SET 3 EXAM 2

Reading time: 15 minutes

Writing time: 120 minutes

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

Note: Formula Sheet is NOT supplied. You will need to use your own!

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, **one** bound reference, **one** approved CAS calculator and one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book.

Instructions

- Complete all multiple-choice questions by circling your choice on the book.
- Complete all extended-response questions in the spaces provided.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1 - Multiple-Choice Questions

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

Let $f(x) = \sqrt{9 - x^2}$ and $g(x) = \dfrac{1}{\sqrt{3 - x}}$.
The maximal domain of $f(x)g(x)$ is

A. $[-3, 3]$

B. $[-3, \infty)$

C. $\mathbb{R} \setminus \{3\}$

D. $[-3, 3)$

E. $\left(-\infty, \dfrac{26}{9}\right]$

Question 2

If $f'(x) = -2\sin(2x)$ and $f(0) = 1$, it follows that

A. $f(x) = 5 - 4\cos(2x)$

B. $f(x) = 4\cos(2x) - 3$

C. $f(x) = \cos(2x)$

D. $f(x) = 2 - \cos(2x)$

E. $f(x) = 1 - \sin(2x)$

Question 3

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that will take the graph of $y = \frac{1}{2}x^2 + x + 1$ to the graph of $y = x^2$ is

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Question 4

The maximal domain of the function with rule $f(x) = \log_e\left((a - x)^4\right)$, where $a \in \mathbb{R}$, is

A. $(-\infty, a)$

B. $\mathbb{R} \setminus \{a\}$

C. (a, ∞)

D. $(-\infty, a]$

E. $\mathbb{R}$

Question 5

The quartic polynomial function $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ passes through the points $(-3, 61)$, $(-2, 11)$, $(2, 31)$, $(-1, 1)$, and $(1, 5)$.

A matrix equation that can be solved to give the values of a, b, c, d, and e is

A. $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 5 \\ 61 \\ 31 \\ 1 \\ 11 \end{bmatrix}$$

B. $$\begin{bmatrix} 16 & 8 & 4 & 2 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 81 & -27 & 9 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 61 \\ 1 \\ 31 \end{bmatrix}$$

C. $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 31 \\ 61 \\ 11 \end{bmatrix}$$

D. $$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 31 \\ 61 \\ 11 \end{bmatrix}$$

E. $$\begin{bmatrix} 81 & -27 & 9 & -3 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 61 \\ 1 \\ 31 \\ 5 \\ 11 \end{bmatrix}$$

Question 6

Let $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$.

Then, for all $x \in \mathbb{R}$, the expression $\dfrac{d}{dx}\left(\log_e\left(\dfrac{2f(x)}{g(x)}\right)\right)$ is equal to

A. $\dfrac{2f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)}$

B. $\dfrac{2\left(f'(x)g(x) - f(x)g'(x)\right)}{\left(g(x)\right)^2} \log_e\left(\dfrac{2f(x)}{g(x)}\right)$

C. $\dfrac{2f'(x)g(x) - f(x)g'(x)}{f(x)g(x)}$

D. $\dfrac{f'(x)}{f(x)} - \dfrac{g'(x)}{g(x)}$

E. $\dfrac{f'(x)g(x) + f(x)g'(x)}{f(x)g(x)}$

Question 7

A biased coin is tossed 25 times. If the probability of the coin landing on heads is 0.7, then the probability, correct to four decimal places, of observing tails more than 10 times is

A. 0.0018

B. 0.9995

C. 0.0978

D. 0.9982

E. 0.1894

Question 8

The cubic function $p: \mathbb{R} \to \mathbb{R}$, where $p(x) = ax^3 + bx^2 + cx$, will have no stationary points exactly when

A. $a = 0$ and $b = 0$

B. $b^2 < 3ac$

C. $b^2 > 4ac$

D. $b^2 = 3ac$

E. $b^2 < 4ac$

Question 9

The average value of the function $g(x) = x^2 \sin(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is closest to

A. 1.234

B. 1.571

C. 1.142

D. 0.727

E. 1.793

Question 10

A continuous random variable Y is normally distributed with mean 164 and variance 4. If the random variable Z has the standard normal distribution, then $\Pr(Y > 160)$ is equal to

A. $\Pr(Z > -1)$

B. $1 - \Pr(Z < 1)$

C. $\Pr(Z > 3)$

D. $\Pr(Z < 2)$

E. $1 - \Pr(Z > -2)$

Question 11

The discrete random variable X has the following probability distribution.

x	0	1	2	3	4
$\Pr(X = x)$	p	0.15	0.15	0.45	0.05

It follows that

A. $p = 0.25$

B. the mean value of X is $p + 2$

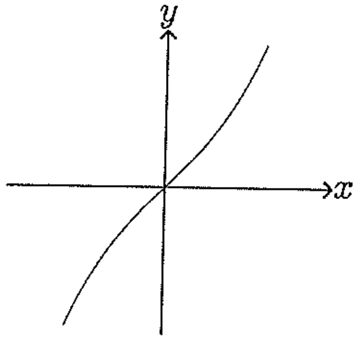
C. the median value of X is 2.5

D. the standard deviation of X is 1.6

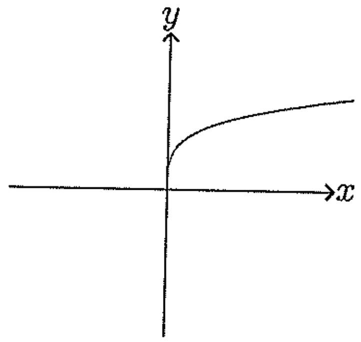
E. the variance of X is 5.6

Question 12
Each of the following graphs represent a relation between x and y. Which of these represents a relation that has no inverse?

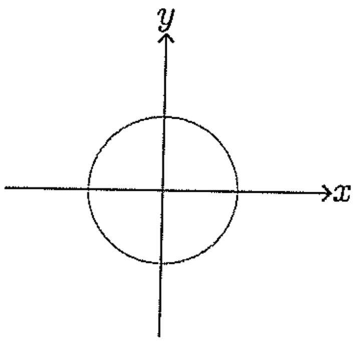
A.



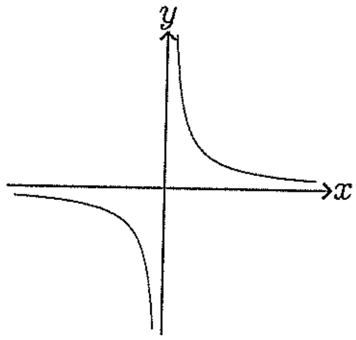
B.



C.



D.



E. None of these graphs represent a relation with no inverse

Question 13
If X is normally distributed with a mean of 0, and $\Pr(X > 2.5) = 0.1$, then the variance of X is closest to

A. 1.95

B. -1.28

C. 3.81

D. 1.28

E. 1.64

Question 14

If $f(x) = \sin(x)$ and $g(x) = \cos(x)$, then for all $x \in \mathbb{R}$, it is true that

A. $f(g(-|x|) = -f(g(|x|))$

B. $f(-g(-x)) = -f(g(x))$

C. $-f(-g(x - \pi)) = f(g(x))$

D. $f(g(\pi - x)) = f(g(x))$

E. $g(f(-x)) = -g(f(x))$

Question 15

Let $f(x) = \begin{cases} 4 - (x-2)^2 & \text{if } x < 1 \\ 2x + 1 & \text{if } x \geq 1 \end{cases}$

Which of the following statements is true?

A. $f(x)$ is not defined at $x = 1$

B. $f(x)$ is always positive

C. $f(x) - |f(x)| = 0$ for all $x < 0$

D. $f(x)$ is differentiable everywhere

E. $\frac{d}{dx}(f(x)) = \frac{d}{dx}(|f(x)|)$ for all $x \geq 0$

Question 16

Let $p : \mathbb{R} \to \mathbb{R}$, where $p(x) = -x^3 + 3x^2 - 3x + 2$. It follows that

A. the graph of p intersects the graph of p^{-1} at exactly 3 distinct points

B. the graph of p intersects the graph of p^{-1} at exactly 2 distinct points

C. the graph of p intersects the graph of p^{-1} at exactly 1 distinct point

D. the graph of p does not intersect the graph of p^{-1}

E. p does not have an inverse

Question 17

If two events A and B in a given sample space are such that $\Pr(A) \neq 0$ and $\Pr(B \mid A) = \frac{1}{2}\Pr(B' \mid A)$, then

A. $\Pr(A \cap B) = \frac{1}{3}\Pr(A)$

B. $\Pr(A) = \Pr(A \cap B)$

C. $\Pr(B) = \frac{1}{2}\Pr(B')$

D. $\Pr(A \cap B') = \frac{1}{3}\Pr(A)$

E. $\Pr(A \cap B) = \frac{2}{3}\Pr(A)$

Question 18

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 - 2$.
$|-f(x)|$ has positive gradient for

A. $x \in (0, \infty)$

B. $x \in (-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$

C. $x \in [-\sqrt{2}, 0) \cup [\sqrt{2}, \infty)$

D. $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

E. $x \in (-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$

Question 19

If $x + 1$ is a factor of $x^3 - 2ax^2 + 3x - 7$, then

A. $a = \frac{26}{7}$

B. $a = -\frac{11}{2}$

C. $a = -\frac{3}{2}$

D. $a = -2$

E. $a = \frac{5}{2}$

Question 20

A region with area A is enclosed by the graphs of $y = 2 - kx^2$ and $y = \frac{x^2}{k}$, where $k > 0$.
Correct to two decimal places, the maximum value of A is

A. 1.63

B. 0.45

C. 1.00

D. 2.67

E. 2.59

Question 21

The tangent to the graph of $y = \frac{1}{\sqrt{x}}$ at the point $(1, 1)$ is used to approximate the value of $\frac{1}{\sqrt{0.7}}$ using the relationship $f(1+h) \approx f(1) + hf'(1)$ for a small value of h. To the nearest percent, this approximation is

A. 29% less than the exact value

B. 29% more than the exact value

C. 15% more than the exact value

D. 4% more than the exact value

E. 4% less than the exact value

Question 22

The simultaneous linear equations $3kx + (4-k)y = 12$ and $kx - y = k - 3$ will have infinitely many solutions exactly when

A. $k \in \mathbb{R} \setminus \{0, 7\}$

B. $k \in \{0, 7\}$

C. $k \in \mathbb{R} \setminus \{0\}$

D. $k = 0$

E. $k = 7$

SECTION 2 - Extended-Response Questions

Instructions for Section 2

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The function f is defined by $f : [0, 3\pi] \to \mathbb{R}$, where $f(x) = \sin(x) + \frac{x}{2}$.

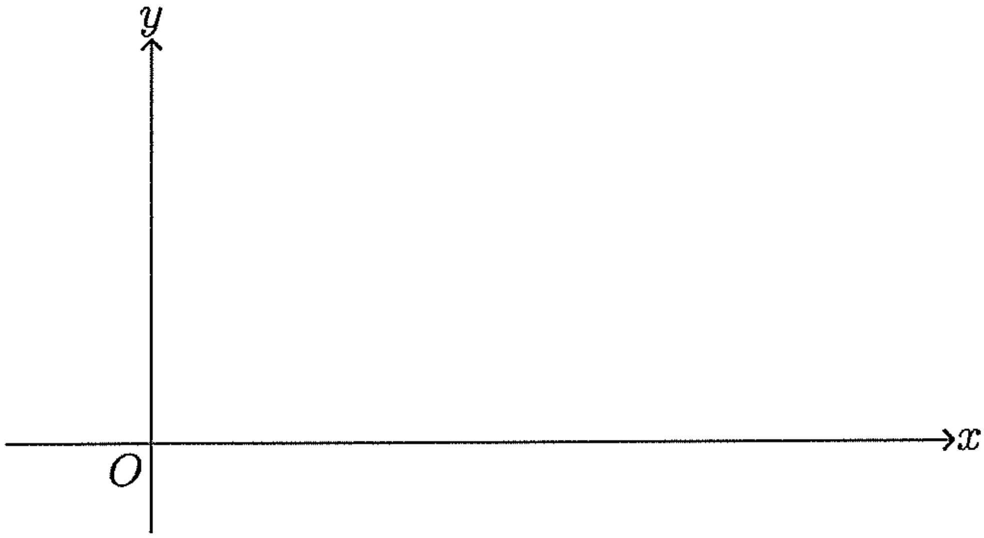
a. **i.** Find the general solution to the equation $\cos(x) = -\frac{1}{2}$ for $x \in \mathbb{R}$.

ii. Find the values of x for which $f'(x) = 0$.

iii. Hence, find the coordinates of the stationary points of the graph of $y = f(x)$.

$2 + 2 + 2 = 6$ marks

b. Sketch the graph of $y = f(x)$ and the line $y = \frac{x}{2}$ on the set of axes below. Label the endpoints and stationary points of f with their exact coordinates.



3 marks

The points $A(a, f(a))$ and $B(b, f(b))$, where $a \in [0, \pi]$ and $b \in [2\pi, 3\pi]$, lie on the graph of $y = f(x)$. The line that passes through the points A and B is tangent to the graph of f at both of these points, and has a gradient of m.

c. **i.** Show that $m = \dfrac{\sin(b) - \sin(a)}{b - a} + \dfrac{1}{2}$.

ii. Find b in terms of a.

iii. Hence, or otherwise, find the value of m.

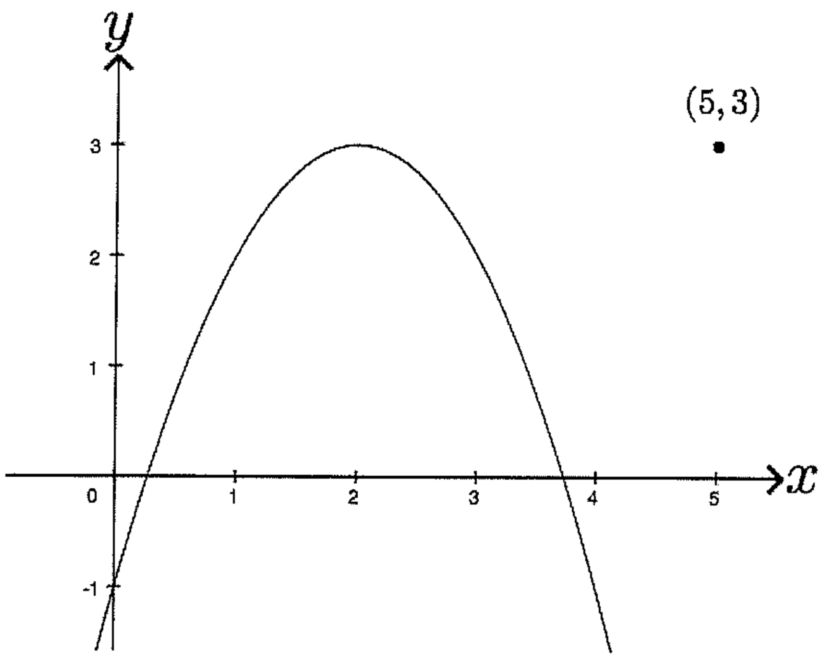
$2+1+2=5$ marks

d. Using the value of m found in **part c.iii.**, find the values of a and b, and hence find the equation of the line through A and B.

3 marks

Question 2

A graph of the parabola with equation $y = -x^2 + 4x - 1$ is shown below.



The distance of any point (x, y) on the parabola from the point $(5, 3)$ is given by $\sqrt{D(x)}$, where $D(x)$ is a polynomial function of x.

a. Show that $\frac{d}{dx}\left(\sqrt{D(x)}\right) = 0$ when $D'(x) = 0$.

2 marks

b. **i.** Find an expression for $D(x)$ in the form $ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, and e are non-zero integers.

ii. Find $D'(x)$.

$2 + 1 = 3$ marks

P is the point on the parabola $y = -x^2 + 4x - 1$ that is closest to the point $(5, 3)$.

c. Solve the equation $D'(x) = 0$ for x, and hence show that the y-coordinate of P is 2.

2 marks

d. **i.** Find the equation of the line that passes through the point P and the point $(5, 3)$.

ii. Show that this line is normal to the parabola at the point P.

iii. This line also intersects the parabola $y = -x^2 + 4x - 1$ at the point Q. Find the x-coordinate of Q, and hence find the area of the region enclosed by the line through P and Q and the parabola $y = -x^2 + 4x - 1$.

$2 + 3 + 3 = 8$ marks

Question 3

Retro Trains schedules a train to depart at 8:02 am every day from Flinders Cross Station. On each day, the actual departure time of this train varies according to the following probability density function:

$$p(t) = \begin{cases} \dfrac{6t}{k^3}(k-t) & \text{if } 0 \le t \le k \\ 0 & \text{otherwise} \end{cases}$$

The random variable T represents the time, in minutes, after 8:00 am that this particular train departs Flinders Cross each day.

a. Write down the expected value of T, in terms of k.

1 mark

For cost reasons, Retro requires that the standard deviation of T is at least 2 minutes.

b. **i.** Find an expression for the standard deviation of T, in terms of k.

ii. Find the value of k for which the standard deviation of T is 2 minutes.

iii. Hence, find the smallest value of $E(T)$, in minutes, correct to one decimal place.

$2 + 1 + 1 = 4$ marks

Thomas goes to Flinders Cross Station every day to catch the train.

On a particular week, the time after 8:00 am that he arrives at the station each day is normally distributed with a mean of 2 minutes and a standard deviation of 80 seconds. During this week, the train departs at 8:05 am on each day.

Assume that, if the train is at the station when he arrives, he will board the train immediately.

c. Find, correct to four decimal places, the probability that he will miss the train on any given day during this week.

2 marks

On another week, Thomas decides that if he misses the train on one day, he will be sure to catch the train the next day. Otherwise, the probability of him catching the train is 0.95.

d. If he misses the train on the first day of this week, find, correct to four decimal places,

i. the probability that he will catch the train on all of the next four days.

ii. the probability that he will miss the train on the fifth day of this week.

$1 + 2 = 3$ marks

Once the train departs from Flinders Cross, it will travel for Y minutes before arriving at Thomas' stop. The probability density function for Y is given by

$$f(y) = \begin{cases} \frac{2}{9}(23 - y) & \text{if } 20 \leq y \leq 23 \\ 0 & \text{otherwise} \end{cases}$$

When Thomas arrives at his stop, he then travels to work on foot, and must arrive there no later than 8:30 am. It takes Thomas 3 minutes to walk to work from his stop, and he must run to work if he has less than 3 minutes to make it to work on time.

e. If, on one day, Thomas boards the train at Flinders Cross, and it departs at 8:06 am, find the probability that he will have to run to work, correct to two decimal places.

2 marks

Question 4

Consider the following three functions

$$f : \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \to \mathbb{R},\ f(x) = \sin(2x) + 2\sin(x)$$

$$g : \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \to \mathbb{R},\ g(x) = \cos(x)$$

$$p : \mathbb{R} \to \mathbb{R},\ p(x) = 2(2x - 1)(x + 1)$$

a. **i.** Find $f'(x)$.

ii. Using the fact that $\cos(2x) = 2\cos^2(x) - 1$, show that $f'(x) = p(g(x))$.

1 + 3 = 4 marks

b. **i.** Write down the range of g.

ii. State the range of values of x for which $p(x) > 0$.

iii. Hence, show that $f'(x) > 0$ for all $x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$1 + 1 + 2 = 4$ marks

c. **i.** Find, correct to three decimal places, the value of x for which $f(x) = 1$.

ii. Using the result of **part b.iii.**, explain why there is exactly one solution to the equation $f(x) = c$ for all values of c in the range of f.

$1 + 2 = 3$ marks

d. Let $h: \left[\frac{\pi}{3}, d\right] \to \mathbb{R}$ be a function with the same rule as f. Find the range of values of d for which the function h does not have an inverse.

3 marks

END OF EXAMINATION