THE
HEFFERNAN
GROUP

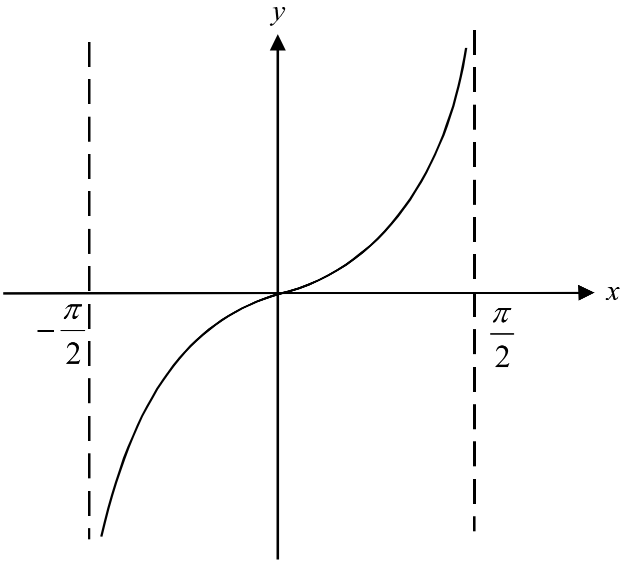
P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 9836 5021
Fax 9836 5025

SPECIALIST MATHS
TRIAL EXAMINATION 2
2001
SOLUTIONS

Question 1

a. When $x = 0,\ y = 0$ So, the y-intercept is zero. **(1 mark)**

b.



(1 mark) for asymptotes
(1 mark) for shape of graph

c. $f(x) = x\sec x$

$$= x(\cos x)^{-1}$$

So, $f'(x) = x \times -1(\cos x)^{-2} \times -\sin x + (\cos x)^{-1}$

$$= \frac{x\sin x}{\cos^2 x} + \frac{1}{\cos x}$$

$$= \frac{x\sin x + \cos x}{\cos^2 x}$$ **(1 mark)**

d. A stationary point occurs when $f'(x) = 0$.

If $f'(x) = 0$, then $x\sin x + \cos x = 0$

When $x = 0$, we have $0 \times \sin 0 + \cos 0 = 1$

So, $f'(0) \neq 0$ and so we do not have a stationary point at $x = 0$. **(1 mark)**

e. The gradient is a minimum when $f''(x) = 0$

Now, $\cos^3 x \neq 0$, so $x + x\sin^2 x + 2\sin x\cos x = 0$ **(1 mark)**

Use a graphics calculator to show that this occurs when the graph of $y = x + x\sin^2 x + 2\sin x\cos x$ crosses the y-axis. This occurs when $x = 0$.

The gradient of the graph of $y = f(x)$ is a minimum at $(0,0)$. **(1 mark)**

f. To verify: $\dfrac{f''(x)}{\sec^3 x} - \dfrac{\cos x}{\text{cosec}^2 x} f(x) = x + \sin(2x)$

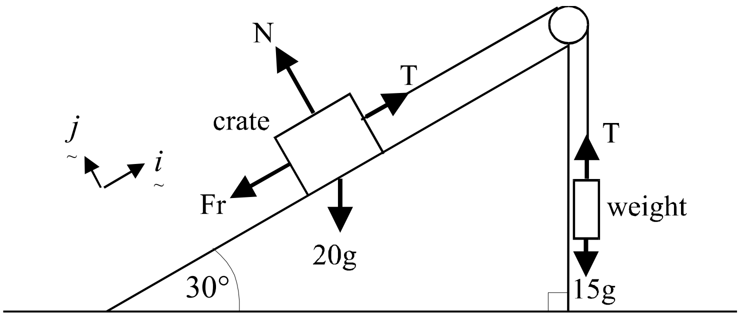
$$\begin{aligned}\text{Left side } &= \frac{f''(x)}{\sec^3 x} - \frac{\cos x}{\text{cosec}^2 x} f(x)\\ &= \frac{x + x\sin^2 x + 2\sin x\cos x}{\cos^3 x} \times \frac{\cos^3 x}{1} - \cos x \sin^2 x \times \frac{x}{\cos x} \qquad \textbf{(1 mark)}\\ &= x + x\sin^2 x + 2\sin x \cos x - x\sin^2 x\\ &= x + 2\sin x\cos x\\ &= x + \sin(2x)\\ &= \text{right side}\end{aligned}$$

Have verified **(1 mark)**

Total 9 marks

Question 2

a.



(1 mark)

b. Resolving around the weight, we get $T = 15g$ ____________(A) **(1 mark)**

Resolving around the crate, we get

$$\underset{\sim}{R} = (T - 20g\sin 30° - Fr)\underset{\sim}{i} + (N - 20g\cos 30°)\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$$

So, $T = 10g + Fr$ and $N = \dfrac{20g\sqrt{3}}{2}$ **(1 mark)**

$= 10g + \mu N$ $\qquad = 10\sqrt{3}g$ ____________(B)

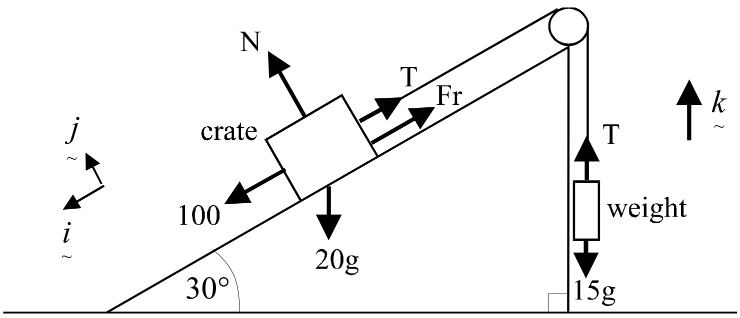
Substitute (A) and (B) into $T = 10g + \mu N$

We obtain

$$\begin{aligned}15g &= 10g + \mu 10\sqrt{3}g\\ \mu &= \frac{5g}{10\sqrt{3}g}\\ &= \frac{\sqrt{3}}{6}\end{aligned}$$

(1 mark)

c. Draw a diagram showing all the forces involved.



Resolving around the crate, we obtain

$$\underset{\sim}{R} = (100 + 20g\sin 30° - T - \mu N)\underset{\sim}{i} + (N - 20g\cos 30°)\underset{\sim}{j}$$

Also, $\underset{\sim}{R} = m\underset{\sim}{a}$

$= 20a \times \underset{\sim}{i}$

So, equating components in the $\underset{\sim}{i}$ direction, we obtain

$100 + 10g - T - \mu N = 20a$ __________(A) **(1 mark)**

Equating components in the $\underset{\sim}{j}$ direction, we obtain

$N - 20g\cos 30° = 0$

so, $N = 10\sqrt{3}g$

In (A), this gives $T = 100 + 10g - \frac{\sqrt{3}}{6}(10\sqrt{3}g) - 20a$

$T = 5g + 100 - 20a$ ________(C) **(1 mark)**

Resolving around the weight, we obtain

$\underset{\sim}{R} = (T - 15g)\underset{\sim}{k}$

Also, $\underset{\sim}{R} = m\underset{\sim}{a}$

$\underset{\sim}{R} = 15a\underset{\sim}{k}$

So, $T - 15g = 15a$

$T = 15a + 15g$ __________(D) **(1 mark)**

Equating (C) and (D) we obtain

$$5g + 100 - 20a = 15a + 15g$$
$$-35a = 10g - 100$$
$$a = \frac{10(g - 10)}{-35}$$
$$= 0.06 \text{ (to 2 decimal places)}$$

So the weight accelerates upwards at at 0.06 m/s^2. **(1 mark)**

Total 8 marks

Question 3

a. $u = 0 + 5i$

So, $r = \sqrt{0^2 + 5^2}$ and $\text{Arg } u = \frac{\pi}{2}$ since on an Argand diagram, u is located on the imaginary axis.

$= 5$

So, $u = 5\text{cis}\frac{\pi}{2}$ **(1 mark)**

b. $v = \bar{u} + |u| - 1 + 6i + \text{Re}\,u$

$= -5i + 5 - 1 + 6i + 0$

$= 4 + i$ **(1 mark)**

c. We have $|z - 5i| = |z - 4 - i|$

so, $|x + yi - 5i| = |x + yi - 4 - i|$ **(1 mark)**

$|x + i(y-5)| = |x - 4 + i(y-1)|$

$\sqrt{x^2 + (y-5)^2} = \sqrt{(x-4)^2 + (y-1)^2}$ **(1 mark)**

$x^2 + y^2 - 10y + 25 = x^2 - 8x + 16 + y^2 - 2y + 1$

$-8y = -8x - 8$

$y = x + 1$ **(1 mark)**

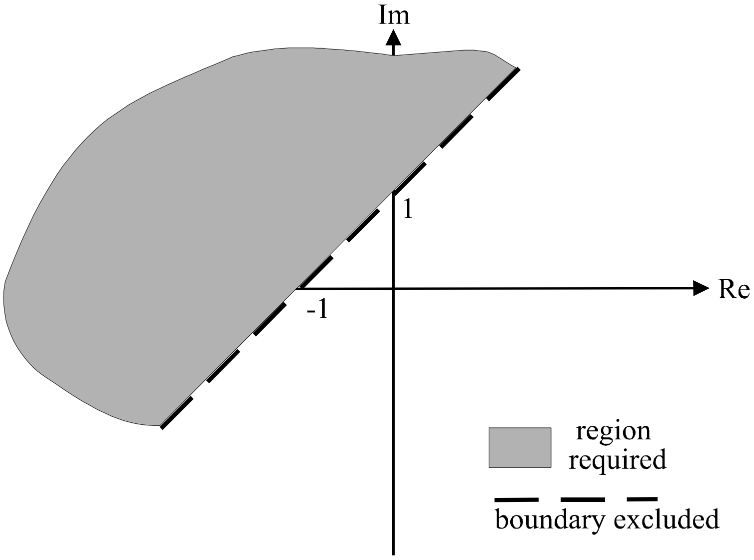
d.

Using our result form part **c.**, we can draw in the line with Cartesian equation $y = x + 1$.
Note that the boundary is not included since we have a less than sign and not a less than or equal to sign.
To decide which side of the line is required, choose a point say $0 + 0i$.
Substitute into $|z - 5i| < |z - 4 - i|$

We obtain $|0 + 0i - 5i| < |0 + 0i - 4 - i|$

$|-5i| < |-4 - i|$

$\sqrt{25} < \sqrt{16 + 1}$ Clearly this statement is not true and so the side from which we chose the point, $0 + 0i$ is not the required side.



(1 mark) for correct required region **(1 mark)** for correct boundary

e. Let $z^3 = 8$

so, $z^3 - 8 = 0$

$$(z-2)(z^2+2z+4) = 0$$

$$z = 2 \text{ or } z = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}i$$

Alternatively, use De Moivre's theorem.

Let $z^3 = 8\text{cis}0$

So, $z_1 = 2\text{cis}0$

and $z_2 = 2\text{cis}\frac{2\pi}{3}$ and $z_3 = 2\text{cis}\frac{-2\pi}{3}$

since the three roots are equally spaced.

Only $z_2 = 2\text{cis}\frac{2\pi}{3} = -1 + \sqrt{3}i$ lies in S.

Remember that the answer is required in Cartesian form.

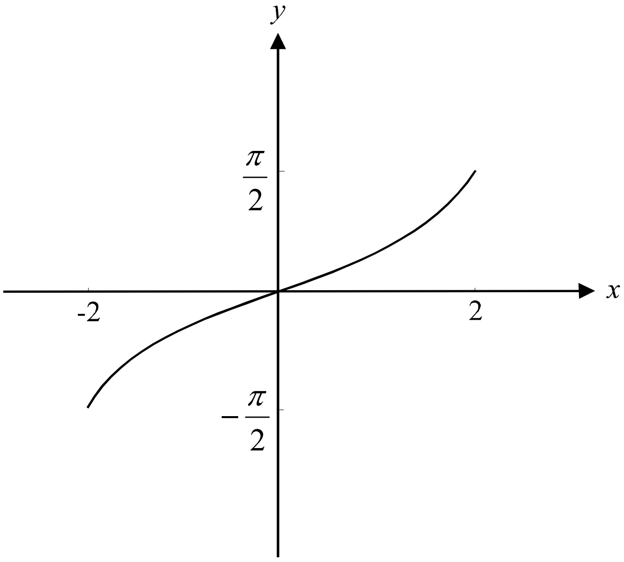
The cube roots of 8 are 2 and $-1 \pm \sqrt{3}i$. **(1 mark)**
Looking at the diagram in part **d.**, we see that the root 2 does not lie in S and the root $-1 - \sqrt{3}i$ doesn't lie in S. The root $-1 + \sqrt{3}i$ is the only root which lies in S. **(1 mark)**

Total 9 marks

Question 4

a. $\text{Sin}^{-1}\frac{x}{a}$ is defined for $x \in [-a, a]$. So in this case, $a = 2$. **(1 mark)**

b. i.



(1 mark)

ii. $f(x) = \text{Sin}^{-1}\frac{x}{2}$

$f'(x) = \frac{1}{\sqrt{4 - x^2}}$ **(1 mark)**

Now, stationary points occur when $f'(x) = 0$, that is $\frac{1}{\sqrt{4-x^2}} = 0$. Now $\sqrt{4 - x^2} \neq 0$ or the function is undefined. A fraction can only equal zero if the numerator equals zero. Clearly $1 \neq 0$ and so the function *f* can have no stationary point. **(1 mark)**

c. $\dfrac{d}{dx}(x\text{Sin}^{-1}\dfrac{x}{2}) = x \times \dfrac{1}{\sqrt{4-x^2}} + 1 \times \text{Sin}^{-1}\dfrac{x}{2}$ (product rule)

$$= \frac{x}{\sqrt{4-x^2}} + \text{Sin}^{-1}\frac{x}{2}$$ **(1 mark)**

d. i. Now from part **c.**, $\dfrac{d}{dx}(x\text{Sin}^{-1}\dfrac{x}{2}) = \dfrac{x}{\sqrt{4-x^2}} + \text{Sin}^{-1}\dfrac{x}{2}$

So, $$\int \frac{d}{dx}(x\text{Sin}^{-1}\frac{x}{2})dx = \int \frac{x}{\sqrt{4-x^2}}dx + \int \text{Sin}^{-1}\frac{x}{2}dx$$

So, $$x\text{Sin}^{-1}\frac{x}{2} + c_1 = \int \frac{x}{\sqrt{4-x^2}}dx + \int \text{Sin}^{-1}\frac{x}{2}dx$$

Rearranging, we obtain, $\displaystyle\int \text{Sin}^{-1}\frac{x}{2}dx = x\text{Sin}^{-1}\frac{x}{2} + c_1 - \int \frac{x}{\sqrt{4-x^2}}dx$ **(1 mark)**

Now, $\displaystyle\int \frac{x}{\sqrt{4-x^2}}dx = \int \frac{1}{\sqrt{u}} \times -\frac{1}{2}\frac{du}{dx}dx$ where $u = 4 - x^2$ and $\dfrac{du}{dx} = -2x$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$= -\frac{1}{2} \times 2 \times u^{\frac{1}{2}} + c_2$$

$$= -\sqrt{4-x^2} + c_2$$

So, $\displaystyle\int \text{Sin}^{-1}\frac{x}{2}dx = x\text{Sin}^{-1}\frac{x}{2} + \sqrt{4-x^2}$ **(1 mark)**

Note that we equate the constants of antidifferentiation to zero since we were asked for “an” antiderivative.

ii. So area required $= \displaystyle\int_0^1 \text{Sin}^{-1}\frac{x}{2}dx$ **(1 mark)**

$$= \left[x\text{Sin}^{-1}\frac{x}{2} + \sqrt{4-x^2} \right]_0^1$$

$$= \left\{ (\text{Sin}^{-1}\frac{1}{2} + \sqrt{3}) - (0 + \sqrt{4}) \right\}$$

$$= \text{Sin}^{-1}\frac{1}{2} + \sqrt{3} - 2$$

$$= \frac{\pi}{6} + \sqrt{3} - 2 \text{ square units}$$ **(1 mark)**

e. Volume required $= \pi \int_0^{\frac{\pi}{2}} x^2 dy$ **(1 mark)**

Now, $y = \text{Sin}^{-1} \dfrac{x}{2}$

So, $\dfrac{x}{2} = \sin y$ where $x \in [-2, 2]$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$$x = 2\sin y$$
$$x^2 = 4\sin^2 y$$

So, volume required $= \pi \int_0^{\frac{\pi}{2}} 4\sin^2 y\, dy$ **(1 mark)**

$$= \pi \int_0^{\frac{\pi}{2}} (2 - 2\cos(2y))dy \qquad \text{since } 2\sin^2\theta = 1 - \cos(2\theta)$$

$$= \pi[2y - \sin(2y)]_0^{\frac{\pi}{2}}$$
$$= \pi\{(\pi - \sin\pi) - (0 - \sin 0)\}$$
$$= \pi(\pi - 0 - 0 + 0)$$
$$= \pi^2 \text{ cubic units}$$ **(1 mark)**

Total 12 marks

Question 5

a. distance from origin $= |\underset{\sim}{r}(t)|$

$$= \sqrt{(t + \frac{1}{t})^2 + (t - \frac{1}{t})^2}$$
$$= \sqrt{t^2 + 1 + 1 + \frac{1}{t^2} + t^2 - 1 - 1 + \frac{1}{t^2}}$$
$$= \sqrt{2t^2 + \frac{2}{t^2}}$$ **(1 mark)**

b. Now, $\underset{\sim}{v}(t) = (1 - \frac{1}{t^2})\underset{\sim}{i} + (1 + \frac{1}{t^2})\underset{\sim}{j}$

So, $\underset{\sim}{v}(5) = \frac{24}{25}\underset{\sim}{i} + \frac{26}{25}\underset{\sim}{j}$ **(1 mark)**

Speed at time $t = 5$ is given by $\left|\underset{\sim}{v}(5)\right| = \sqrt{(\frac{24}{25})^2 + (\frac{26}{25})^2}$

$$= \sqrt{\frac{576 + 676}{625}}$$

$$= \frac{\sqrt{1252}}{25}$$

$$= \frac{2\sqrt{313}}{25}$$ **(1 mark)**

c. i. Given that $\underset{\sim}{r}(t) = (t + \frac{1}{t})\underset{\sim}{i} + (t - \frac{1}{t})\underset{\sim}{j}$

$x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$ **(1 mark)**

So, $x^2 = (t + \frac{1}{t})^2$ and $y^2 = (t - \frac{1}{t})^2$

$= t^2 + 2 + \frac{1}{t^2}$ and $= t^2 - 2 + \frac{1}{t^2}$

So, $x^2 - 2 = t^2 + \frac{1}{t^2}$ and $y^2 + 2 = t^2 + \frac{1}{t^2}$

So, $x^2 - 2 = y^2 + 2$

So, $y^2 = x^2 - 4$

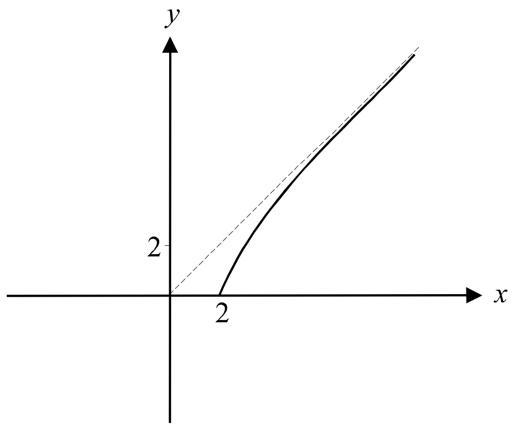
$y = \pm\sqrt{x^2 - 4}$

Note that since $t \geq 1$, and $x = t + \frac{1}{t}$, then $x \geq 2$ So, the domain is $x \in [2, \infty)$ **(1 mark)**

Also, $y = t - \frac{1}{t}$, and $t \geq 1$, so $y \geq 0$. So, the range is $y \in [0, \infty)$ **(1 mark)**

So, the required Cartesian equation is $y = \sqrt{x^2 - 4}$ **(1 mark)**

ii.



(1 mark)

d. Since $\underset{\sim B}{v}(t) = (2 - \frac{2}{t^2})\underset{\sim}{i} + (2 + \frac{2}{t^2})\underset{\sim}{j}, \qquad t > 0$

$$\underset{\sim B}{r}(t) = \int\left((2 - \frac{2}{t^2})\underset{\sim}{i} + (2 + \frac{2}{t^2})\underset{\sim}{j}\right)dt$$

$$= (2t + \frac{2}{t})\underset{\sim}{i} + (2t - \frac{2}{t})\underset{\sim}{j} + \underset{\sim}{c}$$ **(1 mark)**

When $t = 1$, $\underset{\sim B}{r}(t) = 4\underset{\sim}{i}$

So, $4\underset{\sim}{i} = 4\underset{\sim}{i} + \underset{\sim}{c}$

So, $\underset{\sim}{c} = \underset{\sim}{0}$

So, $\underset{\sim B}{r}(t) = (2t + \frac{2}{t})\underset{\sim}{i} + (2t - \frac{2}{t})\underset{\sim}{j}$ **(1 mark)**

The position vector of the first particle is given by

$$\underset{\sim}{r}(t) = (t + \frac{1}{t})\underset{\sim}{i} + (t - \frac{1}{t})\underset{\sim}{j}$$

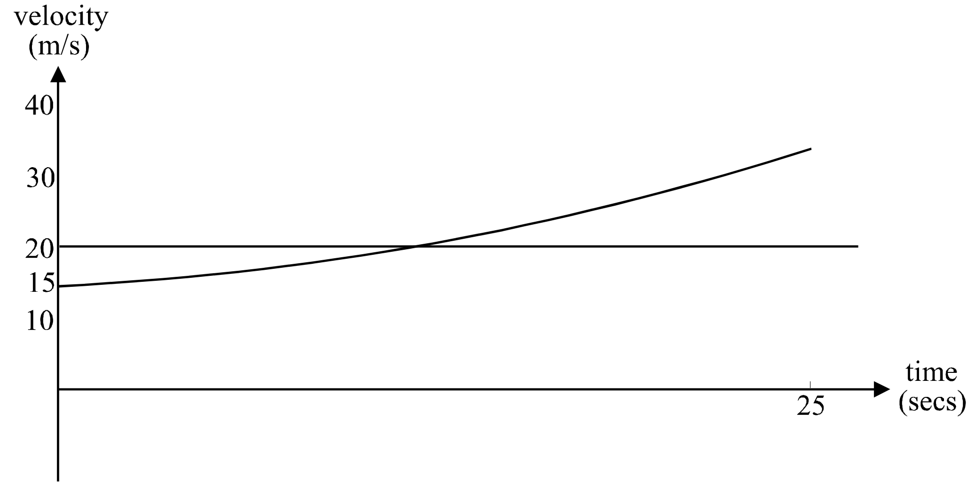
So, $\underset{\sim B}{r}(t) = 2\underset{\sim}{r}(t)$ and hence the 2 position vectors are parallel. **(1 mark)**

Total 11 marks

Question 6

a. At time $t = 0$, Julie gets onto the freeway and so her entry speed is given by $v(0) = 15$ m/s **(1 mark)**

b.



(1 mark)

c. Let Julie catch up to Tom at time T seconds.
We require that the area under the velocity-time graph of Julie and of Tom are the same.

That is, we require that $20T = \int_0^T (0.03t^2 + 15)dt$ **(1 mark)**

$$20T = \left[\frac{0.03t^3}{3} + 15t\right]_0^T$$

$$20T = 0.01T^3 + 15T$$

$$0.01T^3 - 5T = 0$$

$$T(0.01T^2 - 5) = 0$$

$$T = 0 \text{ or } T = \pm 22.36$$

Now $t \geq 0$ so we reject the negative value. The result $T = 0$ relates to the time when Julie enters the freeway, and so the required value of T correct to 2 decimal places is $T = 22.36$ seconds. **(1 mark)**

d. i. $\frac{dv}{dt} = -0.02(v^2 - 625)$

So, $\frac{dt}{dv} = \frac{1}{-0.02(v^2 - 625)}$

$= \frac{-50}{(v-25)(v+25)}$ **(1 mark)**

Now let $\frac{-50}{(v-25)(v+25)} = \frac{A}{(v-25)} + \frac{B}{(v+25)}$

$= \frac{A(v+25) + B(v-25)}{(v-25)(v+25)}$

True iff $-50 \equiv A(v+25) + B(v-25)$

Put $v = -25$ $-50 \equiv -50B$

So, $B = 1$

Put $v = 25$ $-50 \equiv 50A$

So, $A = -1$

So, $\frac{dt}{dv} = \frac{-1}{(v-25)} + \frac{1}{(v+25)}$

So $\int \frac{dt}{dv} dv = -\int \frac{1}{(v-25)} dv + \int \frac{1}{(v+25)} dv$

$t = -\log_e(v-25) + \log_e(v+25) + c$ **(1 mark)**

$t = \log_e \frac{(v+25)}{(v-25)} + c$

Given that $v = 35$ when $t = 0$

We have $0 = \log_e \frac{60}{10} + c$

So $c = -\log_e 6$

So, $t = \log_e \frac{(v+25)}{(v-25)} - \log_e 6$

$t = \log_e \frac{v+25}{6(v-25)}$ **(1 mark)**

ii. Now, $t = \log_e \dfrac{v+25}{6(v-25)}$

So, $$e^t = \frac{(v+25)}{6(v-25)}$$

$$6e^t = 1 + \frac{50}{v-25} \qquad \textbf{(1 mark)}$$

$$6e^t - 1 = \frac{50}{v-25}$$

$$v - 25 = \frac{50}{6e^t - 1}$$

$$v = \frac{50}{6e^t - 1} + 25$$

$$= \frac{50 + 25(6e^t - 1)}{6e^t - 1}$$

$$= \frac{50 + 150e^t - 25}{6e^t - 1}$$

$$\begin{array}{r} 1 \\ v-25\overline{\big)\,v+25} \\ \underline{v-25} \\ 50 \end{array}$$

So $$v = \frac{25(6e^t + 1)}{6e^t - 1}$$ as required **(1 mark)**

Alternatively

$$e^t = \frac{(v+25)}{6(v-25)}$$

$$6e^t(v-25) = v + 25$$

$$6ve^t - 150e^t = v + 25$$

$$6ve^t - v = 150e^t + 25$$

$$v(6e^t - 1) = 25(6e^t + 1)$$

$$v = \frac{25(6e^t + 1)}{6e^t - 1}$$

f. Since $$v = \frac{25(6e^t + 1)}{6e^t - 1}$$

we have, $$v = 25 + \frac{50}{6e^t - 1} \qquad \textbf{(1 mark)}$$

$$\begin{array}{r} 25 \\ 6e^t-1\overline{\big)\,150e^t+25} \\ \underline{150e^t-25} \\ 50 \end{array}$$

As $t \to \infty$, $6e^t - 1 \to \infty$ and so $\dfrac{50}{6e^t - 1} \to 0$. Therefore $25 + \dfrac{50}{6e^t - 1} \to 25$

So, Julie's limiting velocity is 25m/s. **(1 mark)**

Total 11 marks