

$g(20) = 13.36$

$\therefore$ max value is \$13.36

c. Find the average value of the Gold Inc. stock during the first hour of trade.
Give your answer to the nearest cent. 2 marks

Avg value $= \frac{1}{60}\int_0^{60} g(t)\,dt$

$= \$11.34$

d. During the period when the value of the Gold Inc. stock was greater than the value of the Foolsgold stock, find the value of t when the difference in the values was a maximum.

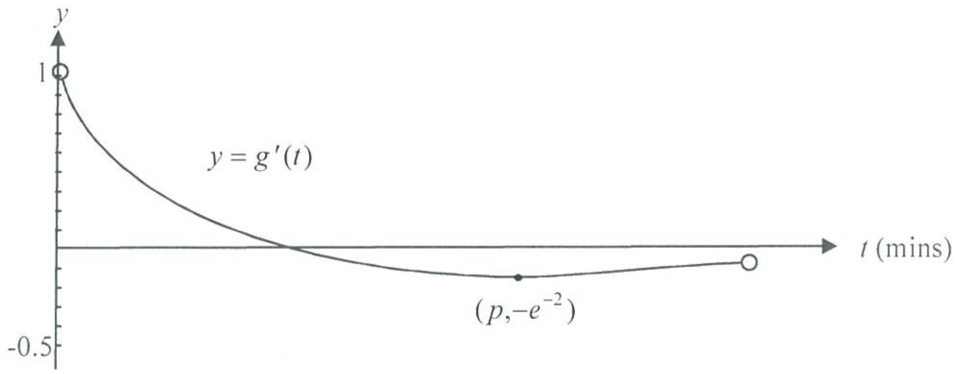
3 marks

$d(t) = g(t) - f(t)$

$d'(t) = 0$ solve for ~~d~~ t

$t = 21$

The graph of the derivative function $g'(t) = \left(1 - \frac{t}{20}\right)e^{-\frac{t}{20}}$, $t \in (0,60)$ is shown below. The graph has a minimum turning point at the point $(p, -e^{-2})$ where p is a positive integer.



e. **i.** Find the value of p. 2 marks

$t = 40$ $\therefore$ $p = 40$

$g''(t) = 0$. solve for t.

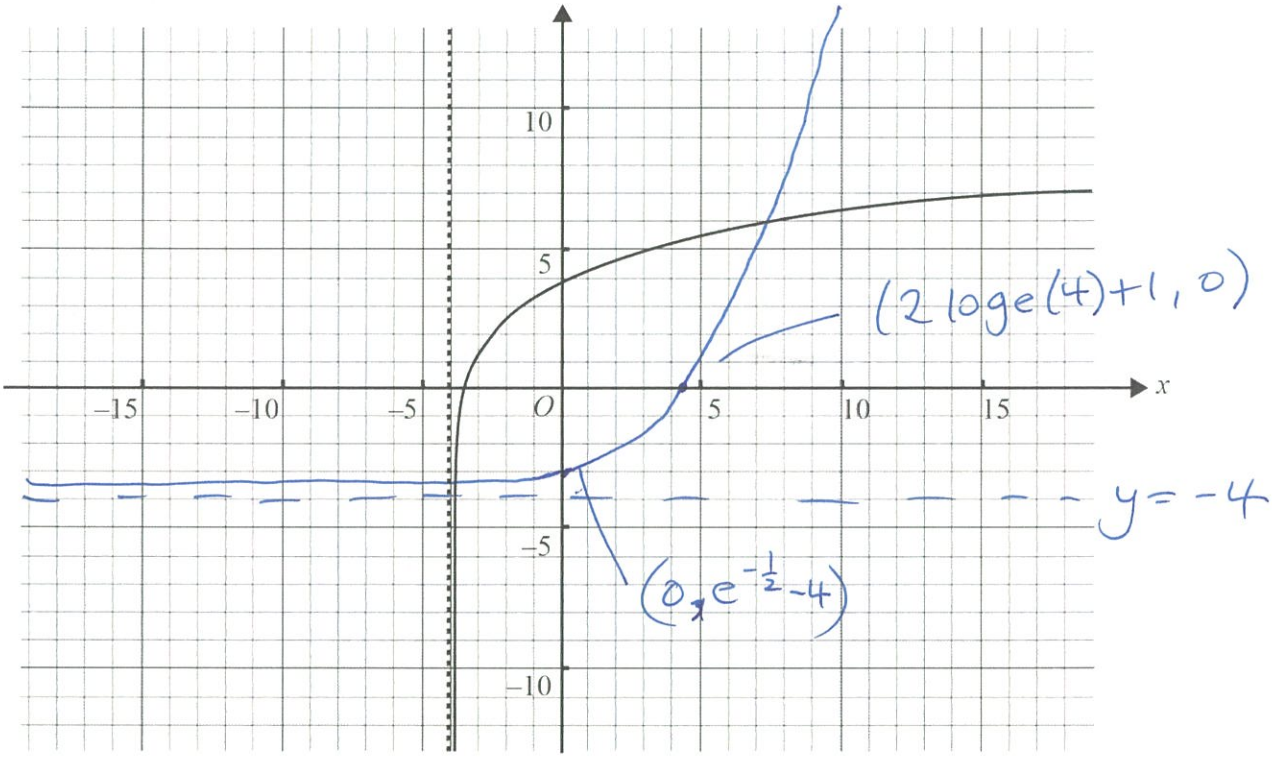
ii. **<u>Hence</u>** find the value of the Gold Inc. stock when the rate at which it was decreasing was a maximum. Give your answer to the nearest cent. 1 mark

stock decreasing max when $t = 40$

$g(40) = \$11.41$

Question 3 (17 marks)

a. Part of the graph of the function $g: (-4, \infty) \rightarrow R$, $g(x) = 2\log_e(x + 4) + 1$ is shown on the axes below.



i. Find the rule and domain of g^{-1}, the inverse function of g.

3 marks

$y = 2\log_e(x+4) + 1$ Swap x & y inverse

$x = 2\log_e(y+4) + 1$

$y = e^{\frac{x}{2}-\frac{1}{2}} - 4$, Domain: R.

ii. On the set of axes above sketch the graph of g^{-1}. Label the axes intercepts with their exact values. ✓ As above.

3 marks

iii. Find the values of x, correct to three decimal places, for which $g^{-1}(x) = g(x)$.

2 marks

$g^{-1}(x) = g(x)$ solve for x

$x = -3.914$ or $x = 5.503$

Calculate the area enclosed by the graphs of g and g^{-1}. Give your answer correct to two decimal places.

$$\int_{-3.914\ldots}^{5.503\ldots} \left(g(x) - g^{-1}(x)\right) dx$$

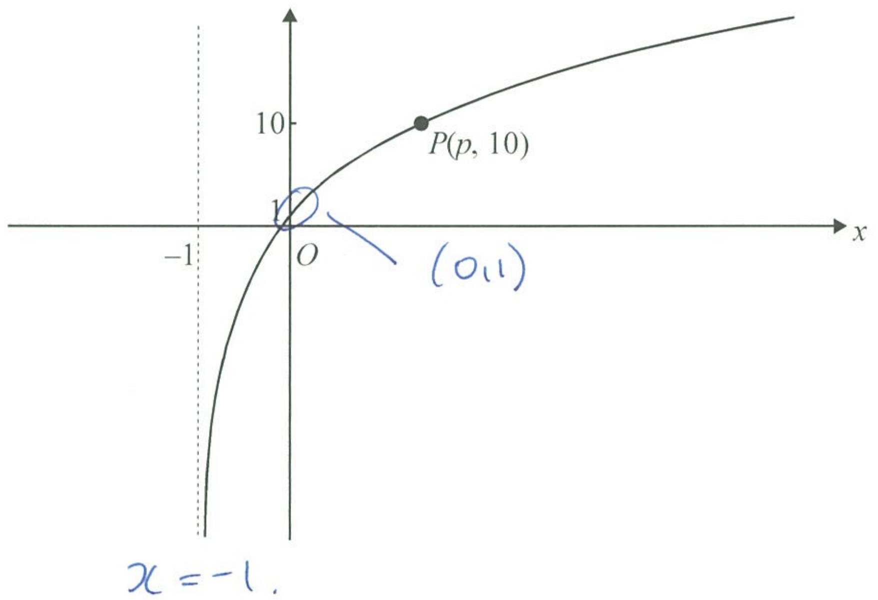
2 marks

Area = 52.63 square units

b. The diagram below shows part of the graph of the function with rule

$$f(x) = k \log_e(x + a) + c, \text{ where } k, a \text{ and } c \text{ are real constants.}$$

- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y-axis intercept at 1.
- The point P on the graph has coordinates $(p, 10)$, where p is another real constant.



i. State the value of a.

1 mark

$a = 1$

ii. Find the value of c.

1 mark

$c = 1$

iii. **Show** that $k = \frac{9}{log_e(p+1)}$ Using i), ii) & (p, 10)

2 marks

$10 = k \log_e(p+1) + 1$

$9 = k \log_e(p+1)$

$k = \frac{9}{\log_e(p+1)}$

iv. Show that the gradient of the tangent to the graph of f at the point P is $\frac{9}{(p+1)log_e(p+1)}$

1 mark

$f(x) = \frac{9}{\log_e(p+1)} \log_e(x+1) + 1$

$f'(x) = \frac{9}{(x+1)\cancel{\log}\log_e(p+1)}$

$f'(p) = \frac{9}{(p+1)\log_e(p+1)}$

v. If the point (–1, 0) lies on the tangent referred to in **part b.iv.**, find the exact value of p.

Using (p, 10) & (–1, 10).

2 marks

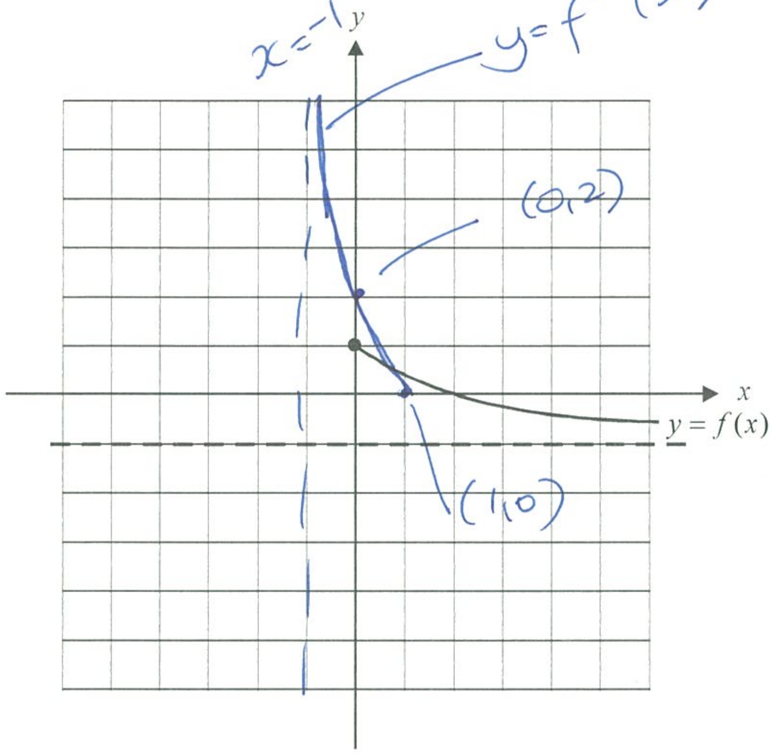
$f'(p) = \frac{10-0}{p+1}$

solve for p $\Rightarrow$ $p = e^{\frac{9}{10}} - 1$.

Question 4 (8 marks)

Let $f:[0,\infty)\to R,\ f(x)=\dfrac{4}{x+2}-1$.

The graph of f is shown below.



a. On the same set of axes, sketch the graph of f^{-1}, the inverse function of f.
Indicate clearly the coordinates of any axes intercepts and the equation of any asymptotes. 2 marks

The graph of f is

- y_1 • dilated by a factor of 2 units from the y-axis and then
- y_2 • reflected in the x-axis and then
- y_3 • translated 3 units vertically upwards

to become the graph of $y=h(x)$. $f(x)=\frac{4}{x+2}-1 \Rightarrow y=\frac{4}{x+2}-1$

b. Write down the rule for h. 2 marks

$y_1=\frac{4}{\frac{x}{2}+2}-1 \Rightarrow y_1=\frac{8}{x+4}-1$

$y_2=-y_1 \Rightarrow -y_1=\frac{8}{x+4}-1 \Rightarrow y_2=1-\frac{8}{x+4}$

$y_3=4-\frac{8}{x+4}$ $\therefore$

$h(x)=4-\frac{8}{x+4}$

Let $q:[0,\infty)\to R,\ q(x)=\dfrac{a}{x+2}-1$ where $a\geq 2$.

c. Find, in terms of a, the coordinates of the x-intercept of the graph of q. 1 mark

x int ($y=0$)

$0=\dfrac{a}{x+2}-1$

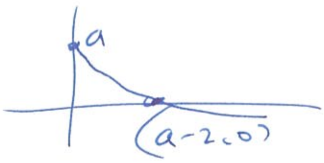
$1=\dfrac{a}{x+2}$

$a=x+2$

$x=a-2$

$(a-2,0)$.

d. Find the area enclosed by the graph of q and the x and y-axes. Give your answer in the form $\log_e\left(\dfrac{u}{v}\right)-a+2$, where u and v are functions of a. 3 marks



Area: $\displaystyle\int_0^{a-2}\left(\frac{a}{x+2}-1\right)dx$.

$= a\log_e(a)-a\log_e(2)-a+2$

$=\log_e a^a-\log_e 2^a-a+2$.

$=\log_e\left(\dfrac{a^a}{2^a}\right)-a+2$ sq. units.

Question 5 (15 marks)

Victoria James is a spy.
She is trapped in a stationary mini-submarine that is being fired on by an enemy ship. The ship is firing 'dolphin' missiles which follow a curved path.
The vertical distance v, in metres, of a missile above the surface of the water (or below if $v < 0$) is given by

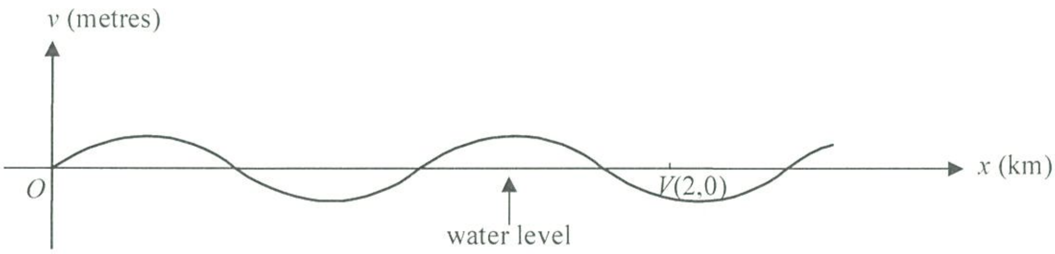
$$v(x) = 3\sin(\pi a x), \quad x \geq 0, \quad a > 0$$

where x kilometres is the horizontal distance of the missile from the ship. The positive constant a can be reset for each missile.

a. What is the maximum distance below the surface of the water that a dolphin missile can reach? 1 mark

3 metres.

The enemy ship is stationary and is located at the point $O(0,0)$. Victoria is located 2 kilometres away at the point $V(2,0)$. The graph below shows the path of the first missile fired at Victoria.



This first missile entered the water for the **second** time at a point that was 0.2 kilometres short of Victoria's position.

b. Show that for this first missile, $a = \frac{5}{3}$. 1 mark

$\frac{2\pi}{n} = \frac{2\pi}{\pi a} = \frac{2}{a}$ $\quad 1.8 \div 3 = 0.6$.

$\frac{2}{a} = 1.2$ – 2nd x intercept

$a = \frac{1.2}{2} \Rightarrow \frac{5}{3}$.

c. Find the acute angle between the horizontal and the tangent to the path of the first missile as it emerges from the water. Give your answer in degrees correct to two decimal places. 2 marks

$v(x) = 3\sin\left(\frac{5\pi x}{3}\right)$ $\quad v'(x) = 5\pi\cos\left(\frac{5\pi x}{3}\right)$

at $x = 1.2$ $\quad v'(x) = 5\pi$.

$\therefore \tan(\theta) = 5\pi \Rightarrow \theta = 86.36°$.

Further missiles are fired.

d. Find the value of a if a missile was to hit Victoria's submarine as the missile entered the water for the second time. 1 mark

$\frac{2}{3} \times 2 = \frac{4}{3}$ $\frac{2\cancel{\pi}}{\cancel{\pi} a} = \frac{4}{3}$

$a = \frac{2}{4/3}$

$a = \frac{6}{4} = \frac{3}{2}$

e. Find the values of a for which a missile will pass over Victoria's submarine at $V(2,0)$ before hitting the water for the first time. 2 marks

$\frac{2}{a} > 4$

$a < \frac{1}{2}, \quad a > 0$

$0 < a < 1/2$.

Victoria leaves her mini-submarine and swims to the enemy ship. She is stationary as she attaches a bomb to the hull of the ship.
Immediately, Victoria starts swimming in a straight line away from the ship.
Victoria's speed, in km/h as she swims away is given by

$$s:[0,d] \to R, \; s(x) = (x+k)^2 - 2^{x+1} + 1$$

where x is Victoria's horizontal distance in km from the ship, k is a positive constant and d is Victoria's distance from the ship when she stops swimming.
Victoria's speed is 2km/h when she is 2km from the ship.

f. Show that $k = 1$. 1 mark

$(2+k)^2 - 2^{2+1} + 1 = 2$.

$4 + 4k + k^2 - 2^3 + 1 = 2$.

$k^2 + 4k - 5 = 0$

$(k+5)(k-1) = 0$

$k = -5$ or $k = 1$.

reject $k = -5$ as $k > 0$

so $\boxed{k = 1}$

g. Find the value of d. Give your answer correct to two decimal places. 2 marks

$s(x) = d$ $\quad$ d = Victoria's distance from ship.

$x > 0$. at ship when she stops swimming

$x = 0$ or $x = 3.25746....$ $\therefore$ $x = 3.26$km.

h. i. Find Victoria's distance from the ship when her speed is a maximum. Give your answer correct to four decimal places. 2 marks

$s'(x) = 0$ $\quad x = -0.5149....$ $\quad x = 2.212432....$

Victoria is 2.2124km from ship at max speed.

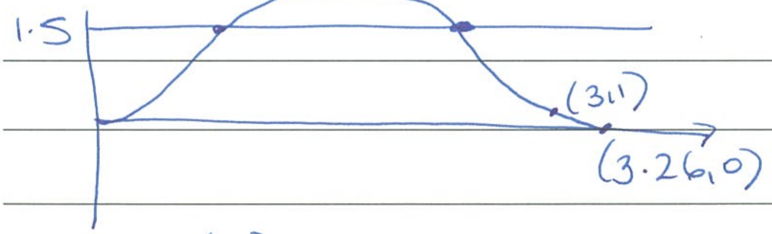
ii. Find the maximum speed Victoria reaches whilst swimming. Give your answer correct to three decimal places. 1 mark

$s(2.212432.....) = 2.050601$

Max speed is 2.051 km/h.

Victoria must detonate the bomb whilst swimming.
To do so, she must be swimming at 1.5km/h or slower and must be at least 3km from the enemy ship.

i. Find the values of x for which Victoria can detonate the bomb. Give your answer as an interval with endpoints expressed correct to two decimal places. 2 marks



$s(x) = 1.5$

$x = -2$, $x = 1.4158$ $\quad x = 2.8054$.

$\therefore$ $x \in [3, 3.26)$. or

$3 \leq x < 3.26$.

END OF EXAMINATION