Question 1

a. Let $z = 1 - 2i$

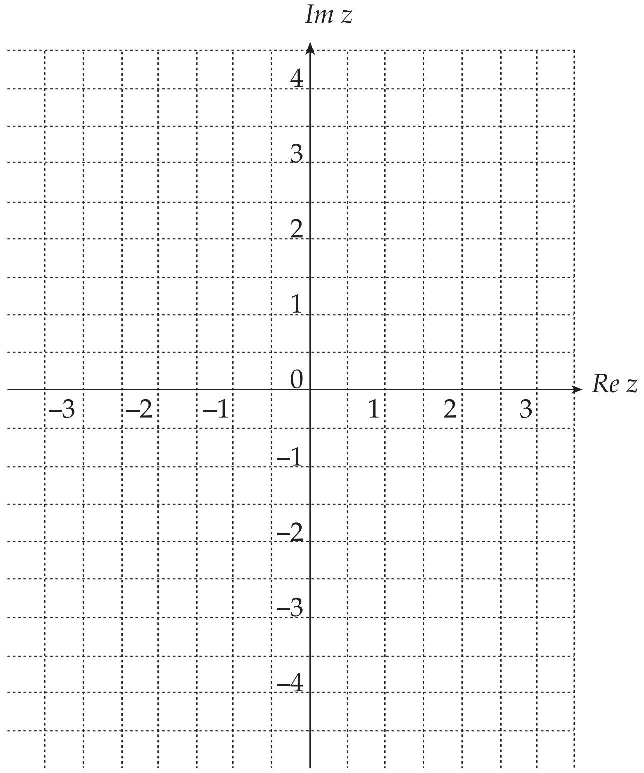
i. Show that $z^2 = -3 - 4i$.

1 mark

ii. Hence, solve the equation $P(z) = 0$, where $P(z) = z^2 - 5z + (7 + i)$.

3 marks

b. Let v and w represent the solutions to $P(z)$. Plot the points v and w on the Argand diagram below, clearly labelling each point.



1 mark

c. Let S be defined by $S = \{z : |z - v| = |z - w|, z \in C\}$

i. Describe S in Cartesian form.

3 marks

ii. Sketch S on the Argand diagram in **b.**

1 mark

Total 9 marks

Question 2

a. **i.** Find $\frac{d}{dx}(x\text{Tan}^{-1}x)$.

1 mark

ii. Hence, show that an antiderivative of $\text{Tan}^{-1}(x)$ is $x\text{Tan}^{-1}(x) - \frac{1}{2}\log_e\left(1 + x^2\right)$.

2 marks

b. A particle moves in a straight line, with a velocity of $\left(\frac{9\text{Tan}^{-1}\sqrt{t}}{\sqrt{t}}\right)$m/s where t>0.

i. Find the exact velocity of the particle at $t = 3$ seconds.

1 mark

ii. Calculate the exact distance the particle moves in the first 3 seconds.

5 marks

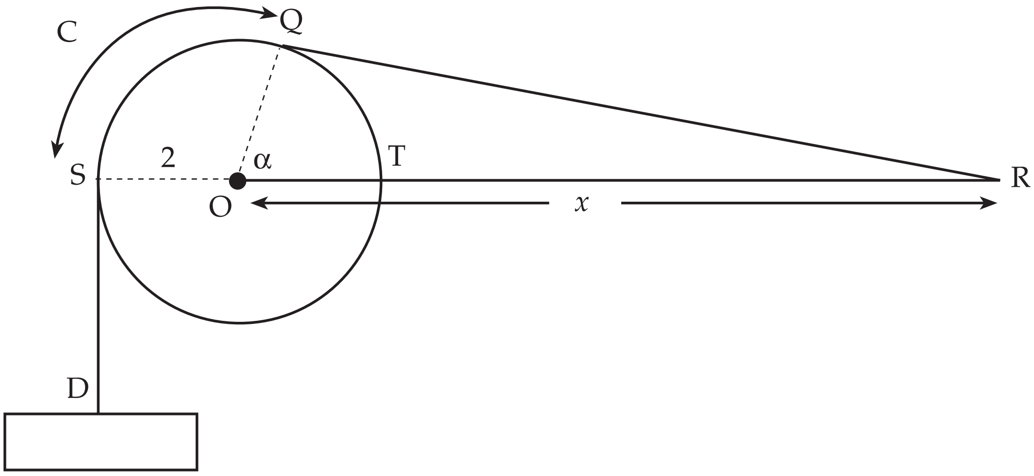
Total 9 marks

Question 3

A crane used for transporting shipping containers uses a hydraulic ram to shift a large wheel in the horizontal plane. This is represented in the diagram below. The length of the ram is x metres. It is attached to the centre of the wheel, O, and a fixed point R. A cable is attached to the point R and makes contact with the wheel between the points Q and S. The length of cable in contact with the wheel is C, which is the arc length between Q and S.

The cable is tangential to the wheel at the points Q and S. ST is a diameter of the wheel.

The radius of the wheel is 2 metres.



Let <QOR = α radians

a. Show that $\cos\alpha = \dfrac{2}{x}$, and express α in terms of x.

2 marks

b. Show that the arc length $C = 2\left(\pi - \text{Cos}^{-1}\left(\frac{2}{x}\right)\right)$.

2 marks

c. Find $\frac{dC}{dx}$.

3 marks

Let V represent the cable length from the points R to S, $V = C + \text{QR}$.

d. Express V in terms of x.

1 mark

The crane begins to lower the container by using the ram to move the wheel closer to R at a constant rate of 3 metres per second.

e. Use calculus to find the rate at which V changes with respect to time when $x = 6$ metres.

4 marks

Total 12 marks

Question 4

When a body is placed into a cooler environment, the rate of the cooling of the body is proportional to the difference between its temperature (T) and that of its surroundings (S). We will assume that the temperature of its surroundings is constant.

a. i. Write down the differential equation which describes this process.

1 mark

a. ii. Solve the differential equation, and express the answer in the form $T = Ae^{Bt} + C$.

2 marks

During her recent holiday to the Victorian Ski Fields Jill spent most evenings at her favourite Ski Lodge. One evening while it was snowing, a thermometer inside the Lodge indicated that the room temperature was 20 °C. Jill moved this thermometer outdoors. In ten minutes the thermometer reading was 15 °C. Ten minutes later the temperature indicated on the thermometer had dropped to 11 °C.

b. **i.** Use the equation found in **a. ii.,** and the above information for the temperature (T °C) and the time (t minutes) to set up simultaneous equations for the arbitrary constants.

3 marks

b. **ii.** Solve these simultaneous equations to show that $A = 25$, $k = \dfrac{\log_e \frac{4}{5}}{10}$ and $S = -5$. Specify the equation which gives the temperature (T) reading on the thermometer in terms of t, where t is measured in minutes after the thermometer is placed outside.

7 marks

c. **i.** What is the outdoor temperature?

1 mark

c. **ii.** When will the thermometer have a reading of reading 0 °C? Give your answer to the nearest second.

2 marks

Total 16 marks

Question 5

Steve constructs model railways. Recently he constructed a small model railway for his son Daly. While travelling on Daly's model railway track, a train represented as a point, has the position vector

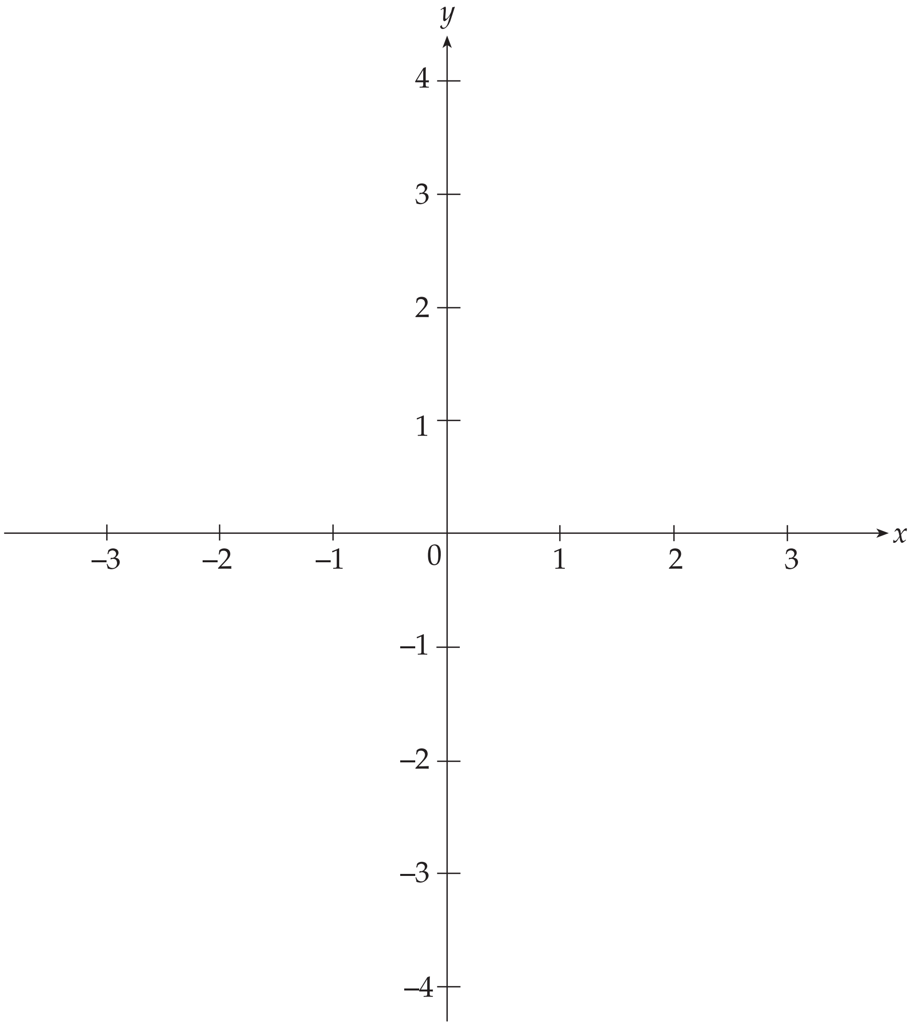
$$\underset{\sim}{r}(t) = 3\cos(2t)\,\underset{\sim}{i} + \sin(4t)\,\underset{\sim}{j}$$

where $\underset{\sim}{r}(t)$ is the displacement, in metres, the train has moved from its starting point in t seconds.

a. **i.** Show that $y^2 = \dfrac{4x^2(9 - x^2)}{81}$ describes the motion of the train in Cartesian form.

3 marks

a. **ii.** Sketch the path of the train on the axes provided.



2 marks

b. Find the acceleration $\ddot{\underset{\sim}{r}}(t)$ of the train as it travels around the track.

2 marks

c. **i.** Express $\underset{\sim}{r}(t) \cdot \ddot{\underset{\sim}{r}}(t)$ in terms of U where $U = \cos 4t$.

4 marks

c. **ii.** Hence, show that the exact values for which $\underset{\sim}{r}(t) \cdot \ddot{\underset{\sim}{r}}(t) = 0$ occur only at $t = \dfrac{\pi + 2n\pi}{4}$ where n is a positive integer or zero.

3 marks

c. **iii.** Evaluate $\ddot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)$.

1 mark

Steve has discovered that when he designs his tracks, the train will derail if the position vector of the train runs perpendicular to its acceleration vector.

d. Will the train derail on the track Steve has built for his son Daly?

3 marks

Total 18 marks