MAV Specialist Mathematics Examination 2 Solutions

Question 1 a. i.

$z^2 = (1-2i)^2$

$= 1^2 - 4i - 4$

$= -3 - 4i$ **[A]**

Question 1 a. ii.

$P(z)=0$

$z^2-5z+7+i=0$

$z=\dfrac{5\pm\sqrt{25-4(7+i)}}{2}$

$=\dfrac{5\pm\sqrt{-3-4i}}{2}$

Using the result from **a. i.** **[M]**

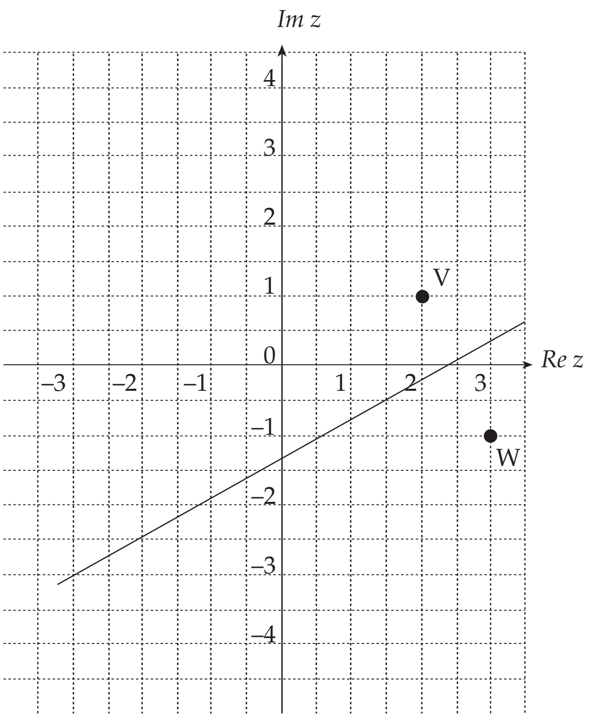
$z=\dfrac{5\pm(1-2i)}{2}$

z = $2 + i$ **[A]**

or z = $3 - i$ **[A]**

Question 1 b.

1 mark for both of their points correctly positioned. **[A]**



Question 1 c. i.

$S=\{z:|z-v|=|z-w|, z\in C\}$

$|x+iy-2-i|=|x+iy-3+i|$ **[M]**

$|x-2+iy-i|=|x-3+iy+i|$ **[M]**

$(x-2)^2+(y-1)^2=(x-3)^2+(y+1)^2$

$x^2-4x+4+y^2-2y+1=x^2-6x+9+y^2+2y+1$

$-4x-2y+5=-6x+2y+10$

$-4y=-2x+5$

$y=\dfrac{x}{2}-\dfrac{5}{4}$ **[A]**

Question 1 c. ii.

On graph, to gain mark should be a straight line with y-intercept (0, –1.25) and x-intercept (2.5, 0) **[A]**

Question 2 a. i.

$\dfrac{d}{dx}(x\text{Tan}^{-1}x) = \text{Tan}^{-1}x+\dfrac{x}{1+x^2}$ **[A]**

Question 2 a. ii.

$\text{Tan}^{-1}x = \dfrac{d}{dx}(x\text{Tan}^{-1}x) - \dfrac{x}{1+x^2}$

$\int \text{Tan}^{-1}x\,dx = \int\left(\dfrac{d}{dx}(x\text{Tan}^{-1}x) - \dfrac{x}{1+x^2}\right)dx$ **[M]**

$= x\text{Tan}^{-1}(x)-\frac{1}{2}\log_e\left(1+x^2\right)$ **[A]**

Question 2 b. i.

$v(t)=\dfrac{9\text{Tan}^{-1}\sqrt{t}}{\sqrt{t}}$

$v(3)=\dfrac{9\text{Tan}^{-1}\sqrt{3}}{\sqrt{3}}$

$=\dfrac{9\times\frac{\pi}{3}}{\sqrt{3}}=\sqrt{3}\pi$ **[A]**

Question 2 b. ii.

$$x=\int_0^3 \frac{9\text{Tan}^{-1}\sqrt{t}}{\sqrt{t}}\,dt$$

Let $u=\sqrt{t}$ **[M]**

When $t=3$, $u=\sqrt{3}$

When $t=0$, $u=0$

$$\frac{du}{dt}=\frac{1}{2\sqrt{t}}$$ **[M]**

$$x=\int_0^3 \frac{9\text{Tan}^{-1}u}{\sqrt{t}}\times\frac{2\sqrt{t}\,du}{dt}\,dt$$

$$=18\int_0^{\sqrt{3}}\text{Tan}^{-1}u\,du$$ **[A]**

$$=18\left[u\text{Tan}^{-1}u-\tfrac{1}{2}\log_e\left(1+u^2\right)\right]_0^{\sqrt{3}}$$

$$=18\left[\left(\sqrt{3}\text{Tan}^{-1}\sqrt{3}-\tfrac{1}{2}\log_e\left(1+\sqrt{3}^2\right)\right)-0\right]$$ **[M]**

$$=18\left[\sqrt{3}\times\frac{\pi}{3}-\log_e(2)\right]$$

$$=6\pi\sqrt{3}-18\log_e(2)$$ **[A]**

Question 3 a.

QR is tangential, therefore <OQR is a right angle **[M]**

$$\cos\theta=\frac{OQ}{OR}$$

$$=\frac{2}{x}$$

$$\theta=\text{Cos}^{-1}\left(\frac{2}{x}\right)$$

Answer given, so need to see working **[M]**

Question 3 b.

Arc Length

$C=r\theta^c$

$\theta^c=\pi-\alpha$

$=\pi-\text{Cos}^{-1}\left(\frac{2}{x}\right)$

r = 2

$C=2\left(\pi-\text{Cos}^{-1}\left(\frac{2}{x}\right)\right)$ Answer given, so need to see working **[M][A]**

Question 3 c.

$$C=2\left(\pi-\text{Cos}^{-1}\left(\frac{2}{x}\right)\right)$$

Let $u=\frac{2}{x}$, $\frac{du}{dx}=\frac{-2}{x^2}$ **[M]**

$$C=2\left(\pi-\text{Cos}^{-1}u\right)$$

$$\frac{dC}{du}=\frac{2}{\sqrt{1-u^2}}$$ **[M]**

$$\frac{dC}{dx}=\frac{dC}{du}\times\frac{du}{dx}$$

$$=\frac{2}{\sqrt{1-\frac{4}{x^2}}}\times\frac{-2}{x^2}$$

$$=\frac{2}{\frac{1}{x}\sqrt{x^2-4}}\times\frac{-2}{x^2}$$

$$=\frac{2}{\sqrt{x^2-4}}\times\frac{-2}{x}$$ **[A]**

$$=\frac{-4}{x\sqrt{x^2-4}}$$

Question 3 d.

Since <OQR is a right angle

$$QR^2=x^2-4$$

$$QR=\sqrt{x^2-4}$$

$$V=2\left(\pi-\text{Cos}^{-1}\left(\frac{2}{x}\right)\right)+\sqrt{x^2-4}$$ **[A]**

Question 3 e.

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ **[M]**

Given $\frac{dx}{dt} = -3$

$V = 2\left(\pi - \text{Cos}^{-1}\left(\frac{2}{x}\right)\right) + \sqrt{x^2 - 4}$

$\frac{dV}{dx} = \frac{-4}{x\sqrt{x^2-4}} + \frac{x}{\sqrt{x^2-4}}$ **[M]**

$\frac{dV}{dx} = \frac{-4+x^2}{x\sqrt{x^2-4}}$

$= \frac{\sqrt{x^2-4}}{x}$

$\frac{dV}{dt} = \frac{\sqrt{x^2-4}}{x} \times -3$ **[M]**

At $x = 6$

$\frac{dV}{dt} = \frac{\sqrt{36-4}}{6} \times -3$

$= -\frac{\sqrt{32}}{2}$

$= -2\sqrt{2}$ **[A]**

Question 4 a. i.

$\frac{dT}{dt} = k(T - S)$ **[A]**

Question 4 a. ii.

$\frac{dT}{dt} = k(T - S)$

$\frac{dt}{dT} = \frac{1}{k(T-S)}$ **[M]**

$t + c = \frac{1}{k}\log_e(T - S)$

$kt + kc = \log_e(T - S)$

$e^{kt+kc} = T - S$

Let $A = e^{kc}$

$T = Ae^{kt} + S$ **[A]**

Question 4 b. i.

At $t = 0$ $\quad A + S = 20$ **[A]**

At $t = 10$ $\quad Ae^{10k} + S = 15$ **[A]**

At $t = 20$ $\quad Ae^{20k} + S = 11$ **[A]**

Question 4 b. ii.

$S = 20 - A$ … (1)

$Ae^{10k} + 20 - A = 15$

$A\left(e^{10k} - 1\right) = -5$ … (2) **[M]**

$Ae^{20k} + 20 - A = 11$

$A\left(e^{20k} - 1\right) = -9$ … (3) **[M]**

(2) + (1) $\quad \frac{A\left(e^{20k}-1\right)}{A\left(e^{10k}-1\right)} = \frac{-9}{-5} \quad k \neq 0$ **[M]**

$5\left(e^{10k}\right)^2 - 5 = 9e^{10k} - 9$

$5\left(e^{10k}\right)^2 - 9e^{10k} + 4 = 0$

$\left(5e^{10k} - 4\right)\left(e^{10k} - 1\right) = 0$

OR

$\frac{A\left(e^{10k}-1\right)\left(e^{10k}+1\right)}{A\left(e^{10k}-1\right)} = \frac{-9}{-5}$ **[M]**

$e^{10k} + 1 = \frac{9}{5}$

$e^{10k} = \frac{4}{5}$

$\left(5e^{10k} - 4\right)$ OR $e^{10k} = \frac{4}{5}$

$k = \frac{\log_e\left(\frac{4}{5}\right)}{10}$ $\quad k = \frac{\log_e\left(\frac{4}{5}\right)}{10}$ **[M]**

Substituting $k = \frac{\log_e\left(\frac{4}{5}\right)}{10}$ into (2)

$A\left(e^{10\frac{\log_e\left(\frac{4}{5}\right)}{10}} - 1\right) = -5$

$\frac{-A}{5} = -5$

$A = 25$ **[M]**

Substituting $A = 25$ into into (1)

$S = -5$

$T = 25e^{\frac{t\log_e\left(\frac{4}{5}\right)}{10}} - 5$ **[M]**

Question 4 c. i.

$T = 25e^{\frac{t\log_e\left(\frac{4}{5}\right)}{10}} - 5$

As $t \to \infty$, $T \to -5$ °C

Outside Temperature is –5 °C **[C]**

Question 4 c. ii.

$0 = 25e^{\frac{t\log_e\left(\frac{4}{5}\right)}{10}} - 5$

$25e^{\frac{t\log_e\left(\frac{4}{5}\right)}{10}} = 5$ **[M]**

$e^{\frac{t\log_e\left(\frac{4}{5}\right)}{10}} = 0.2$

$\frac{t\log_e\left(\frac{4}{5}\right)}{10} = \log_e(0.2)$

t = 72 minutes 8 seconds (to the nearest second) **[C]**

Question 5 a. i.

Alternative 1

$x = 3\cos 2t$ $\qquad$ $y = \sin 4t$

$\frac{x}{3} = \cos 2t$ $\qquad$ $y = 2\sin 2t\cos 2t$ **[M]**

$x^2 = 9\cos^2 2t$

$\frac{x^2}{9} = \cos^2 2t$

$= 1 - \sin^2 2t$

$\sin^2 2t = 1 - \frac{x^2}{9}$ **[M]**

$y = 2\times\sqrt{1-\frac{x^2}{9}}\times\frac{x}{3}$

$= \frac{2x\sqrt{9-x^2}}{9}$

$y^2 = \frac{4x^2\left(9-x^2\right)}{81}$ **[M]**

Alternative 2

$x = 3\cos 2t$

$y^2 = \frac{4x^2\left(9-x^2\right)}{81}$

$RHS = \frac{4x^2\left(9-x^2\right)^2}{81}$

$= \frac{4\times 9\cos^2 2t\times\left(9-9\cos^2 2t\right)}{81}$ **[M]**

$= \frac{4\times 9\cos^2 2t\times 9\left(1-\cos^2 2t\right)}{81}$

$= 4\cos^2 2t\sin^2 2t$

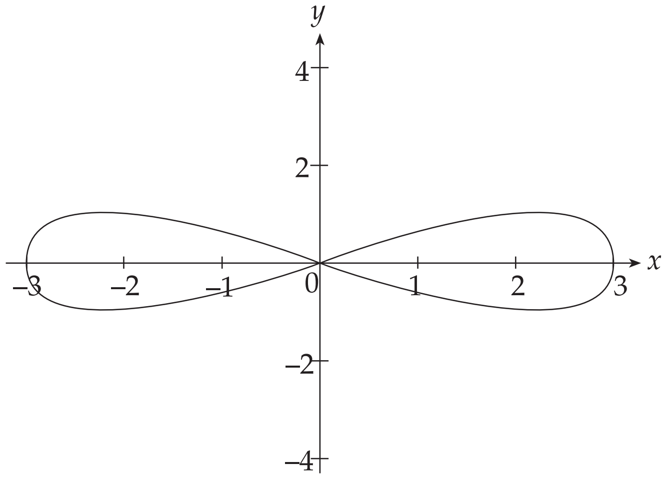
$= \left(2\cos 2t\sin 2t\right)^2$

$= \sin^2 4t$ **[M]**

$LHS = RHS$

$\therefore y^2 = \frac{4x^2\left(9-x^2\right)}{81}$ **[M]**

Question 5 a. ii.



2 marks for complete graph showing both positive and negative components of the graph, correct axial intercepts and range.

1 mark for half the graph, with correct axial intercepts and range.

Question 5 b.

$\underset{\sim}{r}(t) = 3\cos(2t)\,\underset{\sim}{i} + \sin(4t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = -6\sin(2t)\,\underset{\sim}{i} + 4\cos(4t)\,\underset{\sim}{j}$ **[M]**

$\ddot{\underset{\sim}{r}}(t) = -12\cos(2t)\,\underset{\sim}{i} - 16\sin(4t)\,\underset{\sim}{j}$ **[A]**

Question 5 c. i.

$\underset{\sim}{r}(t)\cdot\ddot{\underset{\sim}{r}}(t) = -36\cos^2 2t - 16\sin^2 4t$

$= -18\left(2\cos^2 2t\right) - 16\sin^2 4t$ **[M]**

$= -18(\cos 4t + 1) - 16\left(1 - \cos^2 4t\right)$ **[M]**

$= -18(U + 1) - 16(1 - U^2)$ **[M]**

$= 2(8U^2 - 9U - 17)$ **[A]**

Question 5 c. ii.

$\underset{\sim}{r}(t)\cdot\ddot{\underset{\sim}{r}}(t) = 2(8U^2 - 9U - 17)$

$= 2(8U - 17)(U + 1)$ **[M]**

$\underset{\sim}{r}(t)\cdot\ddot{\underset{\sim}{r}}(t) = 0$ when

$U = \frac{17}{8}$

$U = -1$

$U = \cos 4t$ **[M]**

Disregard $U = \frac{17}{8}$ because it is greater 1

$\cos 4t = -1$

$4t = \pi + 2n\pi$

$t = \frac{\pi + 2n\pi}{4}$ **[M]**

n is a positive integer or zero

Question 5 c. iii.

$\ddot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right) = -12\cos\left(\frac{2\pi}{4}\right)\underset{\sim}{i} - 16\sin\left(\frac{4\pi}{4}\right)\underset{\sim}{j}$

$= -12\cos\left(\frac{\pi}{2}\right)\underset{\sim}{i} - 16\sin(\pi)\,\underset{\sim}{j}$

$= 0\,\underset{\sim}{i} - 0\,\underset{\sim}{j} = \underset{\sim}{0}$ **[A]**

Question 5 d.

The position and acceleration vector will be perpendicular if $\underset{\sim}{r}(t) \cdot \ddot{\underset{\sim}{r}}(t) = 0$ given that neither

$\underset{\sim}{r}(t) = \underset{\sim}{0}$ or $\ddot{\underset{\sim}{r}}(t) = \underset{\sim}{0}$. **[M]**

$\underset{\sim}{r}(t) \cdot \ddot{\underset{\sim}{r}}(t) = 0$

When $t = \dfrac{\pi + 2n\pi}{4}$

$$\ddot{\underset{\sim}{r}}\left(\frac{(\pi + 2n\pi)}{4}\right)$$

$$= -12\cos\left(2\frac{(\pi + 2n\pi)}{4}\right)\underset{\sim}{i} - 16\sin\left(4\frac{(\pi + 2n\pi)}{4}\right)\underset{\sim}{j}$$

$$= -12\cos\left(\frac{(\pi + 2n\pi)}{2}\right)\underset{\sim}{i} - 16\sin(\pi + 2n\pi)\underset{\sim}{j}$$ **[M]**

$= 0\underset{\sim}{i} - 0\underset{\sim}{j} = \underset{\sim}{0}$, for all n when n is a positive integer.

This implies that $\ddot{\underset{\sim}{r}}(t) = \underset{\sim}{0}$ every time that $\underset{\sim}{r}(t) \cdot \ddot{\underset{\sim}{r}}(t) = 0$, so the position and acceleration vector are never perpendicular, and so the train will not derail. **[A]**