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SPECIALIST MATHS TRIAL EXAMINATION 2 2002 SOLUTIONS

Question 1

a. **i.** From the diagram, $\overrightarrow{OQ} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$.

(1 mark)

ii. Similarly, $\overrightarrow{OP} = 7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k}$.

(1 mark)

b. **i.** The vector component of $\overrightarrow{OP}$ perpendicular to $\overrightarrow{OQ}$ is given by

$$\overrightarrow{OP} - \left(\overrightarrow{OP} . \hat{\overrightarrow{OQ}}\right)\hat{\overrightarrow{OQ}}$$ **(1 mark)**

$$= 7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k} - \left\{(7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k}).\frac{1}{25}(7\underset{\sim}{i} + 24\underset{\sim}{j})\right\}\frac{1}{25}(7\underset{\sim}{i} + 24\underset{\sim}{j})$$

$$= 7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k} - \frac{625}{25} \times \frac{1}{25}(7\underset{\sim}{i} + 24\underset{\sim}{j})$$

$$= 7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k} - 7\underset{\sim}{i} - 24\underset{\sim}{j}$$

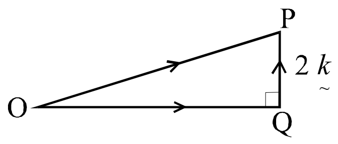
$$= 2\underset{\sim}{k}$$

as required **(1 mark)**

ii. Method 1

From the diagram, QP is the component of $\overrightarrow{OP}$ perpendicular to $\overrightarrow{OQ}$.

(1 mark)



So, $\overrightarrow{PQ} = -2\underset{\sim}{k}$

(1 mark)

Method 2

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$ **(1 mark)**

$$= -7\underset{\sim}{i} - 24\underset{\sim}{j} - 2\underset{\sim}{k} + 7\underset{\sim}{i} + 24\underset{\sim}{j}$$

$$= -2\underset{\sim}{k}$$

(1 mark)

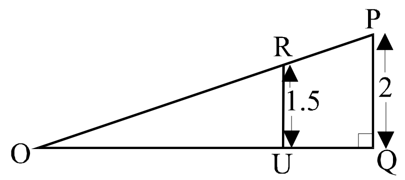
c. Now, $\overrightarrow{RT} = \overrightarrow{RO} + \overrightarrow{OT}$

Now, from the diagram, $\overrightarrow{OT} = 3\underset{\sim}{i} + 25\underset{\sim}{j}$. **(1 mark)**

We need to find $\overrightarrow{RO}$.

Now, ΔORU is similar to ΔOPQ.

So, $\overrightarrow{OR} = \frac{3}{4}(\overrightarrow{OP})$ **(1 mark)**



Now, $\overrightarrow{RT} = \overrightarrow{RO} + \overrightarrow{OT}$

$= -\overrightarrow{OR} + \overrightarrow{OT}$

$= -\frac{3}{4}(\overrightarrow{OP}) + \overrightarrow{OT}$ **(1 mark)**

$= -\frac{3}{4}(7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k}) + 3\underset{\sim}{i} + 25\underset{\sim}{j}$

$= -5.25\underset{\sim}{i} - 18\underset{\sim}{j} - 1.5\underset{\sim}{k} + 3\underset{\sim}{i} + 25\underset{\sim}{j}$

$= -2.25\underset{\sim}{i} + 7\underset{\sim}{j} - 1.5\underset{\sim}{k}$ as required **(1 mark)**

d. We are looking for the angle of deflection $\angle PRT$. We want to find the angle between the vectors $\overrightarrow{RT}$ and $\overrightarrow{RP}$. Since $\overrightarrow{RP}$ runs parallel to $\overrightarrow{OP}$,we seek to find the angle between the vectors $\overrightarrow{RT}$ and $\overrightarrow{OP}$. **(1mark)**

$\overrightarrow{RT} = -2.25\underset{\sim}{i} + 7\underset{\sim}{j} - 1.5\underset{\sim}{k}$

and $\overrightarrow{OP} = 7\underset{\sim}{i} + 24\underset{\sim}{j} + 2\underset{\sim}{k}$

So, $\cos\theta = \frac{-15.75 + 168 - 3}{\sqrt{56.3125}\sqrt{629}}$ **(1 mark)**

$= 0.7930$

$\theta = 37°32'$

So the angle of deflection is $37°32'$ (to the nearest minute) **(1mark)**

Total 13 marks

Question 2

a. 10.00am corresponds to 3 hours after work started so $t = 3$.

Now $\dfrac{dl}{dt} = 2te^{\frac{-t^2}{10}}$

When $t = 3$, $\dfrac{dl}{dt} = 6e^{\frac{-9}{10}}$

$= 2.44$ metres/ hour

So, at 10.00am, the rate at which the path is being laid is 2.44 metres/hour (correct to 2 decimal places).

(1 mark)

b. **i.** We are looking for $\dfrac{dl}{dt}$ to be a maximum. This will occur when $\dfrac{d^2l}{dt^2} = 0$.

Now $\dfrac{dl}{dt} = 2te^{\frac{-t^2}{10}}$

So, $\dfrac{d^2l}{dt^2} = 2t \times \dfrac{-2t}{10} e^{\frac{-t^2}{10}} + 2e^{\frac{-t^2}{10}}$

$= \dfrac{-2t^2}{5} e^{\frac{-t^2}{10}} + 2e^{\frac{-t^2}{10}}$

When $\dfrac{d^2l}{dt^2} = 0$

we have $\dfrac{-2t^2}{5} e^{\frac{-t^2}{10}} + 2e^{\frac{-t^2}{10}} = 0$ **(1mark)**

$e^{\frac{-t^2}{10}}\left(-\dfrac{2t^2}{5} + 2\right) = 0$

Now, $e^{\frac{-t^2}{10}} \neq 0$ for $t \in R$

So, $\dfrac{-2t^2}{5} + 2 = 0$

$-2t^2 + 10 = 0$

$2(5 - t^2) = 0$

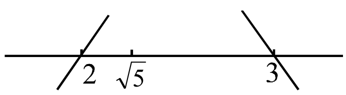
$t = \pm\sqrt{5}$

Since $t \geq 0$, $t = \sqrt{5}$ $(\cong 2.23)$

Check for a maximum.

When $t = 2$, $\dfrac{d^2l}{dt^2} = 0.2681$

When $t = 3$, $\dfrac{d^2l}{dt^2} = -0.6505$



So we have a maximum at $t = \sqrt{5}$. **(1 mark)**

So the fastest rate occurs at $t = \sqrt{5}$ hours.

Now $\sqrt{5}$ hours $= \sqrt{5} \times 60$ minutes

$= 134$ minutes (to the nearest minute)

$= 2$ hour 14 minutes

So the time at which the path is being laid fastest is 9.14am.

(1 mark)

ii. From **part i.,** the fastest rate at which the landscape gardener lays the path occurs at $t = \sqrt{5}$.

So $\dfrac{dl}{dt} = 2\sqrt{5}e^{\frac{-5}{10}}$

$= 2.7$

So, the fastest rate at which the landscape gardener lays the path is 2.7 metres/hour (correct to one decimal place). **(1 mark)**

c. We need to solve, simultaneously, the equations

$\dfrac{dl}{dt} = 2te^{\frac{-t^2}{10}}$ and $\dfrac{dl}{dt} = 2$ for $t > 2$.

Using a graphics calculator we find that the two respective graphs intersect at $(3.5655584,\ 2)$.

So, after 3 hours and 34 minutes (to the nearest minute) or at 10.34am, the rate at which the path is being laid drops below 2 metres per hour.

(1 mark)

d.

$$\int 2te^{\frac{-t^2}{10}}\,dt$$

$$= \int -10\frac{du}{dt}e^u\,dt \quad \textbf{(1 mark)}$$

$$= -10\int e^u\,du$$

$$= -10e^u + c$$

$$= -10e^{\frac{-t^2}{10}} + c$$

(1 mark)

let $u = \dfrac{-t^2}{10}$

$\dfrac{du}{dt} = \dfrac{-2t}{10}$

$= \dfrac{-t}{5}$

e. We need to evaluate

$$\int_0^5 2te^{\frac{-t^2}{10}}\,dt \quad \textbf{(1 mark)}$$

$$= -10\left[e^{\frac{-t^2}{10}}\right]_0^5$$

$$= -10\left\{e^{-2.5} - e^{\circ}\right\}$$

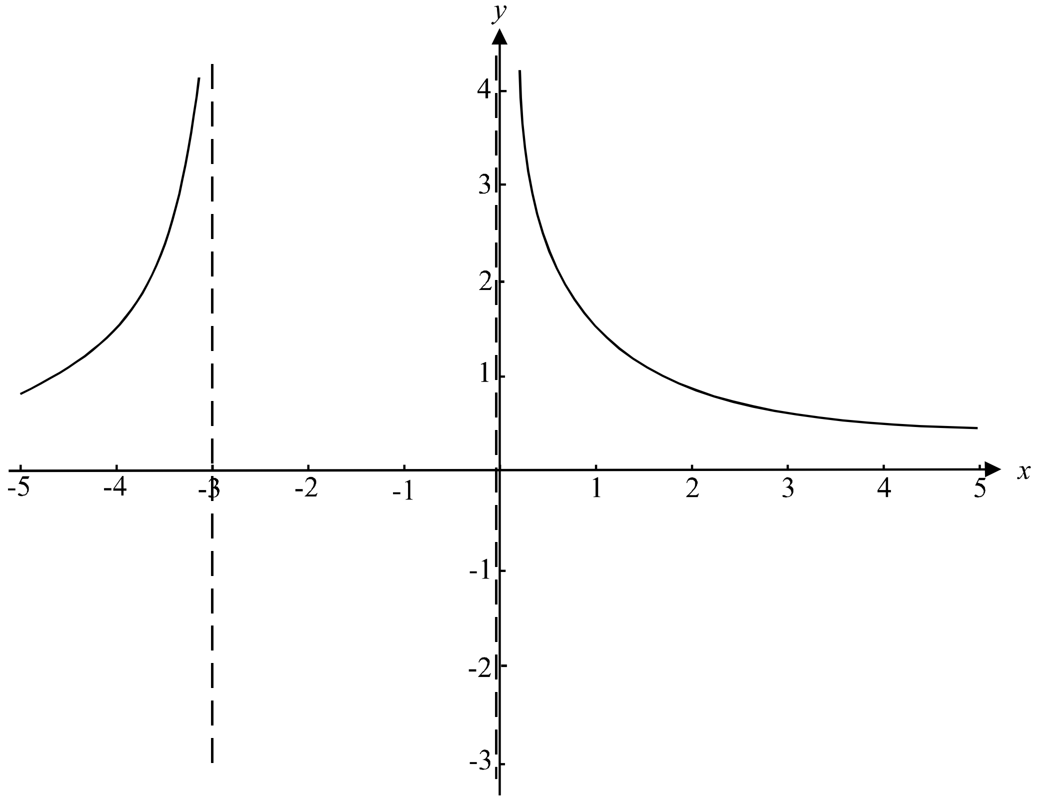
$= 9.2$ metres correct to 1 decimal place

(1 mark)

Total 10 marks

Question 3

a.



(1 mark) asymptotes
(1 mark) shape

b. $d = (-\infty, -3) \cup (0, \infty)$ **(1 mark)**

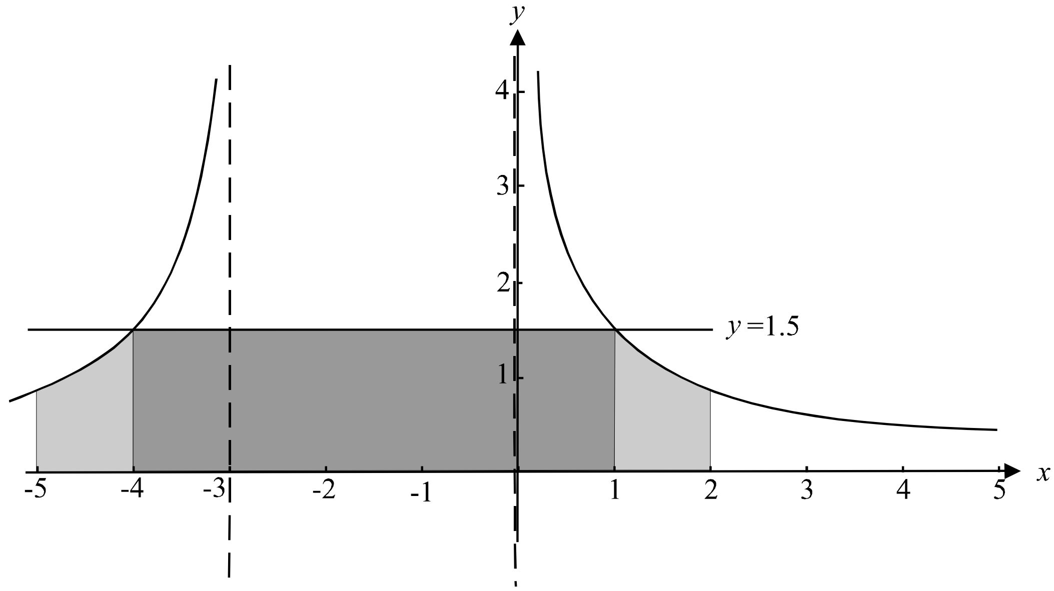
c.

If $f(x) = \dfrac{3}{\sqrt{x(x+3)}}$

$$
\begin{aligned}
\text{Area approximation} &= (3-1)f(2) + (5-3)f(4) \\
&= 2f(2) + 2f(4) \qquad \textbf{(1 mark)} \\
&= 1.897367 + 1.133893 \qquad \text{(correct to 6 decimal places)} \\
&= 3.031260 \\
&= 3.03 \qquad (\text{correct to 3 significant figures})
\end{aligned}
$$

(1mark)

d. The area required is shown in the diagram below broken into 3 sections.



To find the middle, darker area, we need to find the points of intersection between

$$y = \frac{3}{\sqrt{x(x+3)}} \text{ and } y = 1.5$$

So, $1.5 = \frac{3}{\sqrt{x(x+3)}}$

$$1.5\sqrt{x(x+3)} = 3$$
$$\sqrt{x(x+3)} = 2$$
$$x(x+3) = 4$$
$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$
$$x = -4 \text{ or } x = 1$$

So the middle section has an area of $1.5 \times 5 = 7.5$ square units. **(1 mark)**

Total area required $= \int_{-5}^{-4} \frac{3}{\sqrt{x(x+3)}}\,dx + \int_{1}^{2} \frac{3}{\sqrt{x(x+3)}}\,dx + 7.5$ **(1 mark)**

Using a calculator, we obtain

$= 1.177152 + 1.177152 + 7.500000$ (each to 6 decimal places)

$= 9.85$ square units (correct to 3 significant figures)

(1 mark)

e.

$$\text{Volume required} = \pi\int_1^5 y^2\,dx$$
$$= \pi\int_1^5 \frac{9}{x(x+3)}\,dx$$

(1 mark)

Let $\dfrac{9}{x(x+3)} \equiv \dfrac{A}{x} + \dfrac{B}{x+3}$

$$\equiv \frac{A(x+3)+Bx}{x(x+3)}$$

True iff $9 \equiv A(x+3)+Bx$

Put $x=-3$, so $9=-3B$ so $B=-3$

Put $x=0$, so, $9=3A$ so $A=3$

So, $\dfrac{9}{x(x+3)} \equiv \dfrac{3}{x} - \dfrac{3}{x+3}$

So, volume req'd $= \pi\displaystyle\int_1^5\left(\frac{3}{x}-\frac{3}{x+3}\right)dx$ **(1 mark)**

$= \pi\left[3\log_e x - 3\log_e(x+3)\right]_1^5$ **(1 mark)**

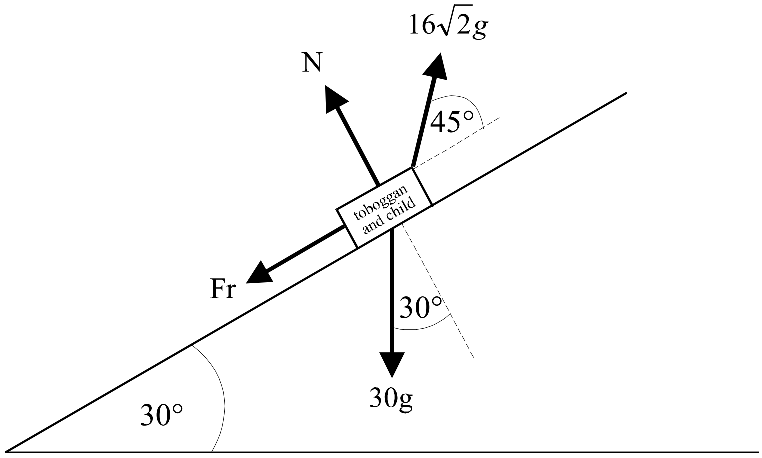
$$= 3\pi\{(\log_e 5 - \log_e 8) - (\log_e 1 - \log_e 4)\}$$
$$= 3\pi(\log_e 5 - \log_e 8 + \log_e 4)$$
$$= 3\pi\log_e\frac{5\times 4}{8}$$
$$= 3\pi\log_e\frac{5}{2} \text{ cubic units}$$

(1 mark)

Total 12 marks

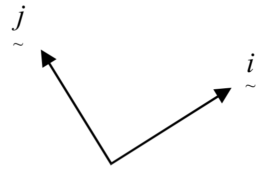
Question 4

a. i.



(2 marks)

ii. Resolve forces, given that $\underset{\sim}{i}$ represents a unit vector parallel with the slope and $\underset{\sim}{j}$ represents a unit vector perpendicular to the slope.



Now, $\underset{\sim}{R} = m\underset{\sim}{a}$ becomes

$$\left(16\sqrt{2}g\cos 45^\circ - 30g\sin 30^\circ - Fr\right)\underset{\sim}{i} + \left(N + 16\sqrt{2}g\sin 45^\circ - 30g\cos 30^\circ\right)\underset{\sim}{j} = 30 \times 0\,\underset{\sim}{i}$$

Note that since we have constant speed, $a = 0$.

(1 mark) for the left hand side of the equation
(1 mark) for the right hand side of the equation

So $\frac{16\sqrt{2}g}{\sqrt{2}} - 15g - Fr = 0$ and $N + \frac{16\sqrt{2}g}{\sqrt{2}} - \frac{30\sqrt{3}g}{2} = 0$

$Fr = g$ so $N = -16g + 15\sqrt{3}g$

$= g(15\sqrt{3} - 16)$

Now, $Fr = \mu N$

So, $\mu N = g$ **(1 mark)**

$$\mu = \frac{g}{g(15\sqrt{3} - 16)}$$

$$= \frac{1}{15\sqrt{3} - 16}$$

as required **(1 mark)**

b **i.** Now, $\underset{\sim}{R} = m\underset{\sim}{a}$

This time,

$$\left(18\sqrt{2}g\cos 45^\circ - 30g\sin 30^\circ - Fr\right)\underset{\sim}{i} + \left(N + 18\sqrt{2}g\sin 45^\circ - 30g\cos 30^\circ\right) = 30a\,\underset{\sim}{i}$$

(1 mark)

So $\frac{18\sqrt{2}g}{\sqrt{2}} - 15g - Fr = 30a$ and $N + 18g - 30g \times \frac{\sqrt{3}}{2} = 0$

$30a = 3g - \mu N$ $N = 15\sqrt{3}g - 18g$

$= 3g - \frac{1}{15\sqrt{3} - 16} \times g(15\sqrt{3} - 18)$ $= g(15\sqrt{3} - 18)$

$= 3g - g\left(\frac{15\sqrt{3} - 18}{15\sqrt{3} - 16}\right)$

$a = 0.72 \text{ m/s}^2$ correct to 2 decimal places

(1 mark)

ii. Since acceleration is constant, we have,

$$s = ut + \frac{1}{2}at^2$$

So, $s = 0 \times 20 + \frac{1}{2} \times 0.72 \times 20^2$

$= 144$ metres

(1 mark)

iii. Again, since acceleration is constant we have

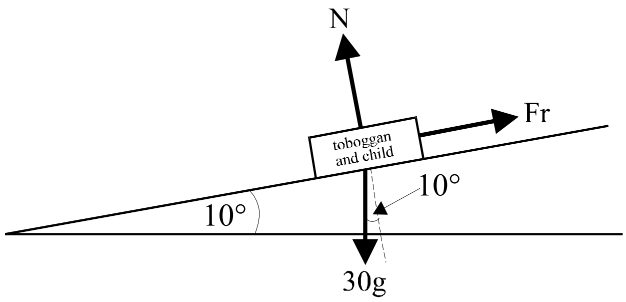
$v = u + at$

$v = 0 + 0.72 \times 20$

$= 14.4$ m/s

(1 mark)

c. **i.**



(2 marks)

ii. Resolving the forces, we have

$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

$$(Fr - 30g\sin 10^\circ)\,\underset{\sim}{i} + \left(N - 30g\cos 10^\circ\right)\underset{\sim}{j} = 0\,\underset{\sim}{i}$$

So, $Fr = 51.05$ and $N = 289.53$ **(1 mark)**

Now at the point of limiting equilibrium, when the toboggan is about to slide,

$Fr = \mu N$

Now, $\mu N = 289.53 \times 0.2$

$= 57.906$

and from above, $Fr = 51.05$.

So $Fr < \mu N$

So, the toboggan is not on the point of sliding down the slope.

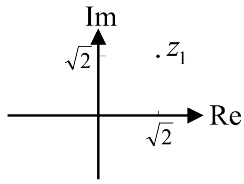
(1 mark)

Total 14 marks

Question 5

a. **i.**

$$z_1 = \sqrt{2} + \sqrt{2}i$$
$$r = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2}$$
$$= \sqrt{2+2}$$
$$= 2$$

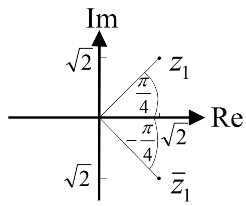


$$\theta = \tan^{-1}\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \tan^{-1} 1$$
$$= \frac{\pi}{4} \text{ since } z_1 \text{ is in the first quadrant.}$$

So $z_1 = 2\text{cis}\frac{\pi}{4}$

(1 mark)

ii. Method 1



$$z_1 = 2\text{cis}\frac{\pi}{4}$$

so $\bar{z}_1 = 2\text{cis}\left(\frac{-\pi}{4}\right)$ since

$\bar{z}_1$, is located by reflecting z_1 in the Real axis.

Method 2

$$z_1 = \sqrt{2} + \sqrt{2}i$$

So, $\bar{z}_1 = \sqrt{2} - \sqrt{2}i$

$$r = \sqrt{2+2}$$
$$= 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)$$
$$= \tan^{-1}(-1)$$
$$= \frac{-\pi}{4} \text{ since } \bar{z}_1 \text{ is in the } 4^{\text{th}} \text{ quadrant.}$$

So, $\bar{z}_1 = 2\text{cis}\left(\frac{-\pi}{4}\right)$

(1 mark)

iii. To show : $\frac{z_1}{z_2} = -i$

$$\text{Left side} = \frac{z_1}{z_2}$$
$$= \frac{2\text{cis}\frac{\pi}{4}}{2\text{cis}\left(\frac{3\pi}{4}\right)}$$
$$= \text{cis}\left(\frac{-\pi}{2}\right)$$

(1mark)

$$= \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)$$
$$= 0 + i \times -1$$
$$= -i$$

= right side Have shown.

(1 mark)

b. Firstly, express z_2 in the form $x+yi$.

$$z_2 = 2\text{cis}\left(\frac{3\pi}{4}\right)$$
$$= 2\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$$
$$= 2\left(-\frac{1}{\sqrt{2}}+i\times\frac{1}{\sqrt{2}}\right)$$
$$= -\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}}i$$
$$= -\sqrt{2}+\sqrt{2}i$$

(1 mark)

Now the equation is given by

$$a(z-z_1)(z-z_2), \quad a\in R$$
$$= a\left(z-\sqrt{2}-\sqrt{2}i\right)\left(z+\sqrt{2}-\sqrt{2}i\right) \quad \textbf{(1 mark)}$$
$$= a(z^2+\sqrt{2}z-\sqrt{2}iz-\sqrt{2}z-2+2i-\sqrt{2}iz-2i-2)$$
$$= a(z^2-2\sqrt{2}iz-4)$$

So z_1 and z_2 could be the roots of any quadratic equation of the form $a(z^2-2\sqrt{2}iz-4)$, $a\in R$.

(1 mark)

c. **i.** We know that $z_3 = cis\theta$

So, $(z_3)^n = (cis\theta)^n$

$= cis(n\theta)$ De Moivre's Theorem

(1 mark)

ii. To show: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

From **part c. i**., we have

$$(z_3)^n = \text{cis}(n\theta)$$

So, $(\text{cis}\theta)^n = \text{cis}(n\theta)$

or, $\text{cis}(n\theta) = (\text{cis}\theta)^n$

So, $\text{cis}(3\theta) = (\text{cis}\theta)^3$ **(1 mark)**

$$\cos(3\theta)+i\sin(3\theta) = (\cos\theta+i\sin\theta)^3$$
$$= \cos^3\theta+3\cos^2\theta\, i\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$$

(1 mark)

Equating imaginary parts, we have,

$\sin 3\theta = 3\cos^2\theta\sin\theta-\sin^3\theta$ as required. **(1 mark)**

Have shown.

Total 11 marks