2000 Specialist Mathematics Exam 2

Suggested Answers and Solutions

Solutions:

Question 1 a i

$f: R^+ \to R$ where $f(x) = \frac{6}{x} - 6 + 3\log_e x$

$$f'(x) = \frac{-6}{x^2} + \frac{3}{x}$$
$$= \frac{3x-6}{x^2}$$

Question 1 a ii

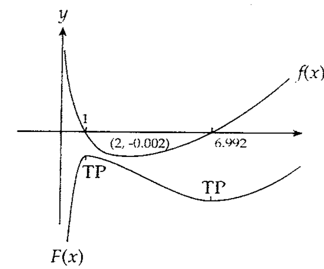
$$f'(x) = \frac{-6}{x^2} + \frac{3}{x}$$
$$f''(x) = \frac{12}{x^3} - \frac{3}{x^2}$$
$$= \frac{12-3x}{x^3}$$

Question 1 b

$$f(x) = \frac{6}{x} - 6 + 3\log_e x$$
$$f(1) = \frac{6}{1} - 6 + 3\log_e(1)$$
$$= 6 - 6 = 0$$

$\therefore f(1) = 0$ is an x - intercept

Question 1 c i



Eqn of graph not required:
$F(x) = 6\log_e x - 6x + 3(\log_e x - x)$

Question 1 c ii

From **a i**

$$f'(x) = \frac{-6}{x^2} + \frac{3}{x}$$
$$= \frac{3x-6}{x^2}, \qquad x \neq 0$$

Turning point occurs when $f'(x) = 0$

$$\frac{3x-6}{x^2} = 0$$
$$3x - 6 = 0$$
$$x = 2$$

$$f(x) = \frac{6}{x} - 6 + 3\log_e x$$
$$f(2) = \frac{6}{2} - 6 + 3\log_e 2$$
$$= 3\log_e 2 - 3$$
$$= 3(\log_e 2 - 1)$$

Turning point: $(2, 3(\log_e 2 - 1))$

Question 1 c iii

Using graphing calcultor

x-intersept at $x = 4.92$

Question 1 d

Using result from **a ii**

$$f''(x) = \frac{12-3x}{x^3}, \qquad x \neq 0$$

Max value of $f'(x)$ occurs when $f''(x) = 0$, i.e. $x = 4$

$$f'(4) = \frac{3 \times 4 - 6}{16}$$
$$= \frac{6}{16}$$
$$= \frac{3}{8}$$

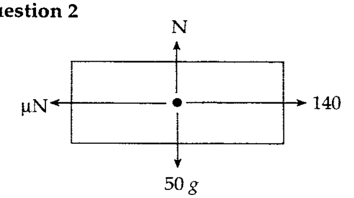
Max gradient at $x = 4$

Max gradient is at $\frac{3}{8}$

Question 1 e

See question **c i**

Question 2

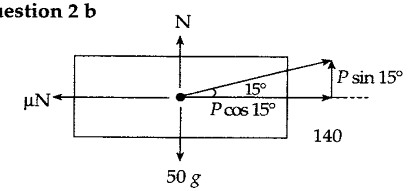


$$140 = \mu N \qquad (1)$$
$$N = 50g \qquad (2)$$

Substituting (2) into (1)

$$140 = \mu 50g$$
$$\mu = \frac{140}{50g}$$
$$\cong 0.286 \text{ (3 dp)}$$

Question 2 b



$$P\sin 15° + N = 50g$$
$$N = 50g - P\sin 15° \qquad (3)$$
$$\mu N = P\cos 15° \qquad (4)$$

Using $\mu = \frac{14}{5g}$ from **2 a**

and substituting (3) into (4)

$$\frac{14}{5g}(50g - P\sin 15°) = P\cos 15°$$

$$\frac{14}{5g} \times 50g - \frac{14}{5g}P\sin 15° = P\cos 15°$$

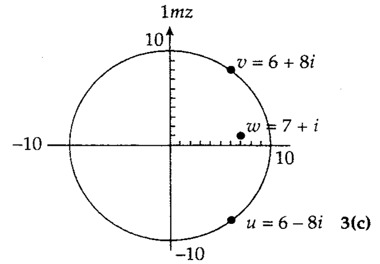
$$140 = \frac{14}{5g}P\sin 15° + P\cos 15°$$

$$140 = P\left[\frac{14}{5g}\sin 15° + \cos 15°\right]$$

$$P = \frac{140}{\frac{14\sin 15°}{5g} + \cos 15°}$$

$= 135$ to the nearest integer

Question 3 a



Question 3 b i

$$v = 6 + 8i$$
$$|v| = \sqrt{6^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100}$$
$$= 10$$

Question 3 b ii

See above—it is a circle with radius 10 and centre (0, 0)

Question 3 c

$$u + i\overline{w} = \overline{w}$$
$$w = 7 + i$$
$$\overline{w} = 7 - i$$

$$u + i\overline{w} = \overline{w}$$
$$u + i(7 - i) = 7 - i$$
$$u = 6 - 8i$$

Question 3 d

Given on above graph as perpendicular bisector of UV, i.e. $y = 0$

Question 3 e

Let $\underset{\sim}{OV} = 6\underset{\sim}{i} + 8\underset{\sim}{j}$

and $\underset{\sim}{OW} = 7\underset{\sim}{i} + \underset{\sim}{j}$

Question 7 [A]

$z = \sqrt{3} + i$

$|z| = 2$

$\arg z = \frac{\pi}{6}$

In its simplest form, z can be written as $2 \text{ cis}\left(\frac{\pi}{6}\right)$ polar form. This is equivalent to $2 \text{ cis}\left(\frac{-11\pi}{6}\right)$

Question 8 [B]

$|z| = |z+2|$ defines the set of points which are equidistant from the origin, 0, and the point $-2+0i$. In other words, the set of points will lie on a line parallel to $Im(z)$ and passing through the point $-1+0i$, that is, a line $\text{Re}(z) = -1$.

Question 9 [E]

$\frac{-1}{\sqrt{1-2x^2}}$ can be transformed to: $\frac{-1}{\sqrt{1-\left(\sqrt{2}x\right)^2}}$

Which is an antiderivative of $\frac{1}{\sqrt{2}}\text{Cos}^{-1}\left(\sqrt{2}x\right)$

Question 10 [C]

$$\int_0^1 \left(\cos^2 x - \sin^2 x\right)dx$$

$$= \int_0^1 \cos 2x dx$$

$$= \left[\frac{\sin 2x}{2}\right]_0^1$$

$$= \frac{\sin(2)}{2} = 0.4546$$

Question 11 [C]

$$\int_1^3 \frac{1}{x^2} . e^{\frac{3}{x}} dx$$

Let $u = \frac{3}{x}$, then $\frac{du}{dx} = \frac{-3}{x^2}$

So, $\frac{1}{x^2} = \frac{-1}{3} \cdot \frac{du}{dx}$

When $x = 1, \ u = 3$
$x = 3, \ u = 1$

So the above integral can be changed to

$$\frac{-1}{3}\int_3^1 e^u . \frac{du}{dx} . dx$$

$$\frac{1}{3}\int_1^3 e^u du$$

Question 12 [E]

$$\frac{4}{x^2 - 4x}$$

$$= \frac{4}{x(x-4)}$$

$$= \frac{A}{x} + \frac{B}{x-4}$$

Hence, $A(x-4) + B(x) \equiv 4$

$A = -1, B = 1$

$$= \frac{1}{x-4} - \frac{1}{x}$$

So an antiderivative of the above function would be:

$\log_e(x-4) - \log_e(x)$

$\log_e\left(\frac{x-4}{x}\right)$

Which is defined for $x>4$

Question 13 [E]

The required solid of revolution is given by:

$$\int_0^{\sqrt{3}} \pi y^2 dx - \int_0^{\sqrt{3}} \pi(1)^2 dx$$

$$\pi\int_0^{\sqrt{3}} \left(y^2 - 1\right)dx$$

$$\pi\int_0^{\sqrt{3}} \left(\frac{4}{1+x^2} - 1\right)dx$$

2000 Specialist Mathematics Exam 1 Solutions

Question 14 [D]

$y = f(x)$ is the derivative function corresponding to $y = F(x)$

i.e. $F'(x) = 0$ at $x = -1$ where

$F'(x)>0$ for $x<-1$ and

$F'(x)<0$ for $x>1$, implying that

$F'(x)$ has a local maximum at $x = -1$.

Also $F'(x) = 0$ for $x = 2$ but since the sign of $F'(x)$ is the same either side of $x = 2$, this point will be a stationary point of inflexion.

Question 15 [D]

$y = \cos(2x - 3)$

$\frac{dy}{dx} = -2\sin(2x-3)$

$\frac{d^2y}{dx^2} = -4\cos(2x-3)$

Options A, B, C cannot be correct, since

$4y + \frac{d^2y}{dx^2} = 0$

since $-2\frac{dy}{dx} = +4\sin(2x-3)$

D is correct and not E.

Question 16 [B]

$g'(x) = x\sqrt{x^2+1}$

$= x\left(x^2+1\right)^{\frac{1}{2}}$

$g(x) = \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} + c$

$g(0) = \frac{1}{3} + c = 1, \quad c = \frac{2}{3}$

$g(x) = \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} + \frac{2}{3}$

Question 17 [B]

No of steps n	x_n	y_n
0	1	2
1	1.1	$2 + (\log_e 1) \times 0.1 = 2$
2	1.2	$2 + (\log_e 1.1) \times 0.1$ $= 2.00953$ $= 2.0095$ (to 4 decimal places)

Question 18 [D]

$a = \frac{dv}{dt} = 5e^{-0.1t}$

$v = -50e^{-.01t} + c$

when $t = 0, v = 0$

$0 = -50 + c, c = 50$

$v = 50 - 50e^{-0.01t}$

when $t = 1$

$v = 50(1 - e^{-0.01})$

$v = 50(0.9516)$

$v = 4.8$ correct to 2 significant figures.

Question 19 [C]

$v = 15 - 10t$ is the relationship between v and t. This is found by considering the v–intercept (0, 15) and the gradient of the line which is –4.

Defining $x = 0$ as ground level,

$x = 15t - 5t^2 + c$

$x = 0$ and $t = 4$

$0 = 60 - 80 + c$, giving $c = 20$

$x = 15t - 5t^2 + 20$

when $t = 0$, $x = 20$m which is the height of the balcony above the ground.

$\overrightarrow{OW} = 7\underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{WV} = 7\underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{i} + 8\underset{\sim}{j}$

$= \underset{\sim}{i} - 7\underset{\sim}{j}$

$\overrightarrow{OW}, \overrightarrow{WV} = \left(7\underset{\sim}{i} + \underset{\sim}{j}\right) + \left(\underset{\sim}{i} - 7\underset{\sim}{j}\right)$

$= 7 - 7 = 0$

$\overrightarrow{OW}$ perpendicular to $\overrightarrow{WV}$

Question 4 a i

$$\frac{d(x\cos x)}{dx} = \cos x - x\sin x$$

Question 4 a ii

$$\frac{d(x\cos x)}{,dx} = \cos x - x\sin x$$

$$x\sin x = \cos x - \frac{d(x\cos x)}{dx}$$

$$\int x\sin x\,dx = \int \cos x\,dx - \int \frac{d(x\cos x)}{dx}\,dx$$

$$= \sin x - x\cos x$$

Question 4 b

$$y = \frac{-x}{50}\ (8 + \sin x)$$

at $x = 10$

$$y = \frac{-10}{50}\ (8 + \sin 10)$$

$= -1.5$ metres

Depth is 1.5 metres

Question 4 b ii

$= \int_0^{60} \frac{x}{50}(8 + \sin x)\,dx$ (treating area as positive)

$$= \frac{1}{50}\int_0^{60} 8x + x\sin x\,dx$$

$$= \frac{1}{50}\left[4x^2 + \sin x - x\cos x\right]_0^{60}$$

$$= \frac{1}{50}\left[\left(4\times 60^2 + \sin 60 - 60\cos 60\right) - (0)\right]$$

$= 289\ \text{m}^2$

Question 4 c i

Volume $= 289.14 \times 150$

$= 43371\ \text{m}^3/\text{sec}$

Nutrient $= \dfrac{\text{Vol}}{500}$

$= \dfrac{43371}{500}$

$= 87.74$

Nutrient required 87 units to nearest unit

Question 4 c ii

Units required $= \dfrac{150}{500}\displaystyle\int_0^{60} \frac{x^2}{1600}(8 + \sin x)\,dx$ (treating area as positive)

$= \dfrac{3}{1600}\displaystyle\int_0^{60} x^2(8 + \sin x)\,dx$

$= 108.6353$

109 units require to the nearest unit

Question 5 a

Javelin reaches the ground when the $\underset{\sim}{k}$ component is zero

$\underset{\sim}{k} \Rightarrow z = 2 + 19.5t + -5t^2$

at $t = 4$ $z = 2 + 19.5 \times 4 - (5 \times 16)$

$= 0$

Question 5 b

$$r(t) = 19.5t\,\underset{\sim}{i} + \left(\frac{\pi t}{2} - 4\sin\left(\frac{\pi t}{8}\right)\right)\underset{\sim}{j} + \left(2 + 19.5t - 5t^2\right)\underset{\sim}{k}$$

$$r(4) = 78\,\underset{\sim}{i} + (2\pi - 4)\,\underset{\sim}{j} + 0\,\underset{\sim}{k}$$

$$(r(4)) = \sqrt{78^2 + (2\pi - 4)^2}$$

$= 78.03$ metres (to nearest cm)

Question 5 c

$$r(t) = 19.5 + \underset{\sim}{i} + \left(\frac{\pi t}{2} - 4\sin\frac{\pi t}{8}\right)\underset{\sim}{j} + \left(2 + 19.5t + -5t^2\right)\underset{\sim}{k}$$

$$\dot{r}(4) = 19.5\,\underset{\sim}{i} + \left(\frac{\pi}{2} - \frac{4\times\pi}{8}\cos\frac{\pi t}{8}\right)\underset{\sim}{j} + (19.5 - 10t)\,\underset{\sim}{k}$$

$$\dot{r}(4) = 19.5\,\underset{\sim}{i} + \left(\frac{\pi}{2} - \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)\right)\underset{\sim}{j} + (19.5 - 40)\,\underset{\sim}{k}$$

$$= 19.5\,\underset{\sim}{i} + \frac{\pi}{2}\,\underset{\sim}{j} - 20.5\,\underset{\sim}{k}$$

Question 5 d

The angle with which the javelin strikes the ground is given by the angle made by the velocity vector $\underset{\sim}{\dot{r}}(4)$

$$\tan\theta = \frac{20.5}{\sqrt{19.5^2 + \frac{\pi^2}{4}}}$$

$\theta = 46.3°$ (or 133.7°)

A longer method would be to find the angle between $\underset{\sim}{\dot{r}}(4)$ and $\underset{\sim}{r}(4)$ using scalar product

$$\underset{\sim}{\dot{r}}(4)\cdot\underset{\sim}{r}(4) = \left|\underset{\sim}{\dot{r}}(4)\right|\left|\underset{\sim}{r}(4)\right|\cos\theta$$

Question 6 a

$v = u + at$

$v = 20$ $a = -9.8$ $u = 0$

$-20 = 0 - 9.8t$

$t = \dfrac{20}{9.8} = 2.04$ seconds

$s = ut + \dfrac{1}{2}at$

$s = 0 + \dfrac{1}{2} \times 9.8 \times \dfrac{-20}{9.8} \times \dfrac{20}{9.8}$

$s = -20.4$ metres

Time in free fall 2.04 second

Distance in free fall 20.4 metres

Question 6 b

$$ma = gm - 0.04\,mg\,v^2$$

$$a = g - 0.04\,g\,b^2$$

$$a = -0.04\,g\left(-\frac{1}{0.4} + v^2\right)$$

$$= -0.04\,g\left(v^2 - 25\right)$$

$$\frac{dv}{dt} = -0.04\,g\left(v^2 - 25\right)$$

Question 6 c i

$$\frac{dt}{dv} = \frac{1}{-0.04g(v^2 - 25)}$$

$$= \frac{-25}{g(v^2 - 25)}$$

Consider $= \dfrac{1}{v^2 - 25} \equiv \dfrac{A}{v-5} + \dfrac{B}{v+5}$

$1 \equiv A(v + 5) + B(v - 5)$

using $v = 5$, $1 = 10A$

$A = \dfrac{1}{10}$

$\therefore\ \dfrac{1}{v^2 - 25} = \dfrac{1}{10}\left(\dfrac{1}{v-5}\right) - \left(\dfrac{1}{v+5}\right)$

$\therefore\ \dfrac{dt}{dv} = \dfrac{-2.5}{g}\left(\dfrac{1}{v-5}\right) - \left(\dfrac{1}{v+5}\right)$

$t + c = \dfrac{-2.5}{g}\ln\dfrac{v-5}{v+5}$

at $t = 0$ $v = 20$

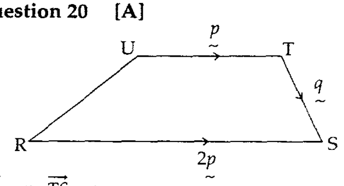
$c = \dfrac{-2.5}{g}\ln(0.6)$

$t = \dfrac{-2.5}{g}\ln\left(\dfrac{v-5}{v+5}\right) + \dfrac{-2.5}{g}\ln(0.6)$

$t = \dfrac{-2.5}{g}\ln\left(\dfrac{v-5}{0.6(v+5)}\right)$

or $t = \dfrac{-1}{0.4g}\ln\left[\dfrac{5(v-5)}{3(v+5)}\right]$

Question 20 [A]



$\overrightarrow{UT} = \underset{\sim}{p}$, $\overrightarrow{TS} = \underset{\sim}{q}$

So $\overrightarrow{RS} = 2\underset{\sim}{p}$

$\overrightarrow{RS} + \overrightarrow{ST} + \overrightarrow{TU} + \overrightarrow{UR} = 0$

$2\underset{\sim}{p} - \underset{\sim}{q} - \underset{\sim}{p} + \overrightarrow{UR} = 0$

hence $\overrightarrow{UR} = \underset{\sim}{q} - \underset{\sim}{p}$

Question 21 [A]

The given vector is:

$-3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$

it has magnitude of 7 units $= \sqrt{9+4+36} = \sqrt{49} = 7$

So a vector with magnitude of 14 units and parallel to the given vector could be

$2\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

or $-2\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

The second one corresponds to A.

Question 22 [C]

$\underset{\sim}{m} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$

$\underset{\sim}{n} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{\hat{n}} = \frac{1}{3}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

The scalar resolute of m in the direction of n is given by

$\underset{\sim}{m} \cdot \underset{\sim}{\hat{n}} = m\cos\theta$

$= \left(-3\underset{\sim}{i} + 4\underset{\sim}{j}\right) \cdot \frac{1}{3}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

$= \frac{1}{3}(3+8)$

$= \frac{11}{3}$

Question 23 [E]

A rhombus has all four sides equal and two pair of parallel sides.

A. Would be sufficient to define a trapezium not a rhombus.

B. Would be sufficient to define a parallelogram only.

C. Would be sufficient to define a rectangle.

D. Would be sufficient to define a rhombus or a square.

E. Is sufficient to define a rhombus only.

Question 24 [A]

$\underset{\sim}{r} = \left(\sin^2 t\right)\underset{\sim}{i} - \left(2\cos^2 t\right)\underset{\sim}{j} \quad 0 \le t \le \frac{\pi}{2}$

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$x = \sin^2 t$

$y = -2\cos^2 t$

$2x - y = 2$ which is a straight line passing through $(0, -2)$ and $(1, 0)$

Question 25 [D]

$\underset{\sim}{r} = \left(\frac{3}{2}\sin(2t)\right)\underset{\sim}{i} - \left(2e^{-2t}\right)\underset{\sim}{j}$

$\underset{\sim}{\dot{r}} = \left(3\cos(2t)\right)\underset{\sim}{i} - \left(4e^{-2t}\right)\underset{\sim}{j}$

at $t = 0$

$\underset{\sim}{\dot{r}} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$

speed is $\left|\underset{\sim}{\dot{r}}\right|$, so at $t = 0$, speed is 5 units.

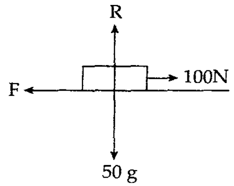
Question 26 [B]

Initial momentum is 5×10 kg m/s

Subsequent momentum is 5×6 kg m/s

Change in momentum is a loss of 20kg.m/s in the direction of the motion.

Question 27 [C]



Since the coefficient of friction is 0.5, the limiting value of friction would be $0.5R$ where $R = 50g$. A horizontal force of 100 Newton is less than the limiting value of friction. So the body will remain at rest with $F = 100N$.

Question 28 [D]

Since the body, B, is in equilibrium the vectors $\underset{\sim}{P}$ and $\underset{\sim}{W}$ together will be equivalent to a vector equal in magnitude but opposite in direction to the vector $\underset{\sim}{T}$. This is expressed by writing

$\underset{\sim}{P} + \underset{\sim}{W} + \underset{\sim}{T} = 0$

This relationship is independent of the angle AB makes with the wall, so long as the body is in equilibrium.

Question 29 [C]

The resultant force acting on the car, F, is given by $F = 900 - 750\,cv$

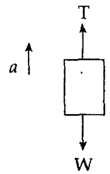
(Note: that the total resistance is dependent upon the mass of the car)

Also $F = ma$

$= 750\,\frac{dv}{dt}$

So $750\,\frac{dv}{dt} = 900 - 750\,c\,v$

Question 30 [A]



Resultant force acting on the lift is F = T – W.

For constant acceleration upwards F>0, so T>W and T is also constant.

For constant velocity, F = 0, so T=W

For constant retardation, F<0, so T<W and T is also a constant.

Diagram A meets all of these criterions.

Question 6 c ii

From **6 c i**

$$
\begin{aligned}
t &= \frac{-2.5}{g}\ln\left(\frac{v-5}{0.6(v+5)}\right)\\
-0.4gt &= \ln\frac{v-5}{0.6(v+5)}\\
e^{-0.4gt} &= \frac{v-5}{0.6(v+5)}\\
0.6ve^{-0.4gt} &= \frac{v-5}{v+5}\\
0.6ve^{-0.4gt}+3e^{-0.4gt} &= v-5\\
3e^{-0.4gt}+5 &= v-0.6ve^{-0.4gt}\\
5\left(1+0.6e^{-0.4gt}\right) &= v\left(1-0.6e^{-0.4gt}\right)\\
v &= \frac{5\left(1+0.6e^{-0.4gt}\right)}{1-0.6e^{-0.4gt}}
\end{aligned}
$$

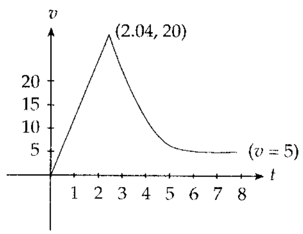
Question 6 d

Limiting velocity occurs for large values of t

when $t \to \infty$ $\quad e^{-0.4gt} \to 0$

$$
\begin{aligned}
V_{\text{limit}} &= \frac{5(1+0)}{1-0}\\
&= 5 \text{ m/s}
\end{aligned}
$$

Question 6 e



note discontinuity at $t = 8$ when $v = 0$