Question 1

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a.

$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r} = \sin t\hat{i} + \cos 2t\hat{j}$ (1 mark)

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b.

$y = \cos 2t$

$y = 1 - 2\sin^2 t$

$y = 1 - 2x^2$

(1 mark)

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c.

$-1 \le \cos 2t \le 1$

$-1 \le y \le 1$

$-1 \le \sin t \le 1$

$-1 \le x \le 1$

Domain : $-1 \le x \le 1$ (1 mark)

Range : $-1 \le y \le 1$ (1 mark)

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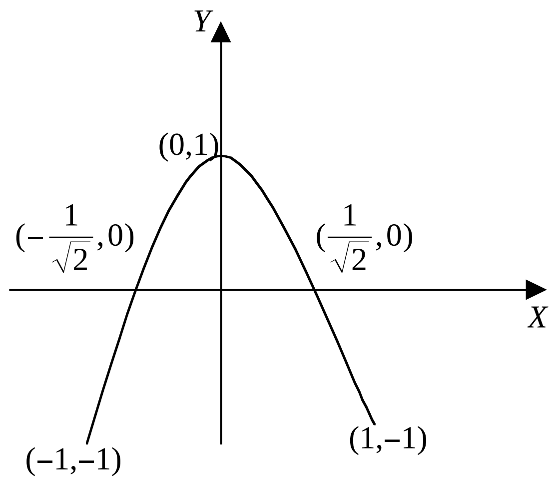
d.



When $x = -1, y = 1 - 2 = -1$

When $x = 1, y = 1 - 2 = -1$

When $y = 0, 1 - 2x^2 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

- shape (1 mark)
- *x* intercepts (1 mark)
- *y* intercepts (1 mark)
- end points (1 mark)

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Question 1 (continued)

e. $r = \sin\frac{\pi}{4}\hat{i} + \cos\frac{\pi}{2}\hat{j}$ $r = \frac{1}{\sqrt{2}}\hat{i}$ (1 mark)	**f.(i)** $\overrightarrow{OF} = 0.8\hat{i} - 0.2\hat{j}$ $\overrightarrow{OA} = \frac{1}{\sqrt{2}}\hat{i}$ $\overrightarrow{AF} = \overrightarrow{AO} + \overrightarrow{OF}$ $\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA}$ (1 mark) $\overrightarrow{AF} = 0.8\hat{i} - 0.2\hat{j} - \frac{1}{\sqrt{2}}\hat{i}$ $\overrightarrow{AF} = 0.09\hat{i} - 0.2\hat{j}$ (1 mark)
f.(ii) New path of travel is along the tangent to the curve. $\dot{r} = \cos t\hat{i} - 2\sin 2t\hat{j}$ (1 mark) Initial point along tangent is when $t = \frac{\pi}{4}$ $\therefore$ a vector $\overrightarrow{\text{AP}}$ which represents the butterfly's new path is $\overrightarrow{AP} = k\left(\frac{1}{\sqrt{2}}\hat{i} - 2\hat{j}\right)$ where k is a constant $\overrightarrow{AP} = k\left(\frac{\sqrt{2}}{2}\hat{i} - 2\hat{j}\right)$ $\overrightarrow{AP} = k(0.7\hat{i} - 2\hat{j})$ (1 mark)	**g.** No, because the position vectors $\overrightarrow{AF}$ and $\overrightarrow{AP}$ are not parallel since $0.09\hat{i} - 0.2\hat{j} \neq k(0.7\hat{i} + 2\hat{j})$ where k is a constant. (2 marks)

Question 2

a.

$g = \frac{k}{R^2}$

$k = gR^2$

(1 mark)

b.

$\vec{F} = m\vec{a}$

$\vec{F} = \frac{-mk}{x^2}$

(1 mark)

c.

$\vec{a} = \frac{-k}{x^2} = \frac{-gR^2}{x^2}$

$\frac{d}{dx}(\frac{1}{2}v^2) = \frac{-gR^2}{x^2}$ (1 mark)

$\frac{1}{2}v^2 = \int -gR^2x^{-2}dx$

$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$ (1 mark)

At the surface of the earth $x = R, v = u$ (1 mark)

$\Rightarrow \frac{1}{2}u^2 = \frac{gR^2}{R} + c$

$\Rightarrow c = \frac{1}{2}u^2 - gR$ (1 mark)

Hence, $\frac{1}{2}v^2 = \frac{gR^2}{x} + \frac{1}{2}u^2 - gR$

$v^2 = \frac{2gR^2}{x} + u^2 - 2gR$

$v^2 = u^2 - 2gR^2(\frac{1}{R} - \frac{1}{x})$ (1 mark)

d.

$v^2 = 2gR - 2gR^2(\frac{1}{R} - \frac{1}{x})$

$v^2 = 2gR - 2gR + \frac{2gR^2}{x}$

$v^2 = \frac{2gR^2}{x}$ (1 mark)

$\text{v} = \sqrt{2\text{gR}^2} \times x^{-\frac{1}{2}}$

$\frac{dx}{dt} = \sqrt{\frac{2gR^2}{x}}$ (1 mark)

$\frac{dt}{dx} = \frac{x^{\frac{1}{2}}}{\sqrt{2gR^2}}$ (1 mark)

Antidiff. both sides w.r.t.x

$t = \frac{2}{3\sqrt{2gR^2}}x^{\frac{3}{2}} + c$ (1 mark)

When $t = 0, x = R$

$0 = \frac{2}{3\sqrt{2gR^2}}R^{\frac{3}{2}} + c$

$\text{c} = -\frac{2R^{\frac{1}{2}}}{3\sqrt{2g}}$

$\Rightarrow t = \frac{2}{3\sqrt{2g} \times R} \times x^{\frac{3}{2}} - \frac{2\sqrt{R}}{3\sqrt{2g}}$

$\Rightarrow t = \frac{2}{3\sqrt{2g}}[\frac{x^{\frac{3}{2}}}{R} - \sqrt{R}]$ (1 mark)

Question 2 (continued)

e.
For body never to return to earth
$x \to \infty$

$v^2 = u^2 - 2gR^2(\frac{1}{R} - \frac{1}{x})$

As $x \to \infty$

$v^2 = u^2 - 2gR^2 \times \frac{1}{R}$

$v^2 = u^2 - 2gR$ (1 mark)

For body not to return to earth, $v > 0$

$\Rightarrow u^2 - 2gR > 0$

$\Rightarrow u^2 > 2gR$

$\Rightarrow u > \sqrt{2gR}$ (1 mark)

$u > \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$

$u > 11.2 \times 10^3$ m/sec

$u > 11.2$ km/sec

$u = 12$ km/sec (1 mark)

Question 3

a.

$\overrightarrow{AB} = \hat{i} + 2\hat{j}$

$\left|\overrightarrow{AB}\right| = \sqrt{5}$

$\overrightarrow{BC} = 2\hat{i} - \hat{j}$

$\left|\overrightarrow{BC}\right| = \sqrt{5}$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = (\hat{i} + 2\hat{j}) \bullet (2\hat{i} - \hat{j})$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = 2 - 2$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$

$\therefore AB$ is perpendicular to BC (1 mark)

$\overrightarrow{\mathrm{DC}} = \overrightarrow{AB}$

$\overrightarrow{DC} = \hat{i} + 2\hat{j}$ (1 mark)

$\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC}$

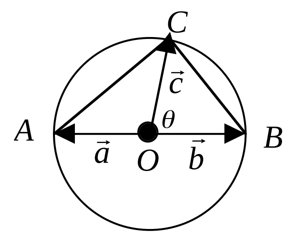
$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$

$\overrightarrow{OD} = \overrightarrow{OC} - \overrightarrow{DC}$

$\overrightarrow{OD} = 5\hat{i} + 2\hat{j} - \hat{i} - 2\hat{j}$

$\overrightarrow{OD} = 4\hat{i}$ (1 mark)

b.



(1 mark)

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$\overrightarrow{AC} = \vec{c} - \vec{a}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$\overrightarrow{BC} = \vec{c} - \vec{b}$

$\overrightarrow{AC} \bullet \overrightarrow{BC} = (\vec{c} - \vec{a}) \bullet (\vec{c} - \vec{b})$ (1 mark)

$\overrightarrow{AC} \bullet \overrightarrow{BC} = \vec{c} \bullet \vec{c} - \vec{c} \bullet \vec{b} - \vec{a} \bullet \vec{c} + \vec{a} \bullet \vec{b}$

$\overrightarrow{AC} \bullet \overrightarrow{BC} = c^2 - bc\cos\theta - ac\cos(180 - \theta)^0 + ab\cos 180^0$

$\overrightarrow{AC} \bullet \overrightarrow{BC} = c^2 - bc\cos\theta + ac\cos\theta - ab$

But, a = b = c (equal radii)

$\therefore \overrightarrow{\mathrm{AC}} \bullet \overrightarrow{BC} = c^2 - c^2\cos\theta + c^2\cos\theta - c^2$

$\therefore \overrightarrow{AC} \bullet \overrightarrow{BC} = 0$ (1 mark)

$\therefore \left|\overrightarrow{\mathrm{AC}}\right|\left|\overrightarrow{BC}\right|\cos\angle ACB = 0$

But $\left|\overrightarrow{AC}\right| \neq 0$ and $\left|\overrightarrow{\mathrm{BC}}\right| \neq 0$

$\therefore \cos\angle ACB = 0$

$\Rightarrow \angle ACB = 90^0$ (1 mark)

Question 4

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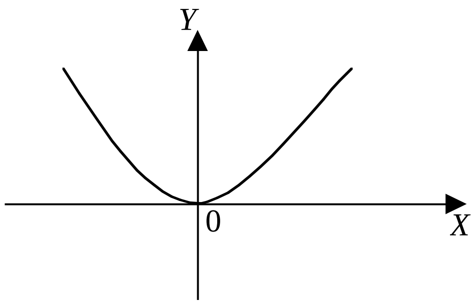
a.



(1 mark)

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b.

$V = \int_0^h \pi x^2 dy$

$V = \int_0^h 10\pi y dy$ (1 mark)

$V = 5\pi y^2]_0^h$

$V = 5\pi h^2$ (1 mark)

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c.

Surface area is a circle.

$S = \pi r^2$ where $r = x$

$\therefore S = \pi x^2$ (1 mark)

$\Rightarrow S = \pi \times 10y$

But $y = h$

$\therefore S = 10\pi h$

(1 mark)

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d.

$\frac{dh}{dt} = \frac{dh}{dv}\frac{dv}{dt}$ (1 mark)

$\frac{dv}{dh} = 10\pi h$ (1 mark)

$\frac{dh}{dt} = \frac{1}{10\pi h} \times -0.002 \times 10\pi h$ (1 mark)

$\frac{dh}{dt} = -0.002$ m/hr

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Question 4 (continued)

e.

$V = 5\pi h^2$

Initially, $V = 80\pi$

$\therefore 80\pi = 5\pi h^2$

$h^2 = 16$

$\therefore h = 4$ m initially, when $t = 0$ (1 mark)

$\frac{dh}{dt} = -0.002$

$h = \int -0.002 dt$

$h = -0.002t + c$ (1 mark)

$4 = c$

$\therefore h = -0.002t + 4$

Pool is empty when $h = 0$ (1 mark)

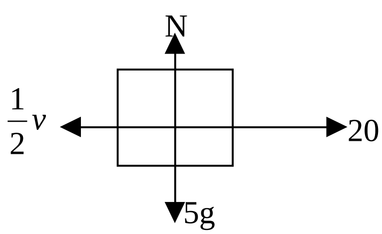
$0.002t = 4$

$t = \frac{4}{0.002}$

$t = 2000$ hrs. (1 mark)

Question 5

a.



(1 mark)

b.

$\vec{F}_R = 20 - \frac{1}{2}v$

(1 mark)

Question 5 (continued)

c.

$m\frac{dv}{dt} = 20 - \frac{1}{2}v$

$5\frac{dv}{dt} = 20 - \frac{1}{2}v$ (1 mark)

$\frac{dv}{dt} = 4 - \frac{v}{10}$

$\frac{dv}{dt} = \frac{40 - v}{10}$

Invert b.s.

$\frac{dt}{dv} = \frac{10}{40 - v}$

Antidiff b.s.w.r.t.v

$t = -10\int \frac{-dv}{40 - v}$ (1 mark)

$t = -10\log_e(40 - v) + c$ where c is a constant

When $t = 0, v = 0$

$0 = -10\log_e 40 + c$

$c = 10\log_e 40$ (1 mark)

$t = 10\log_e 40 - 10\log_e(40 - v)$

$t = 10\log_e \frac{40}{40 - v}$

$\frac{t}{10} = \log_e \frac{40}{40 - v}$

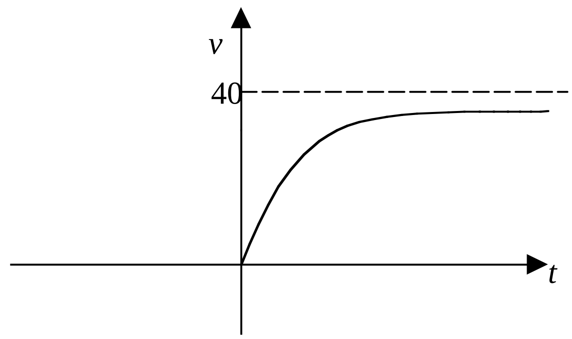
$e^{\frac{t}{10}} = \frac{40}{40 - v}$

$e^{-\frac{t}{10}} = \frac{40 - v}{40}$

$40e^{-\frac{t}{10}} = 40 - v$

$v = 40(1 - e^{-\frac{t}{10}})$ (1 mark)

d.



- shape (1 mark)
- asymptote (1 mark)

e.

As time $\to \infty$ the boat approaches a speed of 40 m/sec (1 mark)

f.

$\frac{dx}{dt} = 40 - 40e^{-\frac{t}{10}}$

$x = \int_0^{60} 40 - 40e^{-\frac{t}{10}}\,dt$ (1 mark)

$x = \left[40t + 400e^{-\frac{t}{10}}\right]_0^{60}$

$x = 2400 + 400e^{-6} - 400$

$x = 2000.99$ m

$x = 2.0$ km (1 mark)

END OF SUGGESTED SOLUTIONS
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