VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

**Victorian Certificate of Education 2002**

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

**STUDENT NUMBER**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Figures | | | | | | | | | Letter |
| Words | | | | | | | | | |

# SPECIALIST MATHEMATICS

# Written examination 2
## (Analysis task)

**Wednesday 6 November 2002**

**Reading time: 11.45 am to 12.00 noon (15 minutes)**
**Writing time: 12.00 noon to 1.30 pm (1 hour 30 minutes)**

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|
| 5 | 5 | 60 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and an approved scientific and/or graphics calculator (memory may be retained).
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

**Materials supplied**
- Question and answer book of 19 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

**Instructions**
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

**Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.**

## Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

Where an **exact** answer is required to a question, appropriate working must be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.

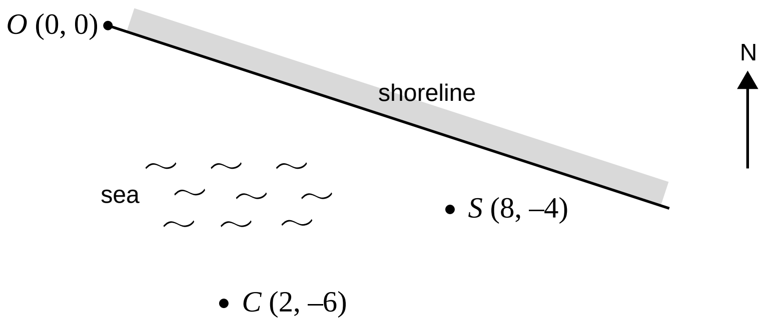
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g$ m/s$^2$, where $g = 9.8$.

Working space

**Question 1**

The diagram shows a straight shoreline with a town at $O$. Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively. Displacements are measured in kilometres.




At 12.00 midday, a cargo ship is at $C(2, -6)$ and a sailing ship is at $S(8, -4)$.

**a.** Find $\overrightarrow{CS}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ and hence determine the distance between the cargo ship and the sailing ship at 12.00 midday. Give your answer correct to the nearest tenth of a kilometre.

2 marks

Let $P(6m, -2m)$ denote a point on the shoreline, where $m > 0$.

**b.** **i.** Express the scalar product $\overrightarrow{OP} \cdot \overrightarrow{PS}$ in terms of $m$.

2 marks

**ii.** Hence, or otherwise, find the coordinates of the closest point on the shoreline to the sailing ship at 12.00 midday.

3 marks

**iii.** To the nearest tenth of a kilometre, how close is the sailing ship to the shoreline at 12.00 midday?

1 mark

Let $t$ be the time in hours after 12.00 midday. The velocities of the cargo ship and the sailing ship are respectively given by $\underset{\sim}{v}_c = 15\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{v}_s = 12\underset{\sim}{i} + (3\sin(t) - 8)\underset{\sim}{j}$.

**c.** **i.** Find the position vector at time $t$ of the cargo ship.

1 mark

**ii.** Find the position vector at time $t$ of the sailing ship.

2 marks

**d.** Show that after two hours, at exactly 2.00 pm, the cargo ship is directly south of the sailing ship.

2 marks

Total 13 marks

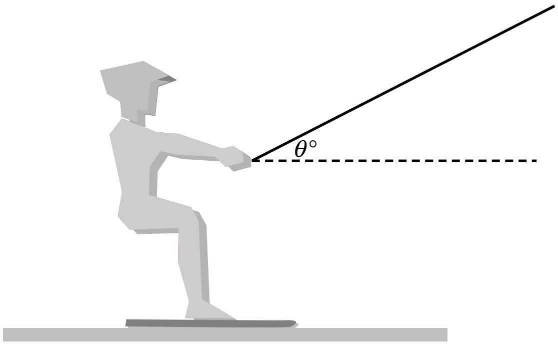
**TURN OVER**

**Question 2**

Kite surfing is a sport in which a surfer on a small board is pulled across the water by means of a kite. The surfer's feet are strapped to the board.

A surfer is pulled in a straight line across level water by a kite. The kite is connected to the surfer by a rope that makes an angle of $\theta°$ to the horizontal. The coefficient of friction between the water and the board is 0.3, and the combined mass of the surfer and board is 80 kg.

A model for this situation is shown in the following diagram, where the term 'kite surfer' refers to the surfer and board.




**a.** On the diagram, indicate and label the following forces which act on the kite surfer.

the weight force, of magnitude $80g$ newtons

the normal reaction, of magnitude $N$ newtons

the friction force, of magnitude $F$ newtons

the tension in the rope, of magnitude $T$ newtons

1 mark

The kite surfer is accelerating at 2 m/s$^2$. Assume that the only forces acting on the kite surfer are those described in part **a**.

**b.** If $\theta = 60$, find $T$ correct to the nearest integer.

4 marks

**Question 2** – continued

Suppose the kite surfer continues to be accelerated across the water at 2 m/s$^2$.

**c.** **i.** Find $\theta$ so that the tension in the rope is a minimum. Give your answer correct to the nearest tenth of a degree.

3 marks

**ii.** What is the minimum tension, correct to the nearest newton?

1 mark

Total 9 marks

Working space

**Question 3**

A dragster racing car accelerates uniformly over a straight line course and completes a ‘standing’ (that is, starting from rest) 400 metres in eight seconds.

**a.** **i.** Find the acceleration (in $m/s^2$) of the dragster over the 400 metres.

1 mark

**ii.** Show that the dragster reaches a speed of 100 m/s at the end of the 400 metre course.

1 mark

At the 400 metre mark, the dragster stops accelerating. At this instant, the dragster's brakes are applied and, in addition, a small parachute opens at the rear to slow the car down.

The retarding force applied by the brakes (including friction) is 5000 N. The retarding force due to the parachute is $0.5v^2$ N where $v$ m/s is the velocity of the car $x$ metres beyond the 400 metre mark. The mass of the dragster (car and driver) is 400 kg.

**b.** **i.** If $a$ m/s$^2$ is the acceleration of the dragster during the retardation stage, write down the equation of motion for the dragster during this stage.

1 mark

**ii.** By choosing an appropriate derivative form for acceleration, show that a differential equation relating $v$ to $x$ is

$$\frac{dv}{dx} = -\frac{(10^4 + v^2)}{800v}$$

2 marks

**iii.** Hence find, to the nearest metre, the distance the dragster takes to stop from the instant the brakes are applied.

3 marks

**c.** Use calculus to find the time, in seconds, taken to bring the dragster to rest from the 400 metre mark.

3 marks

Total 11 marks

**TURN OVER**

Working space

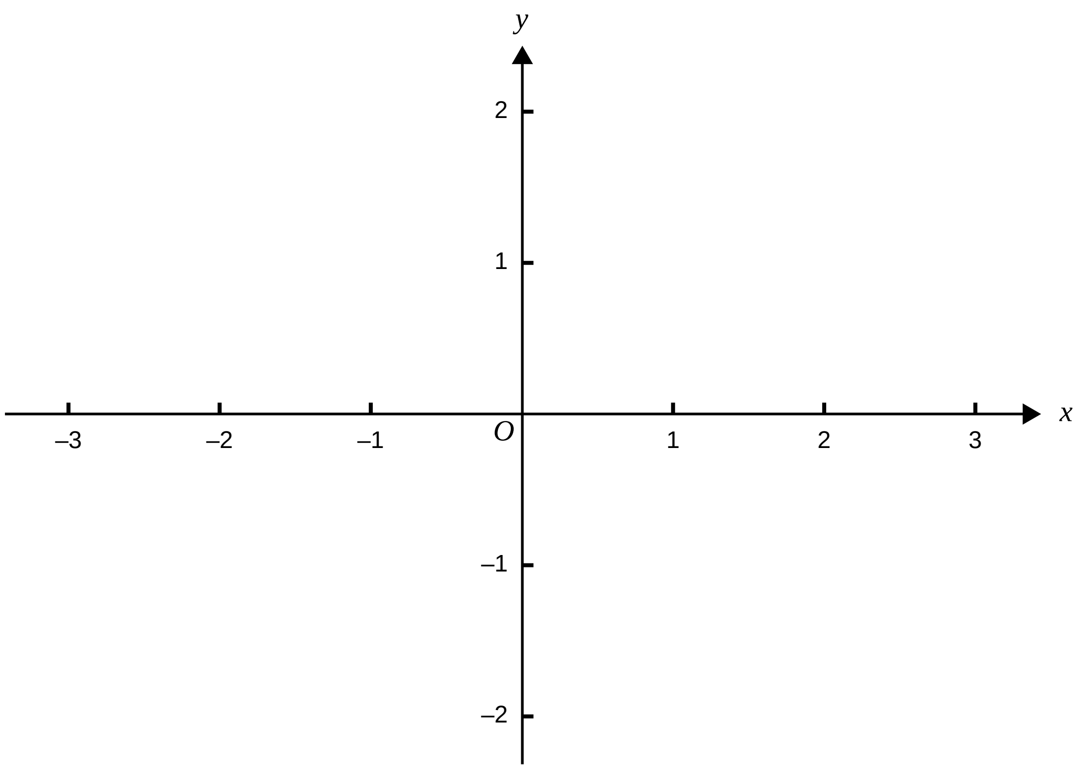
**Question 4**

Consider the function $f: D \to R$ where $f(x) = \log_e(4 - x^2)$ and $D$ is the largest possible domain for which $f$ is defined.

**a.** Find $D$.

1 mark

**b.** On the axes below, sketch the graph of $f$, labelling all the key features.




3 marks

Let $A$ be the magnitude of the area enclosed by the graph of $f$, the coordinate axes and the line $x = 1$.

**c.** Without evaluating $A$, use the graph of $f$ to show that $\log_e 3 < A < \log_e 4$.

2 marks

**Question 4** – continued
**TURN OVER**

**d.** **i.** Differentiate $x \log_e(4 - x^2)$.

2 marks

**ii.** Find an antiderivative of $\dfrac{x^2}{4 - x^2}$.

3 marks

**iii.** Hence find the exact value of $A$ in the form $a + b \log_e c$ where $a$, $b$ and $c$ are integers.

2 marks

Suppose Euler's method is used to solve the differential equation $\frac{dy}{dx} = \log_e(4 - x^2)$, with a step size of 0.05 and initial condition $y = 0$ when $x = 0$.

**e.** **i.** Use Euler's method to express $y_{20}$ in terms of $y_{19}$.

2 marks

**ii.** Given that $y_{19} = 1.2464$, find $y_{20}$, giving your answer to four decimal places.

1 mark

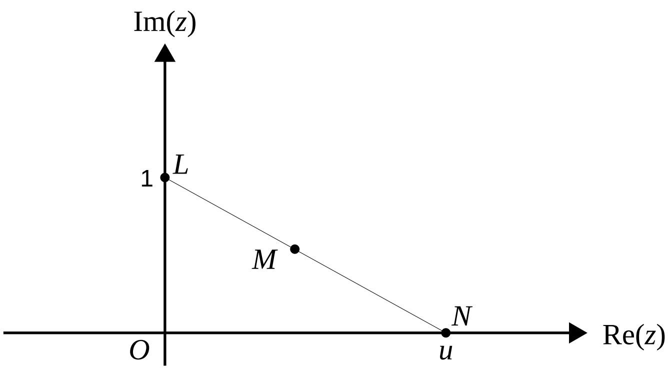
**iii.** Why is $y_{20}$ an estimate of $A$?

1 mark

Total 17 marks

**Question 5**

In the Argand diagram below, $L$ is the point $i$, $N$ is the point $u + 0i$, and $M$ is the midpoint of $LN$.




Let $z = x + yi$ satisfy $|z - i| = |z - u|$.

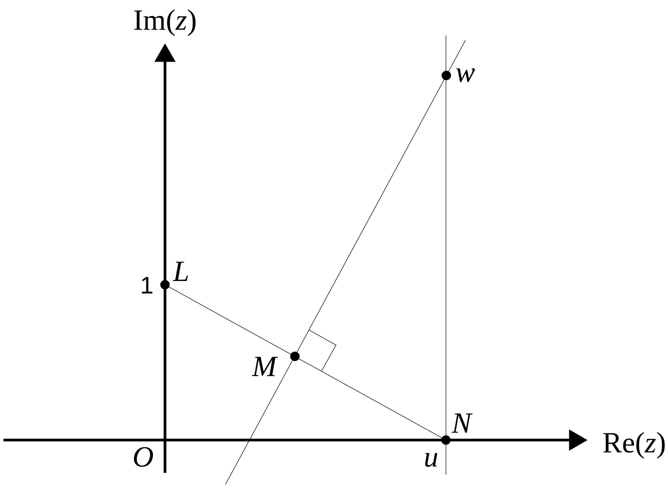
**a.** **i.** Why does $z$ lie on the perpendicular bisector of $LN$?

1 mark

**ii.** Show that $2y = 2ux - u^2 + 1$.

2 marks

Let $w$ denote the point of intersection of the perpendicular bisector of $LN$ and the line defined by $\text{Re}(z) = u$ as shown in the Argand diagram below.




**b.** **i.** Show that $w = u + \frac{1}{2}(u^2 + 1)i$ .

2 marks

**ii.** As $u$ moves along the positive $\text{Re}(z)$ axis, $w$ moves along a curve. Find the Cartesian equation of this curve and sketch it on the diagram above.

2 marks

**c.** Show that the perpendicular bisector of $LN$ is tangent to the curve at $w$.

3 marks

Total 10 marks

Working space

**END OF QUESTION AND ANSWER BOOK**

# SPECIALIST MATHEMATICS

## Written examinations 1 and 2

### FORMULA SHEET

**Directions to students**

| |
|---|
| Detach this formula sheet during reading time. |
| This formula sheet is provided for your reference. |

# Specialist Mathematics Formulas

## Mensuration

| | |
|---|---|
| area of a trapezium: | $\frac{1}{2}(a+b)h$ |
| curved surface area of a cylinder: | $2\pi rh$ |
| volume of a cylinder: | $\pi r^2 h$ |
| volume of a cone: | $\frac{1}{3}\pi r^2 h$ |
| volume of a pyramid: | $\frac{1}{3}Ah$ |
| volume of a sphere: | $\frac{4}{3}\pi r^3$ |
| area of a triangle: | $\frac{1}{2}bc\sin A$ |
| sine rule: | $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ |
| cosine rule: | $c^2 = a^2 + b^2 - 2ab\cos C$ |

## Coordinate geometry

| | |
|---|---|
| ellipse: | $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ |
| hyperbola: | $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ |

## Circular (trigometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$ $\qquad$ $\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ $\qquad$ $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ $\qquad$ $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ $\qquad$ $\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$ $\qquad$ $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

| function | $\mathrm{Sin}^{-1}$ | $\mathrm{Cos}^{-1}$ | $\mathrm{Tan}^{-1}$ |
|---|---|---|---|
| domain | $[-1, 1]$ | $[-1, 1]$ | $R$ |
| range | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $[0, \pi]$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

## Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad$ $-\pi < \operatorname{Arg} z \le \pi$

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ $\qquad$ $\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)

## Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\quad$ $\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\quad$ $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$ $\quad$ $\int \frac{1}{x} dx = \log_e(x) + c, \text{ for } x > 0$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\quad$ $\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\quad$ $\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\quad$ $\int \sec^2(ax)\, dx = \frac{1}{a}\tan(ax) + c$

$\frac{d}{dx}\left(\mathrm{Sin}^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\frac{d}{dx}\left(\mathrm{Cos}^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$ $\quad$ $\int \frac{-1}{\sqrt{a^2-x^2}}\, dx = \mathrm{Cos}^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\frac{d}{dx}\left(\mathrm{Tan}^{-1}(x)\right) = \frac{1}{1+x^2}$ $\quad$ $\int \frac{a}{a^2+x^2}\, dx = \mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

mid-point rule: $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)dx \approx \frac{1}{2}(b-a)\left(f(a) + f(b)\right)$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\,f(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\quad$ $s = ut + \frac{1}{2}at^2$ $\quad$ $v^2 = u^2 + 2as$ $\quad$ $s = \frac{1}{2}(u+v)t$

## Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

$$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$$

## Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

**END OF FORMULA SHEET**