2002 Specialist Mathematics
Written Examination 1 (facts, skills and applications)
Suggested answers and solutions

Part I (Multiple-choice) Answers

1. E	2. C	3. D	4. A	5. E
6. A	7. B	8. D	9. B	10. C
11. A	12. E	13. C	14. A	15. B
16. E	17. D	18. B	19. D	20. A
21. B	22. A	23. C	24. C	25. B
26. D	27. E	28. C	29. C	30. D

1. $\angle QRP = 110°$ [E]

Using the Sine Rule for the triangle QRP.

$\frac{x}{\sin(110°)} = \frac{6}{\sin(40°)}$

2. $y = \frac{x^2+4}{x^2}$ [C]

$= 1 + \frac{4}{x^2}$

This graph has the same vertical asymptote as $y = \frac{4}{x^2}$, that is the y-axis, $x = 0$.

The line $y = 1$ will be the horizontal asymptote (instead of $y = 0$ for $y = \frac{4}{x^2}$).

There are no turning points.

3. $4x^2 - y^2 + 1 = 0$ [D]

$y^2 - 4x^2 = 1$

y-intercepts given by $x = 0$

$y^2 = 1$

$y = \pm 1$

This rules out A (only one y-intercept), B, C.

$y^2 = 4x^2 + 1$

$= 4\left(x^2 + \frac{1}{4}\right)$

$y^2 = 4x^2\left(1 + \frac{1}{4x^2}\right)$

As x becomes bigger $y^2 \to 4x^2$

Not asymptotic behaviour for E, so D is the correct response.

4. $f(x) = \sec^2(x) + \operatorname{cosec}^2(x)$ [A]

$= \frac{1}{\cos^2(x)} + \frac{1}{\sin^2(x)}$

$= \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)\sin^2(x)}$

$= \sec^2(x)\operatorname{cosec}^2(x)$

The two functions are identical and so will be identical over the same domain.

5. $\operatorname{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)\operatorname{Cos}^{-1}\left(\frac{1}{2}\right)$ [E]

$\operatorname{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\operatorname{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

The product of the two must be $\frac{\pi^2}{12}$

6. $z = x + iy$ [A]

The only pure imaginary number will be

$z - \bar{z} = (x + iy) - (x - iy)$

$= 2iy$

7. $z = -2i$ [B]

$z^2 = -4$

$|z^2| = 4$

$\operatorname{Arg}(z^2) = \pi \quad$ since $-\pi < \operatorname{Arg}(z^2) \le \pi$

8. If $z - 2 + i$ is a factor of $P(z)$ [D]

then $P(2 - i) = 0$. Then, since $P(z)$ has all real coefficients $P(2 + i)$ must be 0 also.

In other words $z - 2 - i$ is a factor of $P(z)$.

The other factor must be of the form $z - a$ where a is a real number.

9. The shaded region has both boundaries included. So $z = 1$ must be allowed as is $z = 3$. [B]

This prevents E from being correct.

The expression $z\bar{z} = (x + iy)(x - iy)$

$= x^2 + y^2$

The circle centre (0, 0) and radius is given as $x^2 + y^2 = 9$.

For this reason, D and A are incorrect, as is C.

The correct result is: B

10. $\frac{3x+4}{(x-4)^2}$ can be expressed as partial fractions in the form: [C]

$\frac{3x+4}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Note: the following is not necessary

$3x + 4 = A(x - 4) + B$

Let $x = 4$

$16 = B$

$3x + 4 = A(x - 4) + 16$

let $x = 0$

$4 = -4A + 16$

$A = 3$

$\frac{3x+4}{(x-4)^2} = \frac{3}{x-4} + \frac{16}{(x-4)^2}$

11. $\frac{d}{dx}\left(x\operatorname{Cos}^{-1}(x)\right) = \operatorname{Cos}^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$ [A]

$\operatorname{Cos}^{-1}(x) = \frac{d}{dx}\left(x\operatorname{Cos}^{-1}(x)\right) + \frac{x}{\sqrt{1-x^2}}$

$\int \operatorname{Cos}^{-1}(x)dx = x\operatorname{Cos}^{-1}(x) + \int \frac{x}{\sqrt{1-x^2}}dx$

12. $\int \frac{e^{2x}}{2e^{2x}-1}dx$ where $2e^{2x} - 1 > 0$ [E]

and $e^{2x} > \frac{1}{2}$

let $u = e^{2x}$, $\frac{du}{dx} = 2e^{2x}$

$\int \frac{\frac{1}{2}\frac{du}{dx}}{2u-1}.dx$

$= \frac{1}{4}\int \frac{2}{2x-1}du$

$\frac{1}{2}\int \frac{du}{2u-1}$

$\frac{1}{4}\log_e(2u-1)$

$\frac{1}{4}\log_e(2e^{2x}-1)$

13. $\int_0^4 (2x+3)\sqrt{2x+1}\,dx$ [C]

let $u = 2x + 1$

when $x = 0$, $u = 1$

$x = 4$, $u = 9$

$2x + 3 = u + 2$

$\frac{du}{dx} = 2 \quad dx = \frac{du}{2}$

Then the original integral can be transformed to:

$\int_1^9 (u+2)\sqrt{u}.\boxed{\frac{dx}{du}.du}$

equivalent to dx

$\int_1^9 (u+2)\sqrt{u}.\frac{du}{2}$

$\frac{1}{2}\int_1^9 (u+2)\sqrt{u}.du$

14. Using graphics calculator: $y_1 = \dfrac{x^2 - 2x}{2\cos(x)}$ **[A]**

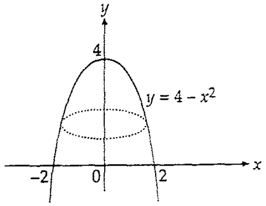
Set window between $x = -1$ to $x = 1$

Shows the graph. Then go to Calculate

$\int f(x)dx$

Enter $x = 0$ (lower limit) and
$x = 1$ (upper limit).
Check that MODE is in radians.
Value as given on calculator is
-0.43 (approx).

15. **[B]**



$\delta V \approx \pi x^2 \delta y$

$\dfrac{\delta V}{\delta y} \approx \pi x^2$

$V = \int_0^4 \pi x^2 dy$

$= \pi \int_0^4 (4 - y)dy$

$= \pi \left[4y - \dfrac{y^2}{2} \right]_0^4$

$= \pi[16 - 8]$

$= 8\pi$

16. $y = e^{kx}$ **[E]**

$\dfrac{d^2y}{dx^2} = k^2 e^{kx}, \quad \dfrac{dy}{dx} = ke^{kx}$

$\dfrac{d^2y}{dx^2} = 5\dfrac{dy}{dx} - 6y$

$k^2 e^{kx} = 5ke^{kx} - 6e^{kx}$

$k^2 = 5k - 6$

$k^2 - 5k + 6 = 0$

$(k-3)(k-2) = 0$

$k = 3$ or $k = 2$

17. $V = \dfrac{4}{3}\pi r^3$ **[D]**

$\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$

$\dfrac{dV}{dt} = \dfrac{d}{dr}\left(\dfrac{4}{3}\pi r^3\right) \cdot \dfrac{dr}{dt}$

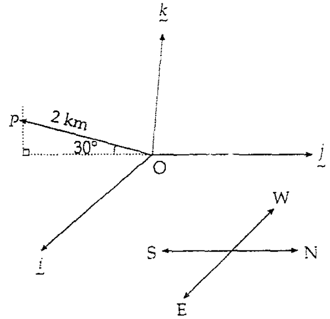
$= 4\pi r^2 \cdot \dfrac{dr}{dt}$

We know $\dfrac{dr}{dt} = 3$, so when $r = 8$

$\dfrac{dV}{dt} = 4\pi(64)(3)$

$= 768\pi$

18. **[B]**



Since the bushwalker travels due South, there will be no $\underset{\sim}{i}$ component in the position vector.

$\overrightarrow{OP} = 2\cos(30°)(-\underset{\sim}{j}) + 2\sin(30°)\underset{\sim}{k}$

$= -\sqrt{3}\,\underset{\sim}{j} + \underset{\sim}{k}$

19. $\underset{\sim}{P} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$ **[D]**

If there is a vector $\underset{\sim}{Q}$ perpendicular to $\underset{\sim}{P}$

then $\underset{\sim}{P} \cdot \underset{\sim}{Q} = 0$

Such a vector could be, from those limited in the question,

$-2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

or $\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

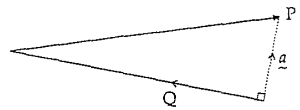
but NOT $\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

The first two vectors have magnitude 3.
So a vector with magnitude 4 could be

$\dfrac{4}{3}\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

Or $\dfrac{4}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

20. **[A]**



$\underset{\sim}{P} = -\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{Q} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

Vector resolute of $\underset{\sim}{P}$ in the direction oı $\underset{\sim}{Q}$ is in fact $-\underset{\sim}{Q}$.

Vector resolute of $\underset{\sim}{P}$ perpendicular to $\underset{\sim}{Q}$ is such that:

$-\underset{\sim}{Q} + \underset{\sim}{a} = \underset{\sim}{P}$

$-\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k} + \underset{\sim}{a} = -\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{a} = \underset{\sim}{j} + \underset{\sim}{k}$

21. If P, Q and S lie on a straight line, then $\underset{\sim}{PQ}$ can be expressed as $k\,\underset{\sim}{QS}$ **[B]**

when $k \in R$.

$\underset{\sim}{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$

$\underset{\sim}{QS} = \underset{\sim}{s} - \underset{\sim}{q}$

$\left(\underset{\sim}{q} - \underset{\sim}{p}\right) = k\left(\underset{\sim}{s} - \underset{\sim}{q}\right)$

22. $\underset{\sim}{v}(t) = \sin(t-1)\underset{\sim}{i} + 5\underset{\sim}{j} - 3e^{(1-t)}\underset{\sim}{k}$ **[A]**

$\dfrac{d\underset{\sim}{v}}{dt} = \cos(t-1)\underset{\sim}{i} + 3e^{(1-t)}\underset{\sim}{k}$

When $t = 1$

$\dfrac{d\underset{\sim}{v}}{dt} = \cos(0)\underset{\sim}{i} + 3e^0\underset{\sim}{k}$

$= \underset{\sim}{i} + 3\underset{\sim}{k}$

23. $\underset{\sim}{r}(t) = (3t-6)\underset{\sim}{i} - (t^2 - 6t - 16)\underset{\sim}{j}$ **[C]**

Particle reaches maximum height when the $\underset{\sim}{j}$ component is maximum, that is

when $t^2 - 6t - 16$ is maximum.

Differentiating with respect to t,

$-2t + 6 = 0$

$t = 3$

Alternatively

$\underset{\sim}{v}(t) = 3\underset{\sim}{i} - (2t - 6)\underset{\sim}{j}$

maximum height will be reached when the vertical component of velocity is zero.

$\Rightarrow 2t - 6 = 0$

$t = 3$

24. $\underset{\sim}{a}(t) = 2\sin(t)\underset{\sim}{i}$ [C]

$\underset{\sim}{v}(t) = -2\cos(t)\underset{\sim}{i} + \underset{\sim}{c}$

$2\underset{\sim}{i} + 2\underset{\sim}{j} = -2\cos(\pi)\underset{\sim}{i} + \underset{\sim}{c}$

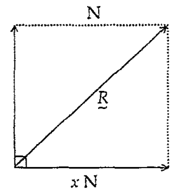
$= +2\underset{\sim}{i} + \underset{\sim}{c}$

$\underset{\sim}{c} = 2\underset{\sim}{j}$

$\underset{\sim}{v}(t) = -2\cos(t)\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{v}(0) = -2\underset{\sim}{i} + 2\underset{\sim}{j}$

25. [B]



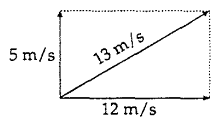
$\text{Mag}(\underset{\sim}{R}) = \sqrt{x^2 + 25}$

$\sqrt{x^2 + 25} = 8$

$x^2 = 64 - 25$

$x = \sqrt{39}$

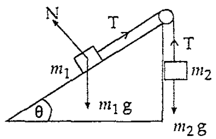
26. [D]



The resultant velocity is 13m/s as shown above.
The magnitude in kg m/s of the momentum will be 2 x 13 = 26.

27. Note that the particle is stationary implying that the resultant forces parallel to the plane are zero (as are the resultant forces perpendicular to the plane). [E]
All diagrams are comparable with the last condition.
The only diagram which is incompatible with the first condition is E.

28. [C]



Since the system is in equilibrium,

$T = m_2 g$

$m_1 g\cos(90° - \theta) = T$

$m_1 g\sin\theta = T = m_2 g$

$\frac{m_1}{m_2} = \frac{1}{\sin\theta}$

29. The ball is dropped from rest, so at $t = 0$, $v = 0$. [C]
This rules out A and B.
When it hits the floor, the direction of the velocity changes instantaneously.
This rules out E.
Since it rises to only half its original height, the magnitude of the rebound velocity has to be less than the magnitude of the velocity on impact with the floor.
This rules out D.

30. If P litres is the amount of oxygen in the cylinder at a given time, then we know that 10 litres is the amount of oxygen coming in per minute. 10 litres of mixed oxygen and air are being withdrawn per minute. So the amount of oxygen being withdrawn per minute will be $10 \times \frac{P}{150} = \frac{P}{15}$. [D]

Net change in oxygen will be $\frac{dP}{dt} = 10 - \frac{P}{15}$

Initial conditions is that for $t = 0$,
$P = 20\%$ of $150 = 30$

Part 2: Short-answers

1. a. Centre of ellipse is at (–1, 2)

$\frac{(x+1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$

When $x = -1$, $y = 5$

$0 + \frac{(5-2)^2}{b^2} = 1$

$b^2 = 9$

When $y = 2$, $x = 0$

$\frac{(0+1)^2}{a^2} + 0 = 1$

$a^2 = 1$

Equation of ellipse is

$\frac{(x+1)^2}{1} + \frac{(y-2)^2}{9} = 1$

b. A suitable transformation on the circle $x^2 + y^2 = 1$ to produce this ellipse would be a sketch or dilation of a factor of 3 in the vertical (y) direction and then a translation of (–1, 2) away from (0, 0).

2. From the diagram the truck has travelled 17 × 60 m exactly before it passes the car which has just then pulled to a stop.
The distance the car has travelled is calculated from the area of the trapezium:

$24 \times T + \frac{24 \times (60 - T)}{2}$

$24T + 12 \times 60 - 12T$

$12(T + 60)$

$\therefore 12(T + 60) = 17 \times 60$

$T + 60 = 17 \times 5$

$T + 60 = 85$

$T = 25$

Answer 25 seconds

3. $\int \frac{\sin\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}dx$

let $u = \cos\left(\frac{x}{2}\right)$

$\frac{du}{dx} = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$

$\sin\left(\frac{x}{2}\right) = -2\frac{du}{dx}$

$\int \frac{-2\frac{du}{dx}}{u^2}dx$

$= \int \frac{-2}{u^2}du$

$= 2u^{-1} + c$

A suitable antiderivative will be $\frac{2}{\cos\left(\frac{x}{2}\right)} + c$

4. From the information given about the gradient function $y = f'(x)$, the original function will have stationary points at $x = 1$ and $x = 3$.
At $x = 1$, $f'(x)$ goes from positive to positive, so at $x = -1$ there will be a stationary point of inflexion.
At $x = 3$, $f'(x)$ goes from positive to negative, so at $x = 3$ there will be a maximum point.
(Note: from the above information, we do not know where the graph $y = f'(x)$ cuts the y-axis or x-axis).
A second function meeting all these conditions could be drawn 'parallel' to the first graph. This is shown in dotted line: it would be above or below the first graph.