2001 Specialist Mathematics Exam 2

Solutions

Question 1

a. i.

$\underset{\sim}{r}(t) = 8\cos 2t\ \underset{\sim}{i} + 6\sin 2t\ \underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = -16\sin 2t\ \underset{\sim}{i} + 12\cos 2t\ \underset{\sim}{j}$

$\underset{\sim}{r}(0) = 8\,\underset{\sim}{i}$ **[A1]**

$\dot{\underset{\sim}{r}}(0) = 0\,\underset{\sim}{i} + 12\,\underset{\sim}{j}$

$\underset{\sim}{r}(0) \cdot \dot{\underset{\sim}{r}}(0) = 0 + 0 = 0$ **[A1]**

$\therefore$ at $t = 0$, $\underset{\sim}{r}(t)$ is perpendicular to $\dot{\underset{\sim}{r}}(t)$

ii.

$\underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t) = 0$

$0 = -128\sin 2t\cos 2t + 72\sin 2t\cos 2t$ **[M1]**

$0 = -64\sin 4t + 36\sin 4t$

$0 = \sin 4t$ **[C1]**

$4t = 0, \pi, 2\pi, 3\pi, \ldots$

$t = 0, \frac{\pi}{4}, \frac{\pi}{2}$ all within the specified domain

[A1]

b. i.

$x = 8\cos 2t$ $\qquad$ $y = 6\sin 2t$

$\frac{x}{8} = \cos 2t$ $\qquad$ $\frac{y}{6} = \sin 2t$

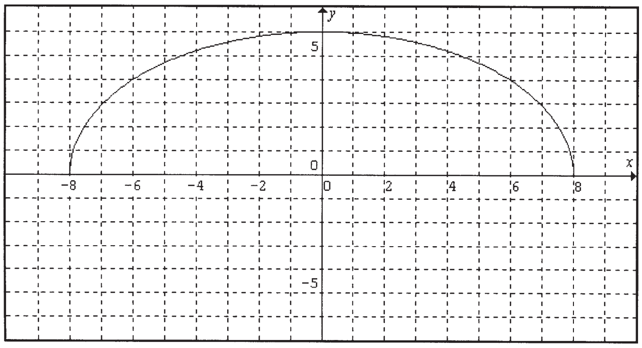
$\frac{x^2}{64} = \cos^2 2t ----(1)$ $\qquad$ $\frac{y^2}{36} = \sin^2 2t ----(2)$

[M1]

$\frac{x^2}{64} + \frac{y^2}{36} = \cos^2 2t + \sin^2 2t$

$\frac{x^2}{64} + \frac{y^2}{36} = 1$ **[A1]**

ii.



Recognition of domain **[A1]**

Shape **[A1]**

c.

$\ddot{\underset{\sim}{r}}(t) = -32\cos t\ \underset{\sim}{i} - 24\sin 2t\ \underset{\sim}{j}$ **[M1]**

$\left|\ddot{\underset{\sim}{r}}(t)\right| = \sqrt{32^2\cos^2 2t + 24^2\sin^2 2t}$

$= \sqrt{1024\cos^2 2t + 576\sin^2 2t}$

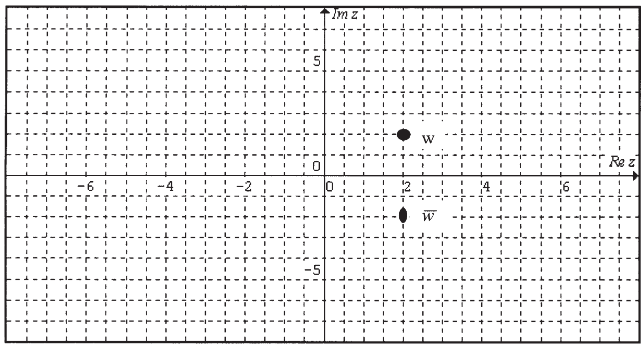
$= \sqrt{448\cos^2 2t + 576(\sin^2 2t + \cos^2 2t)}$

$= \sqrt{448\cos^2 2t + 576}$ **[A1]**

$|r(t)|_{max} = 32$ at $t = 0$ and $t = \frac{\pi}{2}$ **[A1]**

$\left|\ddot{\underset{\sim}{r}}(t)\right|_{min} = 24 \quad t = \frac{\pi}{4}$

Question 2



a. both correct **[A1]**

b. $T = \{z : |z - 2| \le 2\}$ **[A1]**

c.

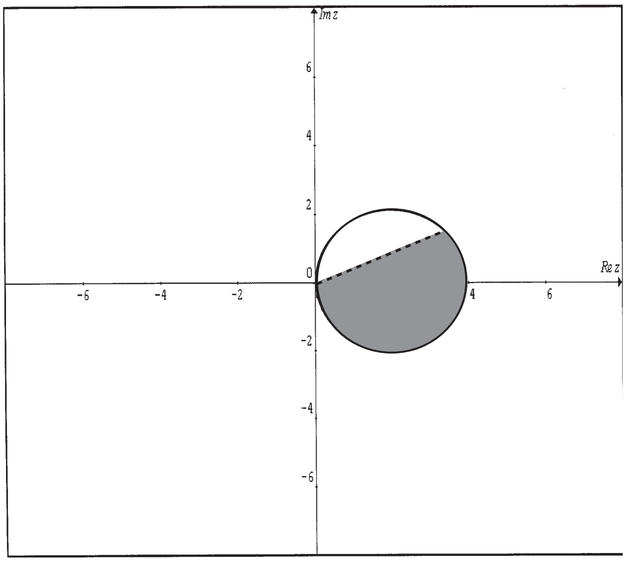
$$|z-2| \le 2$$
$$z = 3+\sqrt{3}i$$
$$\left|3+\sqrt{3}i-2\right| \le 2$$
$$\left|1+\sqrt{3}i\right| \le 2$$
$$\sqrt{1+3} \le 2$$

[A1]

d.

$$|v| = \sqrt{9+3} = 2\sqrt{3}$$
$$\text{Arg}\, v = \text{Tan}^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$$
$$v = 2\sqrt{3}\text{cis}\frac{\pi}{6}$$

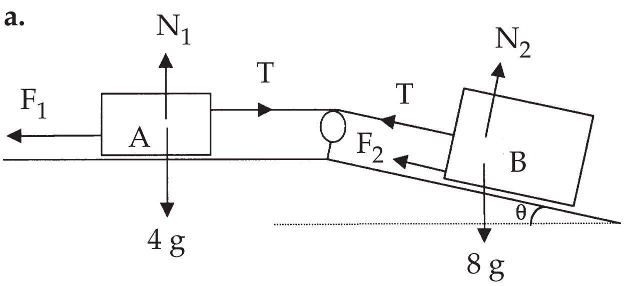
[A1] for Argument
[A1] for Modulus



[A1] correct graphs [A1] correct shading

Question 3

a.



[M1] for most forces
[A1] for all forces correctly allocated

b.

$$T = 0.1N_1 \text{ ----------- } (1)$$
$$T + 0.1N_2 = 8g\sin\theta \text{ ----- } (2)$$
$$N_1 = 4g \text{ ------------ } (3)$$
$$N_2 = 8g\cos\theta \text{ ---------- } (4)$$

[M2]

Using (1) and (3)

$$T = 0.1\times 4\times g = 0.4g$$

Substituting T and (4) into (2)

$$0.4g + 0.8g\cos\theta = 8g\sin\theta$$ **[M1]**

$$0.1 + 0.2\cos\theta = 2\sin\theta$$

Solving using a graphing calculator

$\theta = 9°$

to the nearest degree **[A1]**

Question 4

a. i.

$$f(x) = \frac{12}{\sqrt{x^2+4}} - \frac{x}{3}$$

$$f'(x) = \frac{-12x}{\left(\sqrt{x^2+4}\right)^3} - \frac{1}{3}$$ **[M1]**

$$f''(x) = -\frac{12\left(\sqrt{x^2+4}\right)^3 - 36x^2\sqrt{x^2+4}}{\left(x^2+4\right)^3}$$ **[M1]**

$$= -\frac{12\sqrt{x^2+4}\left(x^2+4-3x^2\right)}{\left(x^2+4\right)^3}$$

$$= -\frac{12\left(-2x^2+4\right)}{\left(x^2+4\right)^{\frac{5}{2}}}$$

$$= \frac{24\left(x^2-2\right)}{\left(x^2+4\right)^{\frac{5}{2}}}$$ **[A1]**

a. ii.

$$f''(x) = 0$$
$$x^2 - 2 = 0$$
$$x^2 = 2$$
$$x = \sqrt{2}$$ **[A1]**

b.

$y = \dfrac{x}{\sqrt{x^2+4}}$

let $u = x^2 + 4$

$\dfrac{du}{dx} = 2x$

$dx = \dfrac{du}{2x}$ **[M1]**

$\int \dfrac{x}{\sqrt{x^2+4}}\,dx$

$= \int \dfrac{1}{2\sqrt{u}}\,du$

$= \sqrt{u}$

$= \sqrt{x^2+4}$ (an antiderivative) **[A1]**

c.

$V = \pi\int_0^4 y^2 dx$

$= \pi\int_0^4 \left(\dfrac{12}{\sqrt{x^2+4}} - \dfrac{x}{3}\right)^2 dx$ **[M1]**

$= \pi\int_0^4 \left(\dfrac{144}{x^2+4} - \dfrac{8x}{\sqrt{x^2+4}} + \dfrac{x^2}{9}\,dx\right)$ **[A1]**

$= \pi\left[\dfrac{x^3}{27} - 8\sqrt{x^2+4} + 72\text{Tan}^{-1}\dfrac{x}{2}\right]_0^4$ **[M1]**

$= \pi\left[\left(\dfrac{4^3}{27} - 8\sqrt{4^2+4} + 72\text{Tan}^{-1}\dfrac{4}{2}\right) - \left(\dfrac{0^3}{27} - 8\sqrt{0^2+4} + 72\text{Tan}^{-1}\dfrac{0}{2}\right)\right]$ **[A1]**

≈ 196

d. i.

$T = 3\sqrt{5-x} - \dfrac{12}{\sqrt{x^2+4}} + \dfrac{x}{3}$ **[A1]**

ii.

Using a graphing calculator maximum thickness is 1.916 m **[A1]**

Question 5

a. i. $\dfrac{d}{dx}\left[\log_e(\cos x)\right] = \dfrac{1}{\cos x}(-\sin x)$ **[M1]**

$= -\tan x$ **[A1]**

ii. $\int(\tan x)dx = -\log_e(\cos x)$ or $\log_e(\sec x)$ **[A1]**

b. $\int(\tan^3 x)dx = \int(\tan x \times \tan^2 x)dx$

$= \int(\tan x(\sec^2 x - 1))dx$ **[M1]**

$= \int(\tan x \sec^2 x - \tan x)dx$

Let $u = \tan x$ **[M1]**

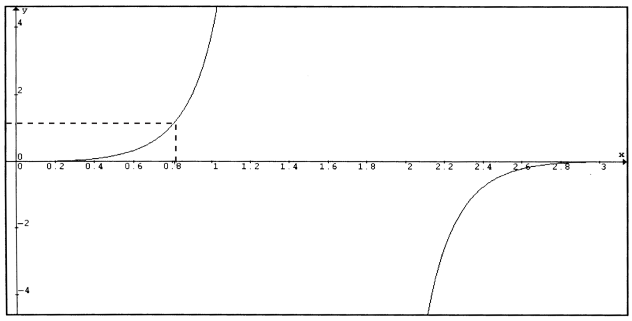
$\dfrac{du}{dx} = \sec^2 x$

$= \int u\dfrac{du}{dx}dx - \int(\tan x)dx$ **[M1]**

$= \dfrac{1}{2}u^2 + \log_e(\cos x) + c$

$= \dfrac{1}{2}\tan^2 x + \log_e(\cos x) + c$ **[A1]**

c.



Correct shape **[A1]**

Scale **[A1]**

d. $A = 1 \times \dfrac{\pi}{4} - \int_0^{\frac{\pi}{4}}(\tan^3 x)dx$ **[M2]**

$= \dfrac{\pi}{4} - \left[\dfrac{1}{2}\tan^2 x + \log_e(\cos x)\right]_0^{\frac{\pi}{4}}$ **[A1]**

$= \dfrac{\pi}{4} - \left[\left(\dfrac{1}{2} + \log_e\left(\dfrac{1}{\sqrt{2}}\right)\right) - (0-0)\right]$

$= \dfrac{\pi}{4} - \dfrac{1}{2} - \log_e\left(\dfrac{1}{\sqrt{2}}\right)$ or $\dfrac{\pi}{4} - \dfrac{1}{2} + \log_e\left(\sqrt{2}\right)$

[A1]

Question 6

a. i. $\frac{1}{A(2000-A} \equiv \frac{a}{A} + \frac{b}{2000-A}$ **[M1]**

$1 \equiv a(2000-A) + bA$

If $A = 2000,\ 1 = 2000b \Leftrightarrow b = \frac{1}{2000}$

If $A = 0,\ 1 = 2000a \Leftrightarrow a = \frac{1}{2000}$ **[M1]**

Hence

$\frac{1}{A(2000-A)} = \frac{1}{2000A} + \frac{1}{2000(2000-A)}$ **[A1]**

a. ii.

$\frac{dA}{dt} = kA(2000-A)$

$\frac{dt}{dA} = \frac{1}{kA(2000-A)}$

$t = \frac{1}{k}\int \frac{1}{2000A} + \frac{1}{2000(2000-A)}\,dA$ **[M1]**

$= \frac{1}{2000k}\int\left(\frac{1}{A} + \frac{1}{2000-A}\right)dA$

$= \frac{1}{2000k}\left(\log_e A - \log_e(2000-A)\right) + c$

$= \frac{1}{2000k}\log_e\left(\frac{A}{2000-A}\right) + c$ **[A1]**

$t = 0,\ A = 20$ **[M1]**

$c = -\frac{1}{2000k}\log_e\left(\frac{20}{1980}\right)$

$t = \frac{1}{2000k}\log_e\left(\frac{A}{2000-A}\right) - \frac{1}{2000k}\log_e\left(\frac{1}{99}\right)$

$t = \frac{1}{2000k}\log_e\left(\frac{99A}{2000-A}\right)$ **[M1]**

$e^{2000kt} = \frac{99A}{2000-A}$ **[M1]**

$2000e^{2000kt} - Ae^{2000kt} = 99A$

$2000e^{2000kt} = A\left(99 + e^{2000kt}\right)$

$A = \frac{2000e^{2000kt}}{99 + e^{2000kt}}$ **[A1]**

b.

$t = 2,\ A = 80,$

$80\left(99 + e^{2000(2k)}\right) = 2000e^{4000k}$ **[M1]**

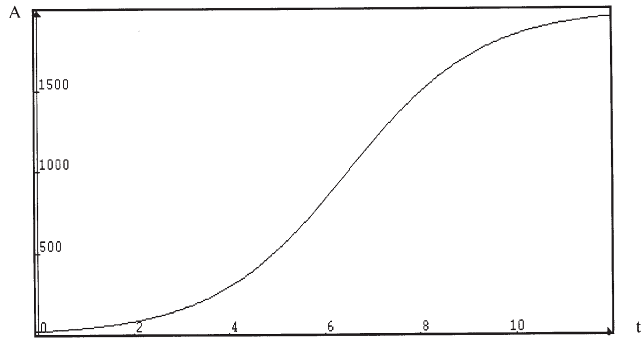
$7920 + 80e^{4000k} = 2000e^{4000k}$

$7920 = 1920e^{4000k}$

$4000k = \log_e\left(\frac{7920}{1920}\right)$

$k = \frac{1}{4000}\log_e\left(\frac{33}{8}\right)$ **[A1]**

c. i.



Correct shape **[A1]**
Scale and axes **[A1]**

c. ii.

The k-value is directly proportional to the rate of infection.

A larger k-value would indicate a more rapid rate of infection. **[A1]**

d.

When A = 1000, t = 6.4854 **[M1]**

Hence 1000 apricots will be infected after 7 days. **[A1]**

$99 \times 1000 = (2000 - 1000)e^{2000kt}$

$99 = e^{2000kt}$

$2000kt = \log_e 99$

$t = \frac{1}{2000k}\log_e 99$

$\simeq 6.48$ days

Hence 1000 apricots are infected after 7 days.

Alternatively students may use a graphing calculator (using a decimal approximation $k \approx 0.000354$)

$$\therefore y = \frac{2000e^{0.708t}}{99 + 2000e^{0.708t}}$$

$y = 1000$ and use intersect function

$x = 6.4902\ldots$

$\therefore t = 7$ days