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SPECIALIST MATHS TRIAL EXAM 1 2002 SOLUTIONS

Part I – Multiple-choice answers

1.	B	7.	B	13.	D	19.	B	25.	D
2.	E	8.	A	14.	E	20.	E	26.	C
3.	D	9.	C	15.	E	21.	E	27.	E
4.	C	10.	B	16.	E	22.	A	28.	D
5.	B	11.	A	17.	D	23.	D	29.	B
6.	E	12.	C	18.	E	24.	E	30.	E

Part I- Multiple-choice solutions

Question 1

The graph has asymptotes given by $x=-\frac{1}{3}$ and $x=2$.

Option A gives us $y=\frac{1}{3x^2-7x+2}$
$=\frac{1}{(3x-1)(x-2)}$

This graph has asymptotes at $x=\frac{1}{3}$ and $x=2$.

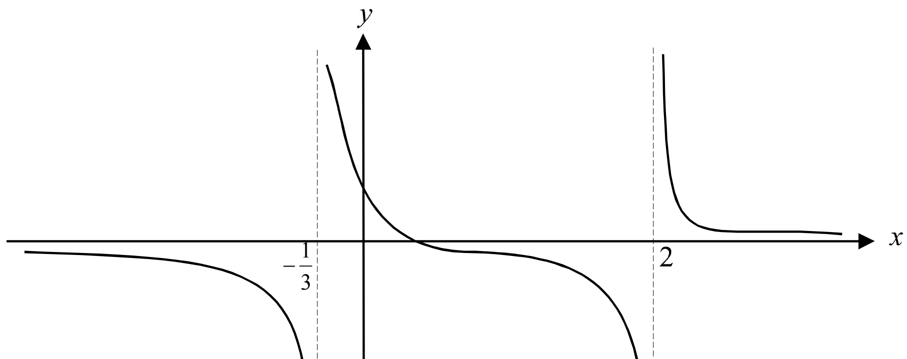
Option B gives us $y=\frac{1}{3x^2-5x-2}$
$=\frac{1}{(3x+1)(x-2)}$

This graph has asymptotes at $x=-\frac{1}{3}$ and $x=2$.

This is the correct option.

Note that option D gives $y=\frac{1}{3x+1}+\frac{1}{x-2}$
$=\frac{x-2+3x+1}{(3x+1)(x-2)}$
$=\frac{4x-1}{(3x+1)(x-2)}$

This graph has the correct asymptotes but the wrong shape as shown on the graph on page 2.



The answer is B

Question 2

The graph is that of a hyperbola which has its centre at $(0,-2)$. Looking along a line given by $y=-2$, we note that $a=4$. Looking at the asymptotes, we note that their gradients are 1 and -1 respectively. So, $\frac{b}{a}=1$ and $-\frac{b}{a}=-1$. Therefore $b=4$.

So the correct equation for this hyperbola is $\frac{x^2}{16}-\frac{(y+2)^2}{16}=1$.

The answer is E.

Question 3

Now $\cot\theta=-\frac{1}{4}$ and θ is a second quadrant angle.

Now, $\cot^2\theta+1=\operatorname{cosec}^2\theta$

So $\frac{1}{16}+1=\operatorname{cosec}^2\theta$

$$\operatorname{cosec}\theta=\sqrt{\frac{17}{16}}$$

(positive since $\sin\theta$ and hence $\operatorname{cosec}\theta$ is positive in the second quadrant)

$$\frac{1}{\sin\theta}=\frac{\sqrt{17}}{4}$$

$$\sin\theta=\frac{4}{\sqrt{17}}$$

So $\frac{16}{17}+\cos^2\theta=1$

$$\cos^2\theta=\frac{1}{17}$$

$$\cos\theta=\frac{-1}{\sqrt{17}}$$

(negative since θ is a second quadrant angle)

So
$$\begin{aligned}\sin 2\theta&=2\sin\theta\cos\theta\\&=2\times\frac{4}{\sqrt{17}}\times\frac{-1}{\sqrt{17}}\\&=\frac{-8}{17}\end{aligned}$$

The answer is D.

Question 4

$$y = \mathrm{Sin}^{-1}\left(\frac{1}{x^2}\right) \text{ and } x \geq 1$$

Let $u = \dfrac{1}{x^2}$

So $y = \mathrm{Sin}^{-1}(u)$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

Now $u = x^{-2}$

$$\frac{du}{dx} = -2x^{-3}$$

So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$= \frac{1}{\sqrt{1-u^2}} \cdot \frac{-2}{x^3}$$

$$= \frac{1}{\sqrt{1-\frac{1}{x^4}}} \cdot \frac{-2}{x^3}$$

$$= \frac{1}{\sqrt{\frac{x^4-1}{x^4}}} \cdot \frac{-2}{x^3}$$

$$= \frac{x^2}{\sqrt{x^4-1}} \cdot \frac{-2}{x^3}$$

$$= \frac{-2}{x\sqrt{x^4-1}}$$

The answer is C.

Question 5

$$\frac{u}{v} = \frac{2+3i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{2+4i+3i-6}{1+4}$$

$$= \frac{-4+7i}{5}$$

$$= \frac{-4}{5} + \frac{7}{5}i$$

The answer is B.

Question 6

When a complex number is multiplied by i, the effect is to rotate it through an angle of $\dfrac{\pi}{2}^c$ anticlockwise about the origin. Multiplying a complex number by i^3 has the effect of rotating it through an angle of $\dfrac{3\pi^c}{2}$ anticlockwise about the origin.

The answer is E.

Question 7

For $a + ai$, $r = \sqrt{a^2 + a^2}$

$= \sqrt{2a^2}$

$= \sqrt{2}a$

Now, since a is a positive or negative whole number, then r could equal $\sqrt{2}$ or $2\sqrt{2}$ or $10\sqrt{2}$. It cannot equal 2 since that can only be the case when $a = \sqrt{2}$ and a has to be a whole number.

Note that $\theta = \text{Tan}^{-1}(\frac{a}{a})$

$= \text{Tan}^{-1}1$

$= \frac{\pi}{4}$ or $\frac{5\pi}{4}$

So, the polar form of $a + ai$ cannot be $2\text{cis}(\frac{\pi}{4})$.

The answer is B.

Question 8

Since $a, b \in R$, then according to the conjugate root theorem, any non-real roots must occur in conjugate pairs. So, if $1 + 2i$ is a solution then so is $1 - 2i$. So, two factors are $(z - 1 - 2i)$ and $(z - 1 + 2i)$

Now,

$(z - 1 - 2i)(z - 1 + 2i)$

$= z^2 - z + 2iz - z + 1 - 2i - 2iz + 2i + 4$

$= z^2 - 2z + 5$

or $((z - 1) - 2i)((z - 1) + 2i)$

$= (z - 1)^2 - (2i)^2$

$= z^2 - 2z + 5$

The third root must be a real number since if it were a complex number it would need a conjugate pair and there can only be three roots.

Since –10 is the constant term in the equation, the third factor must be $(z - 2)$.

So, $(z^2 - 2z + 5)(z - 2)$

$= z^3 - 2z^2 - 2z^2 + 4z + 5z - 10$

$= z^3 - 4z^2 + 9z - 10$

So $a = -4$ and $b = 9$.

The answer is A.

Question 9

Consider the region of the complex plane described by $\{z : |z - i| \le 2\}$.

Now $z = x + yi$

So $|z - i| \le 2$

becomes $|x + yi - i| \le 2$

$\sqrt{x^2 + (y - 1)^2} \le 2$

$x^2 + (y - 1)^2 \le 4$

The region includes all the points on and inside the circle with centre $(0,1)$ and radius 2 units. This eliminates options A, B and E.

Consider the region of the complex plane described by $\{z : \text{Im}(z) - \text{Re}(z) \ge 1\}$.

Now $z = x + yi$

So $\text{Im}(z) - \text{Re}(z) \geq 1$

becomes $y - x \geq 1$

$y \geq x + 1$

The region includes all the points on and above the line $y = x + 1$. Only option C shows the intersection of these two regions.
The answer is C.

Question 10

$$\int e^{2x} \sqrt{(e^x - 1)(e^x + 1)}\, dx$$
$$= \int e^{2x} \sqrt{e^{2x} - 1}\, dx$$

Let $u = e^{2x} - 1$

$$\frac{du}{dx} = 2e^{2x}$$

We have,

$$\int \frac{1}{2} \frac{du}{dx} u^{\frac{1}{2}} dx$$
$$= \frac{1}{2} \int u^{\frac{1}{2}}\, du$$
$$= \frac{1}{2} u^{\frac{3}{2}} \times \frac{2}{3} + c$$
$$= \frac{u^{\frac{3}{2}}}{3} + c$$
$$= \frac{1}{3}\left(e^{2x} - 1\right)^{\frac{3}{2}} + c$$

An antiderivative is $\frac{1}{3}\left(e^{2x} - 1\right)^{\frac{3}{2}}$.

The answer is B.

Question 11

$$\int \sin 2x \cos^2 x\, dx = \int 2 \sin x \cos x \cos^2 x\, dx$$
$$= 2\int \sin x \cos^3 x\, dx \qquad u = \cos x$$
$$= -2\int \frac{du}{dx} u^3\, dx \qquad \frac{du}{dx} = -\sin x$$
$$= -2\int u^3 du$$
$$= -2\frac{u^4}{4} + c$$
$$= \frac{-\cos^4 x}{2} + c$$

The answer is A.

Question 12

$$\int_0^{\sqrt{7}} \frac{5}{\sqrt{7-x^2}}\,dx = 5\int_0^{\sqrt{7}} \frac{1}{\sqrt{7-x^2}}\,dx$$

$$= 5\left[\text{Sin}^{-1}\frac{x}{\sqrt{7}}\right]_0^{\sqrt{7}}$$

$$= 5\left\{\text{Sin}^{-1}1 - \text{Sin}^{-1}0\right\}$$

$$= \frac{5\pi}{2}$$

The answer is C.

Question 13

$$h'(x) = \frac{\sqrt{\log_e(5x)}}{2x}$$

So, $h(x) = \int \frac{\sqrt{\log_e(5x)}}{2x}\,dx$ Let $u = \log_e(5x)$

$$= \frac{1}{2}\int \frac{du}{dx}\sqrt{u}\,dx \qquad \frac{du}{dx} = \frac{1}{x}$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}u^{\frac{3}{2}} \cdot \frac{2}{3} + c$$

$$= \frac{1}{3}\left(\log_e(5x)\right)^{\frac{3}{2}} + c$$

When $x = \frac{1}{5}$, $h(x) = 0$

So, $0 = \frac{1}{3}\left(\log_e 1\right)^{\frac{3}{2}} + c$

So, $c = 0$ since $\log_e 1 = 0$

So, $h(x) = \frac{1}{3}\left(\log_e(5x)\right)^{\frac{3}{2}}$

The answer is D.

Question 14

Area required is given by

$$\int_a^{\frac{\pi}{2}} 3\cos(x)\,dx + \int_0^a x^3\,dx + \int_{\frac{-\pi}{2}}^0 \left(3\cos(x) - x^3\right)dx$$

The answer is E.

Question 15

$$\text{Volume required} = \pi \int_1^{e^2} x^2 \, dy$$

Now $\quad y = e^{2x}$

So, $\quad \log_e y = 2x$

$$x = \frac{1}{2}\log_e y$$
$$= \log_e \sqrt{y}$$

So $\quad x^2 = \left(\log_e \sqrt{y}\right)^2$

So volume required is given by $\pi \int_1^{e^2} \left(\log_e \sqrt{y}\right)^2 dy$ The answer is E.

Question 16

Now, $\quad y = \log_e(\cot\theta), \quad 0 < \theta < \frac{\pi}{2}$

$$= \log_e(\tan\theta)^{-1}$$

So,
$$\frac{dy}{d\theta} = -1(\tan\theta)^{-2} \times \sec^2\theta \div (\tan\theta)^{-1}$$
$$= \frac{-\sec^2\theta}{\tan^2\theta} \div \frac{1}{\tan\theta}$$
$$= \frac{-1}{\tan^2\theta\cos^2\theta} \times \frac{\tan\theta}{1}$$
$$= \frac{-1}{\tan\theta\cos^2\theta}$$
$$= \frac{-\cos\theta}{\sin\theta\cos^2\theta}$$
$$= \frac{-1}{\sin\theta\cos\theta}$$
$$\frac{d^2y}{d\theta^2} = 1(\sin\theta\cos\theta)^{-2} \times (-\sin\theta\sin\theta + \cos\theta\cos\theta)$$
$$= \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta\cos^2\theta}$$

The answer is E.

So $\quad \frac{d^2y}{d\theta^2} - \frac{dy}{d\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta\cos^2\theta} + \frac{1}{\sin\theta\cos\theta}$

Question 17

For option A, $\quad f(0) = 0, \quad f(0) \neq 2 \quad$ and $\quad f'(2) = 0$
For option B, $\quad f(2) \neq 0, \quad f'(2) = 0$ and $\quad f'(5) = 0$
For option C, $\quad f(5) = 0, \quad f'(0) \neq 0 \quad$ and $\quad f'(2) = 0$
For option D, $\quad f(0) = 0, \quad f'(2) = 0 \quad$ and $\quad f''(5) > 0$
For option E, $\quad f(0) = 0, \quad f(5) = 0 \quad$ and $\quad f''(2) < 0$

Note that at a local maximum, $f'(x) = 0$ and $f''(x) < 0$. At a local minimum, $f'(x) = 0$ and $f''(x) > 0$.

The answer is D.

Question 18

Now, $\frac{dx}{dt}=-2$ since the distance x is reducing as the man approaches the building.

Also $\tan\theta=\frac{20}{x}$

So,
$$\begin{aligned}x&=\frac{20}{\tan\theta}\\&=20(\tan\theta)^{-1}\end{aligned}$$

So,
$$\begin{aligned}\frac{dx}{d\theta}&=-20(\tan\theta)^{-2}\times\sec^2\theta\\&=\frac{-20\sec^2\theta}{\tan^2\theta}\\&=\frac{-20\cos^2\theta}{\cos^2\theta.\sin^2\theta}\\&=\frac{-20}{\sin^2\theta}\end{aligned}$$

So,
$$\begin{aligned}\frac{d\theta}{dt}&=\frac{d\theta}{dx}\cdot\frac{dx}{dt}\\&=\frac{\sin^2\theta}{-20}\cdot-2\\&=\frac{\sin^2\theta}{10}\end{aligned}$$

The answer is E.

Question 19

Now if $\frac{dy}{dx}=f(x)$, $x_0=a$, and $y_0=b$

then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+hf(x_n)$

Now, $x_0=1$ $y_0=0$ and
$$\begin{aligned}\frac{dy}{dx}&=f(x)\\&=\sqrt{4-x^2}\end{aligned}$$

So, $x_1=1+0\cdot1$ and $y_1=0+0\cdot1\times\sqrt{4-1}$

$=1\cdot1$ $\quad=0\cdot1\sqrt{3}$

$x_2=1\cdot1+0\cdot1$ and $y_2=0\cdot1\sqrt{3}+0\cdot1\times\sqrt{4-1.1^2}$

$=1\cdot2$ $\quad=0.3402$ (to 4 decimal places)

The answer is B.

Question 20

acceleration $= v(v+1)$

So, $v\dfrac{dv}{dx} = v(v+1)$

$\dfrac{dv}{dx} = v+1$

$\dfrac{dx}{dv} = \dfrac{1}{v+1}$

$x = \int \dfrac{1}{v+1}\,dv$

$x = \log_e(v+1) + c$

When $x = 0,\ v = 0$

$0 = \log_e(1) + c$

$c = 0$

So $x = \log_e(v+1)$

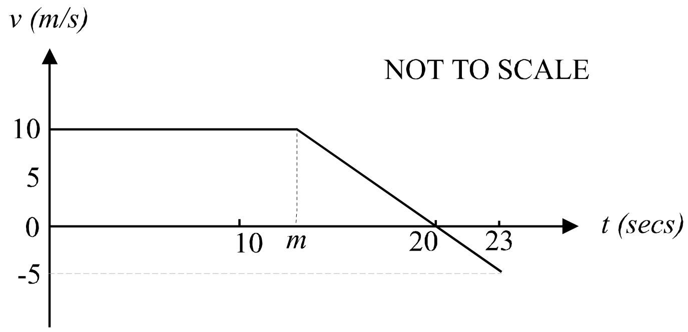
$e^x = v+1$

$v = e^x - 1$

The answer is E.

Question 21

Let m be the latest time when the velocity of the particle was 10 m/s.



Between $t = 0$ and $t = 23$, the displacement of the particle is given by

$$10 \times m + \frac{1}{2} \times 10 \times (20 - m) - \frac{1}{2} \times 5 \times 3$$

$$= 10m + 100 - 5m - 7.5$$

$$= 5m + 92.5$$

So, $5m + 92.5 = 162.5$

$5m = 70$

$m = 14$

The answer is E.

Question 22

$\overrightarrow{AB} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{AC} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

So, $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$$= -\overrightarrow{AB} + \overrightarrow{AC}$$
$$= -2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k} + \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$$
$$= -\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1+16+1}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

The answer is A.

Question 23

$\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}$

$|\underset{\sim}{u}| = \sqrt{5}$ $\quad$ $|\underset{\sim}{v}| = \sqrt{1+1+9}$

$$= \sqrt{11}$$

So $\hat{\underset{\sim}{u}} = \dfrac{1}{\sqrt{5}}\left(2\underset{\sim}{i} - \underset{\sim}{j}\right)$ and $\hat{\underset{\sim}{v}} = \dfrac{1}{\sqrt{11}}\left(\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}\right)$

So $\hat{\underset{\sim}{u}}.\hat{\underset{\sim}{v}} = \dfrac{1}{\sqrt{55}}(2+1)$

$$= \frac{3}{\sqrt{55}}$$

The answer is D.

Question 24

Option A - $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly dependent since $\underset{\sim}{u} - 2\underset{\sim}{v} = \underset{\sim}{0}$

Option B - $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly dependent since $\underset{\sim}{u} = 2\underset{\sim}{v}$ and $\underset{\sim}{w} = \dfrac{1}{3}\underset{\sim}{u}$ and

therefore $\quad \underset{\sim}{u} = 2\underset{\sim}{v}$

becomes $\quad \dfrac{1}{3}\underset{\sim}{u} + \dfrac{2}{3}\underset{\sim}{u} = 2\underset{\sim}{v}$

$$\underset{\sim}{w} + \frac{2}{3}\underset{\sim}{u} - 2\underset{\sim}{v} = \underset{\sim}{0}$$

Option C - $\underset{\sim}{u} = 2\underset{\sim}{v}$ and $\underset{\sim}{w} = \dfrac{1}{3}\underset{\sim}{u}$

$$3\underset{\sim}{w} = \underset{\sim}{u}$$

So $3\underset{\sim}{w} = 2\underset{\sim}{v}$

$$\underset{\sim}{v} = \frac{3}{2}\underset{\sim}{w}$$

So $\underset{\sim}{v}$ is one and a half times the length of $\underset{\sim}{w}$.

Option D – since $\underset{\sim}{v} = \frac{3}{2}\underset{\sim}{w}$, $\underset{\sim}{v}$ is parallel to $\underset{\sim}{w}$.

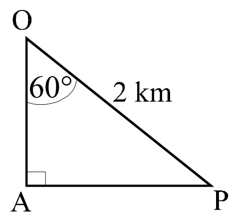
Option E - since $\underset{\sim}{v} = \frac{3}{2}\underset{\sim}{w}$

$$\underset{\sim}{v} - \frac{3}{2}\underset{\sim}{w} = \underset{\sim}{0}$$

So $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly dependent.

So Option E is false.
The answer is E.

Question 25
Now, in ΔAOP,



$$\sin 60^\circ = \frac{AP}{2}$$
$$AP = \sqrt{3}$$

Also, $\cos 60^\circ = \frac{AO}{2}$

$$AO = 1$$

So,
$$\vec{OP} = \vec{OA} + \vec{AP}$$
$$= -\underset{\sim}{j} + \sqrt{3}\,\underset{\sim}{i}$$

Note that the unit vector $\underset{\sim}{j}$ runs in the north direction.

The answer is D.

Question 26
$\underset{\sim}{r} = 5t\,\underset{\sim}{i} + \sin(2t)\,\underset{\sim}{j} + t\,\underset{\sim}{k},\ \ t \ge 0$

So, $\dot{\underset{\sim}{r}} = 5\,\underset{\sim}{i} + 2\cos(2t)\,\underset{\sim}{j} + \underset{\sim}{k}$

So, $\left|\dot{\underset{\sim}{r}}\right| = \sqrt{25 + 4\cos^2(2t) + 1}$

$$= \sqrt{26 + 4\cos^2(2t)}$$

The speed will be a maximum when $\cos(2t) = 1$ and hence $\cos^2(2t) = 1$.
The maximum speed is $\sqrt{30}$.
The answer is C.

Question 27

momentum = mass × velocity

Momentum is a vector quantity.
So momentum $= 2(3\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + \underset{\sim}{k})$

$$= 6\,\underset{\sim}{i} + 4\,\underset{\sim}{j} + 2\,\underset{\sim}{k}$$

The answer is E.

Question 28

Let the direction of the force be in the $\underset{\sim}{i}$ direction.

Now, $\underset{\sim}{R} = m\underset{\sim}{a}$

becomes $\left(3 + t^2\right)\underset{\sim}{i} = 2\dfrac{dv}{dt}\underset{\sim}{i}$

So, $\dfrac{dv}{dt} = \dfrac{1}{2}\left(3 + t^2\right)$

$$v = \frac{1}{2}\int\left(3 + t^2\right)dt$$

$$= \frac{1}{2}\left(3t + \frac{t^3}{3}\right) + c$$

When $t = 0$, $v = 4$

So, $4 = 0 + c$

So, $v = \dfrac{1}{2}\left(3t + \dfrac{t^3}{3}\right) + 4$

When $t = 3$, $v = \dfrac{1}{2}\left(9 + \dfrac{27}{3}\right) + 4$

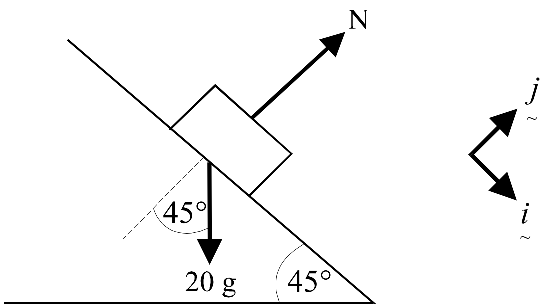
$$= \frac{1}{2}(18) + 4$$

$$= 13\text{m/s}$$

The answer is D.

Question 29

Draw a diagram.



$$\underset{\sim}{R} = m\underset{\sim}{a}$$

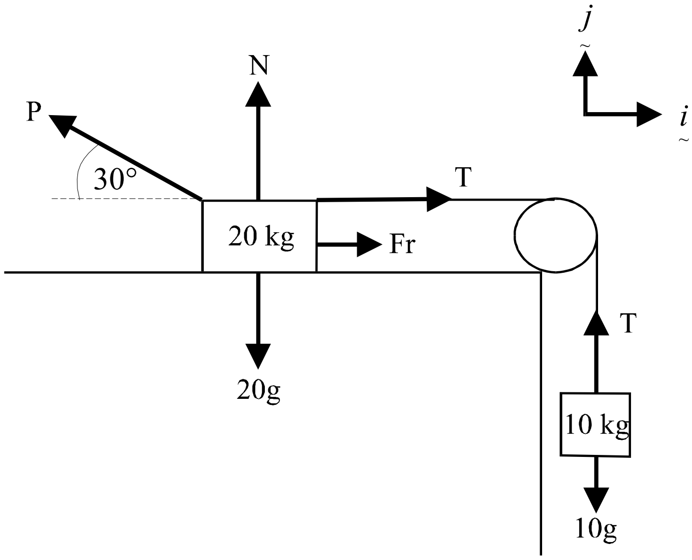
$\left(20g\sin 45^\circ\right)\underset{\sim}{i} + \left(N - 20g\cos 45^\circ\right)\underset{\sim}{j} = 20a\,\underset{\sim}{i}$

So, $20g\sin 45^\circ = 20a$

$$a = \frac{g}{\sqrt{2}}$$

The answer is B.

Question 30
Draw a diagram.



Resolving forces around the 10kg mass, we find that $\underset{\sim}{R} = m\underset{\sim}{a}$

$$(T - 10g)\underset{\sim}{j} = 10a\underset{\sim}{j}$$

So, $$T = 10a + 10g$$

Around the 20kg mass, we have,

$$(T + Fr - P\cos 30^\circ)\underset{\sim}{i} + \left(N + P\sin 30^\circ - 20g\right)\underset{\sim}{j} = 20a \times -\underset{\sim}{i}$$

So, $$10a + 10g + Fr - \frac{\sqrt{3}P}{2} = -20a$$

$$a = \frac{\sqrt{3}P}{60} - \frac{Fr}{30} - \frac{g}{3}$$

The answer is E.

PART II – Solutions

Question 1

i.

$$\begin{aligned} & z^3 - 2z^2 + 5z - 10 \\ &= z^2(z-2) + 5(z-2) \\ &= (z^2+5)(z-2) \\ &= (z^2 - 5i^2)(z-2) \\ &= (z - \sqrt{5}i)(z + \sqrt{5}i)(z-2) \end{aligned}$$

(1 mark)

ii.

$$\begin{aligned} & z^6 - 64 \\ &= (z^3)^2 - (2^3)^2 \\ &= (z^3 - 2^3)(z^3 + 2^3) \\ &= (z-2)(z^2+2z+4)(z+2)(z^2-2z+4) \quad \textbf{(1 mark)} \\ &= (z-2)\left((z^2+2z+1)-1+4\right)(z+2)\left((z^2-2z+1)-1+4\right) \\ &= (z-2)\left((z+1)^2+3\right)(z+2)\left((z-1)^2+3\right) \\ &= (z-2)(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)(z+2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i) \end{aligned}$$

(1 mark)

Question 2

a. $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i}$

So $\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + \underset{\sim}{c}$

When $t = 0$, $\dot{\underset{\sim}{r}}(t) = \underset{\sim}{j}$

So, $\underset{\sim}{j} = 0\underset{\sim}{i} + \underset{\sim}{c}$

So, $\underset{\sim}{c} = \underset{\sim}{j}$

So, $\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + \underset{\sim}{j}$ **(1 mark)**

So, $r(t) = t^2\underset{\sim}{i} + t\underset{\sim}{j} + \underset{\sim}{c}_1$

When $t = 0$, $r(t) = -\underset{\sim}{i} - \underset{\sim}{j}$

So $-\underset{\sim}{i} - \underset{\sim}{j} = \underset{\sim}{c}_1$

So, $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + t\underset{\sim}{j} - \underset{\sim}{i} - \underset{\sim}{j}$

$= (t^2 - 1)\underset{\sim}{i} + (t - 1)\underset{\sim}{j}$ **(1 mark)**

b. Since $\underset{\sim}{r}(t) = (t^2 - 1)\underset{\sim}{i} + (t - 1)\underset{\sim}{j}$

$x = t^2 - 1$ and $y = t - 1$

$t = y + 1$

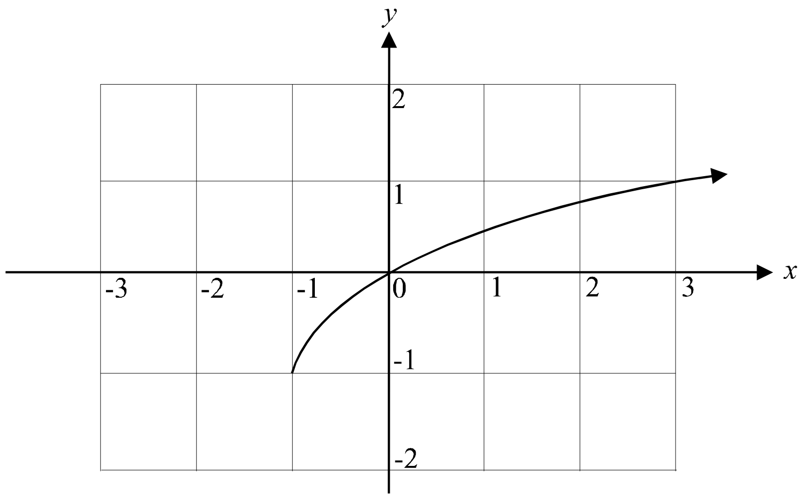
So $x = (y + 1)^2 - 1$

$x + 1 = (y + 1)^2$

$\sqrt{x + 1} = y + 1$

$y = -1 + \sqrt{x + 1}$ **(1 mark)**

We know to take the upper branch of the graph of $x + 1 = (y + 1)^2$ rather than the lower branch since when $t = 0$, $x = -1$ and $y = -1$ when $t = 1$, $x = 0$ and $y = 0$, when $t = 2$, $x = 3$ and $y = 1$ and so on. By continuing to evaluate points on the graph, we see that the upper branch is the path traced out by the particle.



(1 mark)

Question 3

<u>To Prove</u>: $\overrightarrow{BF}.\overrightarrow{FG}=0$

Now, $\overrightarrow{BE}=\frac{1}{2}\underset{\sim}{m}+\frac{1}{2}\underset{\sim}{n}$

$\overrightarrow{EC}=-\frac{1}{2}\underset{\sim}{m}+\frac{1}{2}\underset{\sim}{n}$

and $\overrightarrow{EF}=-\frac{1}{4}\underset{\sim}{m}+\frac{1}{4}\underset{\sim}{n}$

So
$$\begin{aligned}\overrightarrow{BF}&=\overrightarrow{BE}+\overrightarrow{EF}\\&=\frac{1}{2}\underset{\sim}{m}+\frac{1}{2}\underset{\sim}{n}-\frac{1}{4}\underset{\sim}{m}+\frac{1}{4}\underset{\sim}{n}\\&=\frac{1}{4}\underset{\sim}{m}+\frac{3}{4}\underset{\sim}{n}\end{aligned}$$
(1 mark)

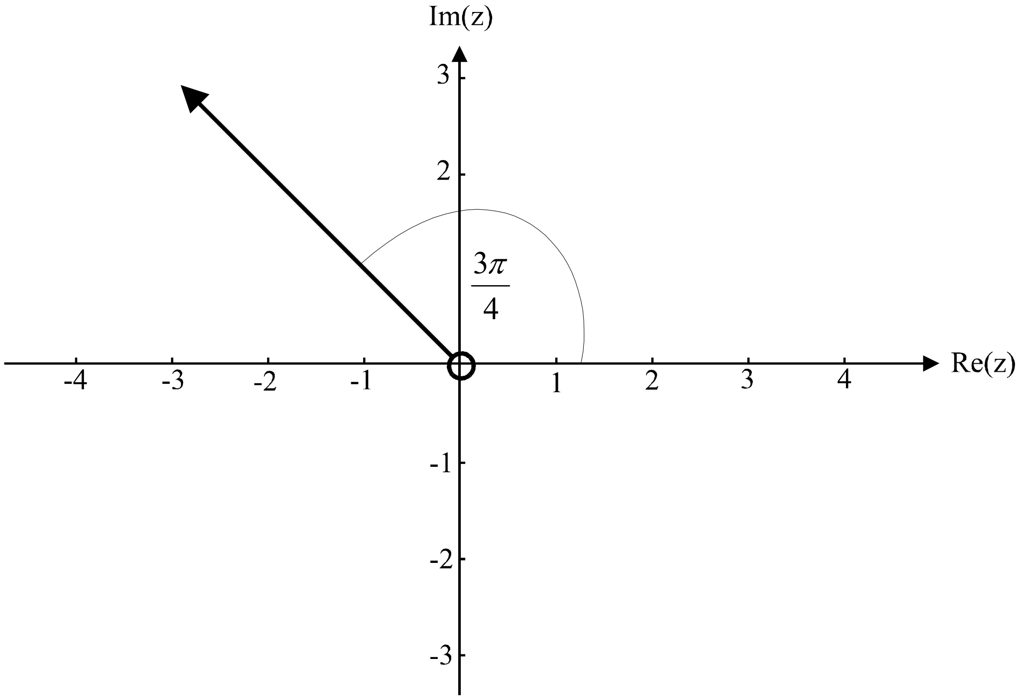
Also,
$$\begin{aligned}\overrightarrow{FG}&=\overrightarrow{FC}+\overrightarrow{CG}\\&=\overrightarrow{EF}+\overrightarrow{CG}\\&=-\frac{1}{4}\underset{\sim}{m}+\frac{1}{4}\underset{\sim}{n}-\frac{1}{2}\underset{\sim}{m}\\&=-\frac{3}{4}\underset{\sim}{m}+\frac{1}{4}\underset{\sim}{n}\end{aligned}$$
(1 mark)

So
$$\begin{aligned}\overrightarrow{BF}.\overrightarrow{FG}&=\left(\frac{1}{4}\underset{\sim}{m}+\frac{3}{4}\underset{\sim}{n}\right).\left(-\frac{3}{4}\underset{\sim}{m}+\frac{1}{4}\underset{\sim}{n}\right)\\&=\frac{-3}{16}\underset{\sim}{m}.\underset{\sim}{m}+\frac{1}{16}\underset{\sim}{m}.\underset{\sim}{n}-\frac{9}{16}\underset{\sim}{m}.\underset{\sim}{n}+\frac{3}{16}\underset{\sim}{n}.\underset{\sim}{n}\\&=\frac{-3}{16}\underset{\sim}{m}.\underset{\sim}{m}-\frac{1}{2}\underset{\sim}{m}.\underset{\sim}{n}+\frac{3}{16}\underset{\sim}{n}.\underset{\sim}{n}\\&=\frac{-3}{16}|\underset{\sim}{m}||\underset{\sim}{m}|\cos 0+\frac{3}{16}|\underset{\sim}{n}||\underset{\sim}{n}|\cos 0\quad\text{since } \underset{\sim}{m}.\underset{\sim}{n}=0 \text{ because } \overrightarrow{AB}.\overrightarrow{BC}=0\\&=\frac{-3}{16}|\underset{\sim}{m}|^2+\frac{3}{16}|\underset{\sim}{n}|^2\\&=\frac{-3}{16}|\underset{\sim}{m}|^2+\frac{3}{16}|\underset{\sim}{m}|^2\quad\text{since } |\underset{\sim}{m}|=|\underset{\sim}{n}| \quad\text{because } ABCD \text{ is a square}\\&=0\end{aligned}$$

Have proved **(1 mark)**

Question 4

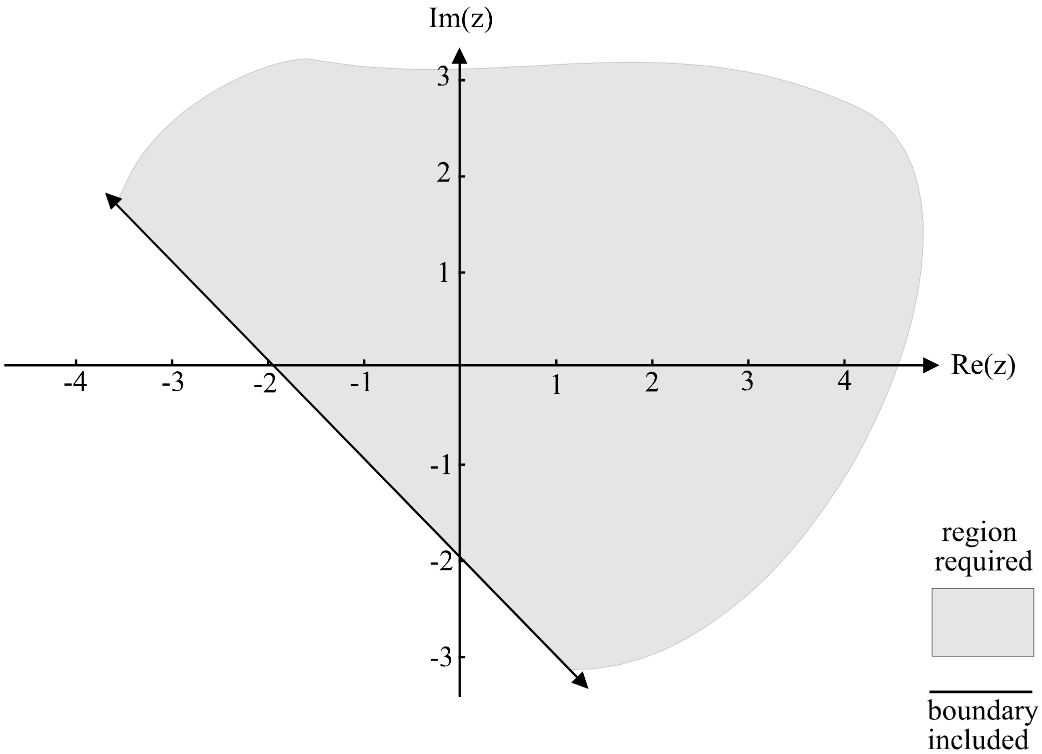
a.



All the points along the ray shown have $Arg z = \frac{3\pi}{4}$. Note that the complex number $0 + 0i$ is excluded.

(1 mark)

b.



(1 mark) for correct straight line
(1 mark) for correct shading

Method 1- geometrically
We are looking for the perpendicular bisector between $(0, 0)$ and $(-2, -2)$ which is $y = -x - 2$. This straight line describes those complex numbers for which the distance between themselves and the origin is the same as the distance between themselves and the point $-2 - 2i$.
The complex numbers indicated above the straight line are the complex numbers for which the distance between themselves and the origin is less than the distance between themselves and the point $-2 - 2i$.
Method 2 – algebraically
Let $\quad z = x + yi$
So, $\quad |z| = |z + 2 + 2i|$
Becomes $\quad |x + yi| = |x + yi + 2 + 2i|$

$$\sqrt{x^2 + y^2} = \sqrt{(x+2)^2 + (y+2)^2}$$
$$x^2 + y^2 = x^2 + 4x + 4 + y^2 + 4y + 4$$
$$y = -x - 2$$

The boundary, which is included, is given by $y = -x - 2$. We also require those points for which $y > -x - 2$ as indicated by the shaded region above.

Question 5

a. The required differential equation is

$$\frac{dm}{dt} = -km \qquad \text{where } k \text{ is a constant}$$ **(1 mark)**

b.

Now, $\quad \dfrac{dm}{dt} = -km$

$$\frac{dt}{dm} = \frac{-1}{km}$$
$$t = \frac{-1}{k}\int \frac{1}{m}\,dm$$
$$t = \frac{-1}{k}\log_e m + c$$

When $\quad t = 0,\ m = 100$

So, $\quad 0 = -\dfrac{1}{k}\log_e 100 + c$

$$c = \frac{1}{k}\log_e 100$$

So $\quad t = -\dfrac{1}{k}\log_e m + \dfrac{1}{k}\log_e 100$

$$= \frac{1}{k}\log_e \frac{100}{m}$$

(1 mark)

When $t = 10,\ m = 50$

So, $10 = \frac{1}{k}\log_e 2$

$k = \frac{1}{10}\log_e 2$

$= 0.0693$ (to 4 decimal places)

So, $t = \frac{1}{0.0693}\log_e \frac{100}{m}$

$0.0693t = \log_e \frac{100}{m}$

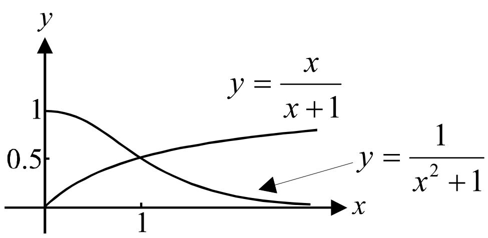
$e^{0.0693t} = \frac{100}{m}$

$m = 100e^{-0.0693t}$

(1 mark)

Question 6

Use your graphics calculator to draw a quick sketch graph.



(1 mark)

The point of intersection of the two graphs is $(1, 0.5)$.
Area required is given by

$$\int_0^1 \left(\frac{1}{x^2+1} - \frac{x}{x+1}\right)dx$$ **(1 mark)**

$$= \int_0^1 \left(\frac{1}{x^2+1} - 1 + \frac{1}{x+1}\right)dx \quad \text{since } \frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$= \left[\text{Tan}^{-1}x - x + \log_e(x+1)\right]_0^1$$ **(1 mark)**

$$= \left\{\left(\text{Tan}^{-1}1 - 1 + \log_e 2\right) - \left(\text{Tan}^{-1}0 - 0 + \log_e 1\right)\right\}$$

$$= \frac{\pi}{4} - 1 + \log_e 2$$

(1 mark)