Year 2003

VCE

Specialist Mathematics Trial Examination 1

Suggested Solutions

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These solutions are suggested solutions only. Teachers and students should carefully read the answers and comments supplied by the Mathematics Examiners.

Suggested Solutions Part I

Question 1 C

Cosine Rule

$a^2 = b^2 + c^2 - 2bc\cos A$

$\Rightarrow a^2 = 8^2 + 7^2 - 2\times 8\times 7\times \cos 110^\circ$

$\Rightarrow a^2 = 64 + 49 - 112\times \cos 110^\circ$

$\Rightarrow a^2 = 113 - 112\times \cos 110^\circ$

$\Rightarrow a = \sqrt{113 - 112\times \cos 110^\circ}$

Since $\sin 20^\circ = -\cos 110^\circ$

Then $a = \sqrt{113 + 112\sin 20^\circ}$

Question 2 C

$y = \dfrac{x^3 + 4x}{4x^2} = \dfrac{x}{4} + \dfrac{1}{x}$

$\Rightarrow x = 0$ is a vertical asymptote

$\Rightarrow y = \dfrac{x}{4}$ is an asymptote

The graph has a turning point in the first quadrant and a turning point in the third quadrant.

Question 3 B

The intersection of the two asymptotes for the hyperbola is (-3,2).

Hence, the equation of the hyperbola is

$\dfrac{(x+3)^2}{9} - \dfrac{(y-2)^2}{4} = 1$

Question 4 E

Let $u = \dfrac{4}{3x} = \dfrac{4}{3}x^{-1}$

$\Rightarrow \dfrac{du}{dx} = -\dfrac{4}{3}x^{-2} = \dfrac{-4}{3x^2}$

$y = Cos^{-1}u$

$\Rightarrow \dfrac{dy}{du} = \dfrac{-1}{\sqrt{1-u^2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx} = \dfrac{-1}{\sqrt{1-\dfrac{16}{9x^2}}}\times\dfrac{-4}{3x^2}$

$= \dfrac{4}{3x^2}\times\dfrac{1}{\sqrt{\dfrac{9x^2-16}{9x^2}}}$

$= \dfrac{4}{3x^2}\times\dfrac{3x}{\sqrt{9x^2-16}}$

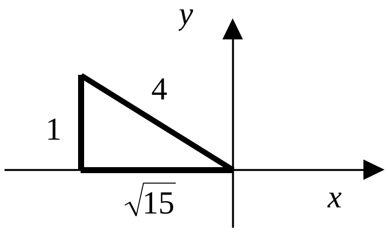
$= \dfrac{4}{x\sqrt{9x^2-16}}$

Question 5 D

$\text{cosec}(x) = 4$

$\Rightarrow \dfrac{1}{\sin x} = 4$

$\Rightarrow \sin x = \dfrac{1}{4}$



$\Rightarrow \tan x = \dfrac{-1}{\sqrt{15}}$ (second quadrant)

$\Rightarrow \cot x = -\sqrt{15}$

Question 6 D

$y = \text{Tan}^{-1}(x)$ domain

R range $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Hence,

$y = a\,\text{Tan}^{-1}(bx) + c$

range $\left(-\dfrac{a\pi}{2} + c, \dfrac{a\pi}{2} + c\right)$

Question 7 E

$z = x + iy$

$\bar{z} = x - iy$

$\Rightarrow z - \bar{z} = 2iy$ which is not a real number

Question 8 D

$\bar{u} = a + bi$

$\Rightarrow \dfrac{1}{\bar{u}} = \dfrac{1}{a+bi} \times \dfrac{a-bi}{a-bi}$

$= \dfrac{a-bi}{a^2 - b^2 i^2}$

$= \dfrac{a-bi}{a^2 + b^2}$

Question 9 A

$\alpha = 1 - 2i$

$\beta = 1 + 2i$

$\alpha + \beta = 2$

$\alpha\beta = 1 - 4i^2 = 5$

so $z^2 - 2z + 5$ is a factor

$\Rightarrow (z + w)(z^2 - 2z + 5) = z^3 + az^2 + bz - 10$

$\Rightarrow w = -2$

$\Rightarrow (z - 2)(z^2 - 2z + 5) = z^3 - 4z^2 + 9z - 10$

$\Rightarrow a = -4$ and $b = 9$

Question 10 D

Checking each alternative.

A. $2cis\left(-150^0\right) = 2\cos(-150^\circ) + i2\sin(-150^\circ) = -\sqrt{3} - i$

B. $2cis\left(\dfrac{7\pi}{6}\right) = 2\cos\left(\dfrac{7\pi}{6}\right) + i2\sin\left(\dfrac{7\pi}{6}\right) = -\sqrt{3} - i$

C. $-2cis\left(\dfrac{\pi}{6}\right) = -2\cos\left(\dfrac{\pi}{6}\right) - i2\sin\left(\dfrac{\pi}{6}\right) = -\sqrt{3} - i$

D. $-2cis\left(\dfrac{5\pi}{6}\right) = -2\cos\left(\dfrac{5\pi}{6}\right) - i2\sin\left(\dfrac{5\pi}{6}\right) = \sqrt{3} + i$

E. $2cis\left(\dfrac{-5\pi}{6}\right) = 2\cos\left(\dfrac{-5\pi}{6}\right) + i2\sin\left(\dfrac{5\pi}{6}\right) = -\sqrt{3} - i$

So D is not correct

Question 11 E

Using TI - 83 calculator

mode RAD

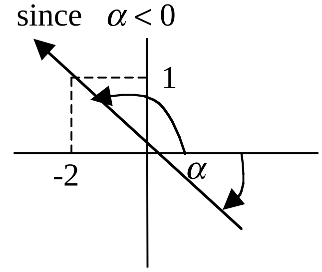
$\text{fnInt}(Y_1, X, 1, 3) = -0.7312$

Question 12 C

$\alpha = \text{Tan}^{-1}\left(-\dfrac{1}{2}\right) = -\text{Tan}^{-1}\left(\dfrac{1}{2}\right)$ since $\alpha < 0$

$-\pi < Arg(z) \le \pi$ so

$Arg(z) = \pi + \text{Tan}^{-1}\left(-\dfrac{1}{2}\right)$



Question 13 A

$\dfrac{d}{dx}(xTan^{-1}(x)) = Tan^{-1}(x) + \dfrac{x}{1+x^2}$

$\Rightarrow \int Tan^{-1}(x)dx + \int \dfrac{x}{1+x^2} = xTan^{-1}(x)$

$\Rightarrow \int Tan^{-1}(x)dx = xTan^{-1}(x) - \int \dfrac{x}{1+x^2}$

$\Rightarrow \int Tan^{-1}(x)dx = x\,\text{Tan}^{-1}(x) - \dfrac{1}{2}\log_e\left(1 + x^2\right)$

Question 14 B

Centre $c(-5,5)$

$c = -5 + 5i$ radius $r = 5$

$|z - c| = r$

$|z + 5 - 5i| = 5$ not D,E

circle also $(z - c)(\bar{z} - \bar{c}) = r^2$

$(z + 5 - 5i)(\bar{z} + 5 + 5i) = 25$

$\Rightarrow \{z : (z + 5 - 5i)(\bar{z} + 5 + 5i) = 25\}$

Question 15 C

Points of intersection

$4\sin^2(2x) = 4\cos^2(2x)$

$\Rightarrow \tan^2(2x) = 1$

$\Rightarrow \tan(2x) = \pm 1$

$\Rightarrow 2x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\Rightarrow x = \frac{\pi}{8}, \frac{3\pi}{8}$

By symmetry (shaded area)

$\text{Area} = 2\int_0^{\frac{\pi}{8}} (4\cos^2(2x) - 4\sin^2(2x))dx$

$= 8\int_0^{\frac{\pi}{8}} (\cos^2(2x) - \sin^2(2x))dx$

$= 8\int_0^{\frac{\pi}{8}} (\cos 4x)dx = 2\sin 4x]_0^{\frac{\pi}{8}}$

$= 2(\sin\frac{\pi}{2} - \sin 0) = 2$

Question 16 D

$\bar{a} = -\bar{i} + t\,\bar{j} + \bar{k}$

$\Rightarrow |\bar{a}| = 4 = \sqrt{1 + t^2 + 1}$

$\Rightarrow 16 = (\sqrt{2 + t^2})^2$

$\Rightarrow t^2 = 14$

$\Rightarrow t = \pm\sqrt{14}$ both answers OK

Question 17 D

Let $u = 3x$

$\frac{du}{dx} = 3$

Terminals $x = \frac{4}{3}$ $u = 4$; $x = 0$ $u = 0$

$\Rightarrow \int_0^{\frac{4}{3}} \frac{dx}{\sqrt{64 - 9x^2}} = \frac{1}{3}\int_0^4 \frac{du}{\sqrt{64 - u^2}}$

$= \frac{1}{3} Sin^{-1}(\frac{u}{8})\Big]_0^4$

$= \frac{1}{3} Sin^{-1}\frac{1}{2} - 0$

$= \frac{1}{3} \times \frac{\pi}{6} = \frac{\pi}{18}$

Question 18 B

$\bar{a} = \frac{1}{2}(\bar{i} - \bar{j} + z\bar{k})$

$\Rightarrow |\bar{a}| = \frac{1}{2}\sqrt{1 + 1 + z^2} = \frac{\sqrt{z^2 + 2}}{2}$

$\cos 135^\circ = -\frac{\sqrt{2}}{2} = \frac{z/2}{|\bar{a}|}$

$\Rightarrow -\frac{\sqrt{2}}{2} = \frac{z}{\sqrt{z^2 + 2}}$ so $z < 0$

$\Rightarrow 2z = -\sqrt{2}(\sqrt{z^2 + 2})$

$\Rightarrow 4z^2 = 2z^2 + 4$

$\Rightarrow z^2 = 2$

$\Rightarrow z = -\sqrt{2}$ since $z < 0$

Question 19 A

$x = \sqrt{t} - 2$

$\Rightarrow \sqrt{t} = x + 2$

$\Rightarrow t = (x+2)^2$

$y = 3t^2$

$\Rightarrow y = 3(x+2)^2 \quad x \geq -2$

Question 20 C

$m = 2kg \quad a = 5m/s^2$

$F = F_1 + F_2 = ma = 10$

$F_1 = -yj$ and $y > 0$

$F_2 = 6\sqrt{2}(-i + j) \times \frac{1}{\sqrt{2}} = 6(-i + j)$

$F_1 + F_2 = -6i + (6 - y)j$

$\Rightarrow 10 = |F_1 + F_2| = -6i - 8j$

$\Rightarrow 6 - y = -8 \Rightarrow y = 14$

Question 21 C

$S = \int_0^{\frac{7}{4}} \frac{dt}{\sqrt{4t+9}}$

Let $u = 4t + 9$

$\frac{du}{dt} = 4$

When $t = \frac{7}{4}, u = 16$ When $t = 0, u = 9$

$\Rightarrow S = \int_0^{\frac{7}{4}} \frac{dt}{\sqrt{4t+9}} = \int_9^{16} u^{-\frac{1}{2}} du = \frac{1}{2} u^{\frac{1}{2}} \Big]_9^{16}$

$\Rightarrow S = \frac{1}{2}(\sqrt{16} - \sqrt{9}) = \frac{1}{2}$

Question 22 C

If $r_A(t) = r_B(t)$

$t^2 - 5t + 6 = 2t - 4$

$\Rightarrow (t-3)(t-2) = 2(t-2)$

$\Rightarrow$ If $t = 2$ i components are equal

$2t - 6 = t^2 - 8t + 15$

$\Rightarrow 2(t-3) = (t-3)(t-5)$

$\Rightarrow$ If $t = 3$ j components are equal

The two boats do not collide. Option (C) is true because to collide both *i* and *j* must be equal at the same time.

Question 23 C

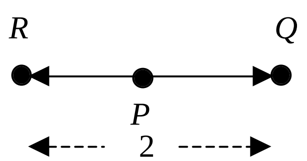
$\overrightarrow{PQ} = -\overrightarrow{PR}$

$\Rightarrow \overrightarrow{PQ} = \overrightarrow{RP}$

They have a point in common P

so $P, Q,$ and R are collinear.

Option (A) is true.



$\Rightarrow \left|\overrightarrow{PQ}\right| = 1$

Option (B) is true.

$\overrightarrow{PQ}.\overrightarrow{QR} = \left|\overrightarrow{PQ}\right|\left|\overrightarrow{QR}\right| \cos 180^\circ = 1 \times 1 \times -1 = -1$

Option (E) is true.

$\overrightarrow{PQ}$ is parallel to $\overrightarrow{RP}$.

Option (D) is true.

$\overrightarrow{PQ}$ is not perpendicular to $\overrightarrow{QR}$.

Option (C) is false.

Question 24 B

$x\frac{dx}{dy} = \sqrt{4x^2 + 9}$

$\Rightarrow \frac{dx}{dy} = \frac{\sqrt{4x^2+9}}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{4x^2+9}}$

Use the Euler PRGM on the TI-83

For $X_0 = 1$ $Y_0 = 2$ $X_N = 2$ $H = 0.5$

Value of $y = 2.3155$

Question 25 A

$x = 1 + \frac{1}{t} = 1 + t^{-1}$

$\frac{dx}{dt} = -t^{-2}$

$y = \sqrt{12 + t^2}$

$\frac{dy}{dt} = 2 \times \frac{1}{2} \times t \times \frac{1}{\sqrt{12+t^2}} = \frac{t}{\sqrt{12+t^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-t^3}{\sqrt{12+t^2}}$

For $t = 2, \frac{dy}{dx} = \frac{-8}{\sqrt{16}} = -2$

Question 26 C

For $x = 0$

$x < 0 \quad \frac{dy}{dx} > 0$

$x > 0 \quad \frac{dy}{dx} < 0$

$x = 0$ is a local maximum

For $x = 4$

$x < 4 \quad \frac{dy}{dx} < 0$

$x > 4 \quad \frac{dy}{dx} < 0$

$x = 4$ is a stationary of inflexion

Question 27 E

$\cos 2x = 0$

$\Rightarrow 2x = \pm\frac{\pi}{2}$

$\Rightarrow x = \pm\frac{\pi}{4}$

$V = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2 dx = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 9\cos^2 2x dx$

$= 18\pi\int_0^{\frac{\pi}{4}} \cos^2 2x dx$

$= 9\pi\int_0^{\frac{\pi}{4}} (1 - \cos 4x) dx$

$= 9\pi\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$

$= 9\pi\left[(\frac{\pi}{4} - \frac{1}{4}\sin\pi) - (0 - 0)\right]$

$= \frac{9\pi^2}{4}$

Note: $\pi xfnInt(Y_1^2, X, 0, \frac{\pi}{4}) \times 2 = 22.206$

Question 28 E

$\tilde{a}(t) = \frac{3\pi}{2}\cos\left(\frac{\pi t}{2}\right)\tilde{i} - \pi\sin\left(\frac{\pi t}{2}\right)\tilde{j}$

Integrating

$\tilde{r}(t) = 3\sin\left(\frac{\pi t}{2}\right)\tilde{i} + 2\cos\left(\frac{\pi t}{2}\right)\tilde{j} + \tilde{c}$

$\tilde{r}(0) = 2\tilde{j} + c = 6\tilde{j}$

$\tilde{c} = 4\tilde{j}$

$\tilde{r}(t) = 3\sin\left(\frac{\pi t}{2}\right)\tilde{i} + (4 + 2\cos\left(\frac{\pi t}{2}\right))\tilde{j}$

$\tilde{r}(1) = 3\tilde{i} + 4\tilde{j}$

$\tilde{p} = m\tilde{r}(1)$

$\Rightarrow \tilde{p} = 2(3\tilde{i} + 4\tilde{j})$

$\Rightarrow \tilde{p} = 6\tilde{i} + 8\tilde{j}$

Question 29 B	**Question 30 B**
$Q = Q(t) = 3(50+2t) + C(50+2t)^n$ $\Rightarrow \dfrac{dQ}{dt} = 6 + 2Cn(50+2t)^{n-1}$ ...(1) Also $\dfrac{dQ}{dt} = 12 - \dfrac{2}{(50+2t)}\left[3(50+2t) + C(50+2t)^n\right]$ $= 12 - 6 - 2C(50+2t)^{n-1}$ $= 6 - 2C(50+2t)^{n-1}$ ...(2) Equating (1) and (2) $6 + 2Cn(50+2t)^{n-1} = 6 - 2C(50+2t)^{n-1}$ $\Rightarrow n = -1$	Displacement = area under the graph $= \int_0^2 (10 - t^2)dt + \frac{1}{2} \times 2 \times 6 + (\frac{1}{2} \times 4 \times -12)$ $= 10t - \frac{1}{3}t^3 \Big]_0^2 + 6 - 24$ $= 20 - \frac{8}{3} + 6 - 24$ $= -\frac{2}{3}$ $\frac{2}{3}$m from starting point

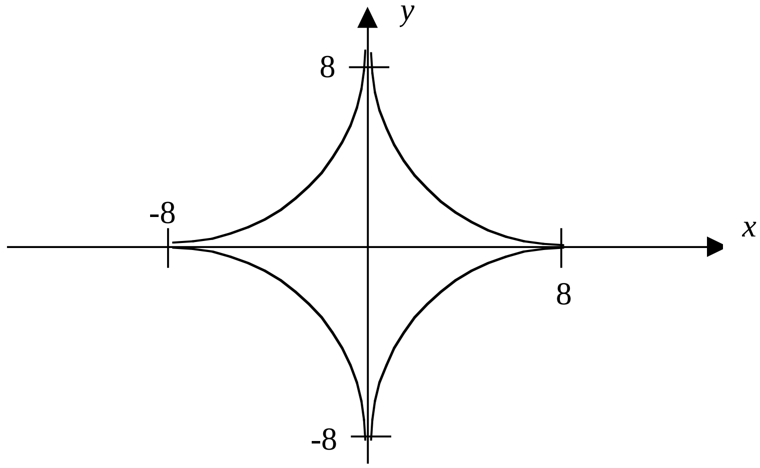
Question 1
i.

Using TI – 83

mode PAR RAD

$X_{1T} = 8(\cos(2T))\wedge 3$
$Y_{1T} = 8(\sin(2T))\wedge 3$

WINDOW
$X_{\min} = -10 \quad X_{\max} = 10$
$Y_{\min} = -10 \quad Y_{\max} = 10$



Question 1
ii.

$x = 8\cos^3 2t$

$\Rightarrow \dfrac{x}{8} = \cos^3 2t$

$\Rightarrow \dfrac{x^{\frac{1}{3}}}{2} = \cos 2t$

$y = 8\sin^3 2t$

$\Rightarrow \dfrac{y}{8} = \sin^3 2t$

$\Rightarrow \dfrac{y^{\frac{1}{3}}}{2} = \sin 2t$

$\sin^2 2t + \cos^2 2t = 1$

$\Rightarrow \dfrac{y^{\frac{2}{3}}}{4} + \dfrac{x^{\frac{2}{3}}}{4} = 1$

$\Rightarrow x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$

$\Rightarrow n = \dfrac{2}{3} \quad a = 4$

Question 1
iii.

$\tilde{r}(t) = 8\cos^3 2t\tilde{i} + 8\sin^3 2t\tilde{j} \qquad t \geq 0$

$\Rightarrow \dot{\tilde{r}}(t) = -48\cos^2 2t\sin 2t\tilde{i} + 48\sin^2 2t\cos 2t\tilde{j}$

$\Rightarrow \left|\dot{\tilde{r}}(t)\right| = \sqrt{48^2\cos^4 2t\sin^2 2t + 48^2\sin^4 2t\cos^2 2t}$

$= \sqrt{48^2\cos^2 2t\sin^2 2t(\cos^2 2t + \sin^2 2t)}$

$= 48\cos 2t\sin 2t$

$= 24\sin 4t$

$c = 24 \quad b = 4$

Question 2 i.

$P(z) = z^3 + (2+3i)z^2 + 5z + 10 + 15i$

$P(-2-3i)$

$= (-2-3i)^3 + (2+3i)(-2-3i)^2 + 5(-2-3i) + 10 + 15i$

$= -(2+3i)^3 + (2+3i)^3 - 10 - 15i + 10 + 15i$

$= 0$

Hence, $z + 2 + 3i$ is a factor.

Question 2 ii.

$z^3 + (2+3i)z^2 + 5z + 10 + 15i = 0$

$\Rightarrow z^2[z + 2 + 3i] + 5[z + 2 + 3i] = 0$ grouping

$\Rightarrow (z + 2 + 3i)(z^2 + 5) = 0$

$\Rightarrow (z + 2 + 3i)(z + \sqrt{5}i)(z - \sqrt{5}i) = 0$

$\Rightarrow z = -2 - 3i \ , \ \pm\sqrt{5}i$

Question 3 i.

When $x = 0,\ y = 1 = \dfrac{c}{32}$

Hence, $c = 32$

$-x^2 + bx + 32 = (8 - x)(x + 4) = 0$

$\Rightarrow -x^2 + bx + 32 = -x^2 + 4x + 32 = 0$

Hence, $b = 4$

Question 3 ii.

$y = \dfrac{x + 32}{-x^2 + 4x + 32} = \dfrac{A}{8 - x} + \dfrac{B}{x + 4}$

$= \dfrac{A(x+4) + B(8-x)}{(8-x)(x+4)}$

Now $x + 32 = A(x+4) + B(8-x)$

If $x = -4,\ 28 = 12B \Rightarrow B = \dfrac{7}{3}$

If $x = 8,\ 40 = 12A \Rightarrow A = \dfrac{10}{3}$

$y = \dfrac{x + 32}{-x^2 + 4x + 32} = \dfrac{10}{3(8-x)} + \dfrac{7}{3(x+4)}$

Question 3 iii.

$\text{Area} = \int_0^4 \dfrac{x + 32}{-x^2 + 4x + 32} dx = p\log_e 2$

$= \int_0^4 \left(\dfrac{10}{3(8-x)} + \dfrac{7}{3(x+4)} \right) dx$

$= \dfrac{1}{3}\left[7\log_e(x+4) - 10\log_e(8-x)\right]_0^4$

$= \dfrac{1}{3}\left[7\log_e 8 - 10\log_e 4 - 7\log_e 4 + 10\log_e 8\right]$

$= \dfrac{1}{3}\left[7\log_e \dfrac{8}{4} + 10\log_e \dfrac{8}{4}\right]$

$= \dfrac{1}{3}\left[7\log_e 2 + 10\log_e 2\right]$

$= \dfrac{17}{3}\log_e 2 = p\log_e 2 \Rightarrow p = \dfrac{17}{3}$

check $\text{fnInt}(Y_1, X, 0, 4) = 3.928$

Question 4

$Let \quad u=\frac{2}{x}=2x^{-1}$

$\frac{du}{dx}=-2x^{-2}=-\frac{2}{x^2}$

$dx=\frac{-x^2du}{2}$

Terminals

$x=\frac{12}{\pi}\Rightarrow u=2\times\frac{\pi}{12}=\frac{\pi}{6}$

$x=\frac{6}{\pi}\Rightarrow u=2\times\frac{\pi}{6}=\frac{\pi}{3}$

$$\int_{\frac{6}{\pi}}^{\frac{12}{\pi}}\frac{4\cos\left(\frac{2}{x}\right)}{x^2}dx=\int_{\frac{\pi}{3}}^{\frac{\pi}{6}}\frac{4\cos(u)}{x^2}.\frac{-x^2du}{2}$$

$$=-2\int_{\frac{\pi}{3}}^{\frac{\pi}{6}}\cos(u)du$$

$$=-2\sin u\Big]_{\frac{\pi}{3}}^{\frac{\pi}{6}}$$

$$=-2\left[\sin\frac{\pi}{6}-\sin\frac{\pi}{3}\right]$$

$$=-2\left[\frac{1}{2}-\frac{\sqrt{3}}{2}\right]$$

$$=\sqrt{3}-1$$

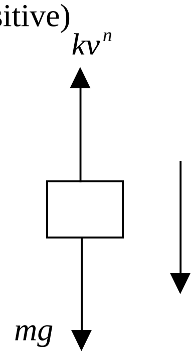
check $\text{fnInt}(Y_1,X,\frac{6}{\pi},\frac{12}{\pi})=0.732$

Question 5 i.

By Newton's Law (taking down as positive)

$\sum F = ma = mg - kv^n$

$\Rightarrow a = g - \dfrac{k}{m}v^n$



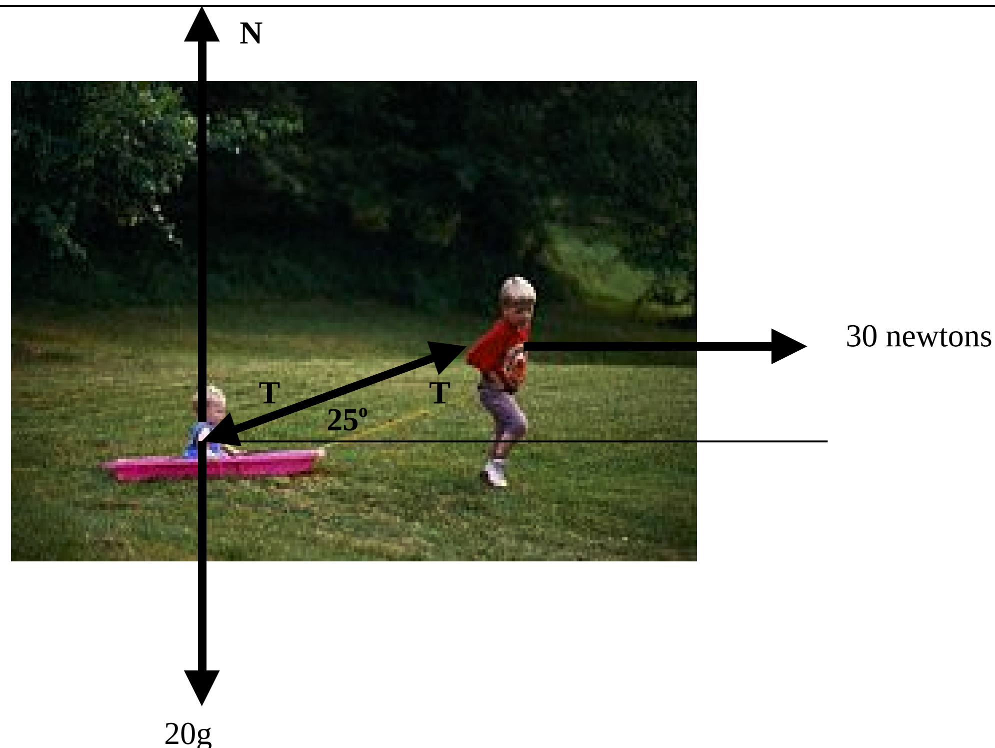
Question 5 ii.

$a = v\dfrac{dv}{dx} = g - \dfrac{k}{m}v^n$

$x = \displaystyle\int_0^{v_f} \frac{vdv}{g - \frac{k}{m}v^n} = \int_0^2 \frac{vdv}{9.8 - \frac{0.5}{2}v^3}$

$\text{fnInt}(X/(9.8 - 0.25X \wedge 3), X, 0, 2) = 0.223 \text{ m}$

Question 6



Boy: $30 - T\cos 25^\circ = 35a$ (1) Girl: $T\cos 25^\circ = 20a$ (2)

Add (1) and (2)

$\Rightarrow 30 = 55a \Rightarrow a = \dfrac{6}{11} m/s^2$

Hence, $T = \dfrac{20a}{\cos 25^\circ} = \dfrac{20 \times \frac{6}{11}}{\cos 25^\circ} = 12.04$ Newtons

END OF SUGGESTED SOLUTIONS
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