Year 2003

VCE

Specialist Mathematics
Trial Examination 2

Detailed Suggested Solutions

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These solutions are suggested solutions only. Teachers and students should carefully read the answers and comments supplied by the Mathematics Examiners.

Question 1

i.

$u = \frac{1}{4}\left(\sqrt{3} - i\right) = \frac{1}{2}cis\left(\frac{-\pi}{6}\right)$

$v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right) = 1 + i$

$\Rightarrow uv = \frac{1}{4}(\sqrt{3} - i)(1 + i)$

$\Rightarrow uv = \frac{1}{4}(\sqrt{3} - i + \sqrt{3}i - i^2)$

$\Rightarrow uv = \frac{1}{4}\left((\sqrt{3} + 1) + (\sqrt{3} - 1)i\right)$

ii.

$uv = \frac{1}{2}cis\left(\frac{-\pi}{6}\right) \times \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$\Rightarrow uv = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$\Rightarrow uv = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{12}\right)$

iii.

$\frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{12}\right) + i\frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{12}\right)$

$= \frac{1}{4}(\sqrt{3} + 1) + i\frac{1}{4}(\sqrt{3} - 1)$

Equating imaginary parts

$\frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{12}\right) = \frac{1}{4}(\sqrt{3} - 1)$

$\Rightarrow \sin\left(\frac{\pi}{12}\right) = \frac{1}{4} \times \frac{2}{\sqrt{2}}(\sqrt{3} - 1) \times \frac{\sqrt{2}}{\sqrt{2}}$

$\Rightarrow \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$

iv.

$\sin\left(\frac{\pi}{12}\right) = \sin 15^\circ$

$= \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

Question 2

i.

$A\ (0,11)$

$B\ \left(\frac{7}{2},19\right)$

ii.

When $x=0,\ y=11 \Rightarrow c=11$

When $x=\frac{7}{2},\ y=19$

$y=ax^2+c$

$\Rightarrow 19=\frac{49}{4}a+11$

$\Rightarrow \frac{49}{4}a=8$

$\Rightarrow a=\frac{32}{49}$

iii.

$y=\frac{32}{49}x^2+11$

$\Rightarrow y-11=\frac{32}{49}x^2$

$\Rightarrow x^2=\frac{49}{32}(y-11)$

$V=\pi\int_a^b x^2 dy$

$\Rightarrow V=\pi\int_{11}^{19}\frac{49}{32}(y-11)dy$

$\Rightarrow V=\frac{49\pi}{32}\left[\frac{1}{2}y^2-11y\right]_{11}^{19}$

$\Rightarrow V=\frac{49\pi}{32}\left[(\frac{1}{2}\times 19^2-11\times 19)-(\frac{1}{2}\times 11^2-11\times 11)\right]$

$\Rightarrow V=49\pi$

iv.

At $A(0,11)$

$y=11=A+1$

$\Rightarrow A=10$

At $B\left(\frac{7}{2},19\right)$

$\Rightarrow y=10+e^{\frac{7k}{2}}=19$

$\Rightarrow e^{\frac{7k}{2}}=9$

$\Rightarrow \frac{7k}{2}=\log_e 9$

$\Rightarrow k=\frac{2}{7}\log_e 9$

v.

$y=10+e^{0.6278x}$

$\Rightarrow x=\frac{1}{0.6278}\log_e(y-10)$

$V=\pi\int x^2dy=\pi\int_{11}^{19}\frac{1}{(0.6278)^2}(\log_e(y-10))^2 dy$

$Y_1=\frac{1}{0.6278}\log_e(y-10)$

$\pi * \text{fnInt}(Y_1^2, X, 11, 19)=158.63$

$(TI-83)$

vi.

$V=\pi\int_{11}^{h}\frac{49}{32}(y-11)dy$

$\Rightarrow V=\frac{49\pi}{32}\left[\frac{1}{2}(y-11)^2\right]_{11}^{h}$

$\Rightarrow V=\frac{49\pi}{64}\left[(h-11)^2-0\right]$

$\Rightarrow V=\frac{49\pi}{64}(h-11)^2$ shown

Question 2

vii.

Given $\frac{dv}{dt} = -k\sqrt{h}$

$V = \frac{49\pi}{64}(h-11)^2$

Differentiate with respect to h

$\Rightarrow \frac{dV}{dh} = \frac{49\pi}{32}(h-11)$

Using a chainrule

$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{49\pi}{32}(h-11)} = \frac{A\sqrt{h}}{h-11}$ where A is a constant

Inverting

$\frac{dt}{dh} = \frac{h-11}{A\sqrt{h}}$

$\Rightarrow \int A dt = \int (h-11)h^{-\frac{1}{2}} dh$

Integrating

$\Rightarrow At = \int (h-11)h^{-\frac{1}{2}} dh$

$\Rightarrow At + c = \int (h^{\frac{1}{2}} - 11h^{-\frac{1}{2}}) dh = \frac{2}{3}h^{\frac{3}{2}} - 22h^{\frac{1}{2}}$

Now when $t = 0, h = 19 \Rightarrow c = \frac{-28\sqrt{19}}{3} \approx 40.683$

When $t = 3, h = 16 \Rightarrow 3A + c = -45\frac{1}{3} \Rightarrow A = 1.55$

When the glass is empty $h = 11$

$At + c = \frac{-44\sqrt{11}}{3} \approx -48.64$

$\Rightarrow -1.55t - 40.683 = -48.64$

$\Rightarrow t = \frac{48.64 - 40.683}{1.55}$

$\Rightarrow t = 5.133$ minutes

$\Rightarrow t = 5.1$ minutes

Question 3

a.

$57.6 \text{ km/hr} = \frac{57.6 \times 1000}{60 \times 60} = 16 \text{ m/s}$

$14.4 \text{ km/hr} = \frac{14.4 \times 1000}{60 \times 60} = 4 \text{ m/s}$

Hence, $u = 16, v = 4, a = -3, t = ?, s = ?$

$v = u + at$

$4 = 16 - 3t$

$3t = 12$

$t = 4$ seconds

$s = \left(\frac{u+v}{2}\right)t$

$s = \left(\frac{16+4}{2}\right) \times 4$

$s = 40$ metres

It takes 4 seconds to reach 14.4 km/hr and travels 40 metres

b.(i)

$ma = -R$

$\Rightarrow 4096a = -24v^3$

$a = v\frac{dv}{dx}$

$\Rightarrow 4096v\frac{dv}{dx} = -24v^3$

$\Rightarrow 512\frac{dv}{dx} = -3v^3$ shown

b.(ii)

$\frac{dv}{dx} = -\frac{3}{512}v^2$

$\int \frac{dv}{v^2} = -\frac{3}{512}\int dx$

$-\frac{3x}{512} = -\frac{1}{v} + c_1$

Now when $v = 16, x = 0$

$c_1 = \frac{1}{16}$

$-\frac{3x}{512} = -\frac{1}{v} + \frac{1}{16}$

$\frac{1}{v} = \frac{3x}{512} + \frac{1}{16} = \frac{3x + 32}{512}$

$v = \frac{512}{3x + 32}$

Now when $v = 4$

$3x + 32 = \frac{512}{4} = 128$

$3x = 96$

$x = 32$ m

Question 3

b.(iii)

Use $a = \frac{dv}{dt} = -\frac{3}{512}v^3$

$\Rightarrow \int \frac{dv}{v^3} = -\frac{3}{512}\int dt$

$\Rightarrow -\frac{3t}{512} = \frac{1}{2}v^{-2} + c_2$

Now when $v = 16, t = 0$

$\Rightarrow c_2 = \frac{1}{512}$

$\Rightarrow \frac{1}{v^2} = \frac{3t}{256} + \frac{1}{256}$

Now when $v = 4$

$\frac{1}{16} = \frac{3t+1}{256}$

$\Rightarrow 3t + 1 = \frac{256}{16} = 16$

$\Rightarrow 3t = 15$

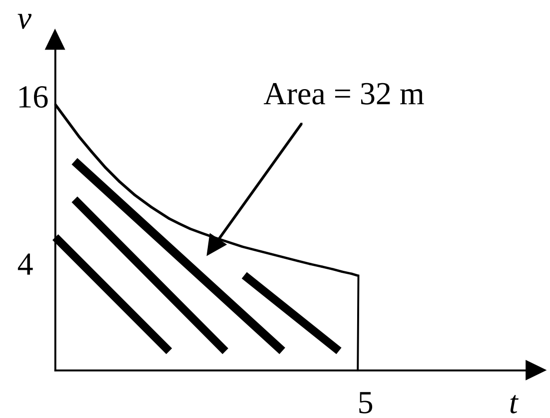
$\Rightarrow t = 5$ secs

b.(iv)

$\frac{1}{v^2} = \frac{3t+1}{256}$

$\Rightarrow v^2 = \frac{256}{3t+1}$

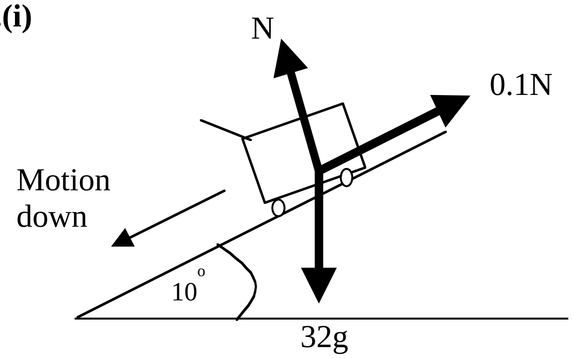
$\Rightarrow v = \frac{16}{\sqrt{3t+1}}$ for $0 \le t \le 5$



Check on TI-83

Question 4

a.(i)



a.(ii)

Forces along the slope

$32a = 32g\sin 10^\circ - 0.1N$ (1)

Forces perpendicular to the slope

$N - 32g\cos 10^\circ = 0$ (2)

$\Rightarrow N = 32g\cos 10^\circ$

Substitute into (1)

$\Rightarrow 32a = 32g\sin 10^\circ - 0.1 \times 32g\cos 10^\circ$

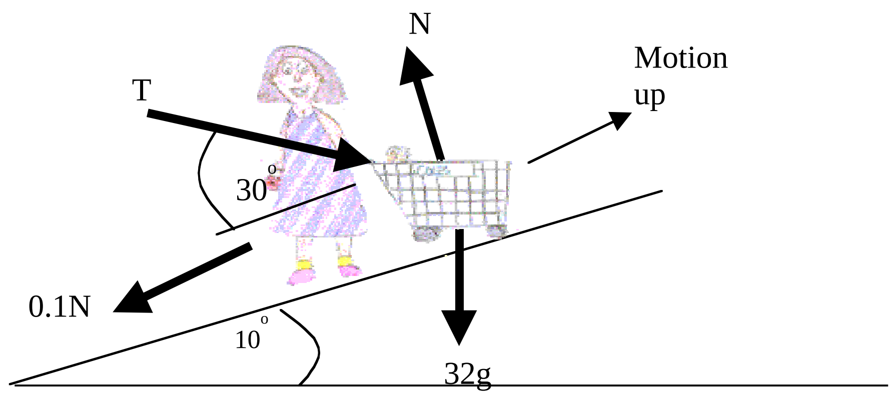
$\Rightarrow a = g(\sin 10^\circ - 0.1\cos 10^\circ)$

$\Rightarrow a = 9.8(\sin 10^\circ - 0.1\cos 10^\circ)$

$\Rightarrow a = 0.74 \ m/s^2$

Question 4

b.(i)



b.(ii)

Forces along the slope. speed is constant so $a = 0$

$T\cos 30^\circ - 0.1N - 32g\sin 10^\circ = 0 \quad (1)$

Forces perpendicular to the slope

$N - T\sin 30^\circ - 32g\cos 10^\circ = 0 \quad (2)$

$\Rightarrow N = T\sin 30^\circ + 32g\cos 10^\circ$

Substitute into (1)

$\Rightarrow T\cos 30^\circ - 0.1(T\sin 30^\circ + 32g\cos 10^\circ) - 32g\sin 10^\circ = 0$

$\Rightarrow T(\cos 30^\circ - 0.1\sin 30^\circ) = 32g\sin 10^\circ + 32g \times 0.1\cos 10^\circ$

$\Rightarrow T = \dfrac{32 \times 9.8(\sin 10^\circ + 0.1\cos 10^\circ)}{\cos 30^\circ - 0.1\sin 30^\circ} = 104.58$ Newtons

Question 5

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i.

$\bar{r}.\bar{k} = 12\sin\left(\frac{\pi t}{3}\right) = 0$

$\Rightarrow \frac{\pi t}{3} = 0, \pi, 2\pi$

$\Rightarrow t = 0, 3$

It take three seconds for the golf ball to hit the ground.

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ii.

$\bar{r}(t) = 8t\bar{i} + 50t\bar{j} + 12\sin\left(\frac{\pi t}{3}\right)\bar{k}$

$\Rightarrow \dot{\bar{r}}(t) = 8\bar{i} + 50\bar{j} + 12 \times \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right)\bar{k}$

$\Rightarrow \dot{\bar{r}}(0) = 8\bar{i} + 50\bar{j} + 4\pi\bar{k}$

$\Rightarrow \left|\dot{\bar{r}}(0)\right| = \sqrt{64 + 2500 + 16\pi^2}$

$\Rightarrow \left|\dot{\bar{r}}(0)\right| = 15.17 \quad m/s$

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iii.

$\bar{r}(3) = 24\bar{i} + 150\bar{j} + 0\bar{k}$

$\Rightarrow \left|\bar{r}(3)\right| = \sqrt{24^2 + 150^2} = 151.91$

$\Rightarrow \left|\bar{r}(3)\right| = 152 \text{ m}$

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iv.

$\dot{\bar{r}}.\bar{k} = \cos\frac{\pi t}{3} = 0$

$\Rightarrow \frac{\pi t}{3} = \frac{\pi}{2}$

$\Rightarrow t = \frac{3}{2}$

$\bar{r}(t) = 8t\bar{i} + 50t\bar{j} + 12\sin\left(\frac{\pi t}{3}\right)\bar{k}$

$\Rightarrow r(\tfrac{3}{2}) = 12i + 75j + 12k$

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Question 6

i.

$5x - 4 > 0$

$\Rightarrow 5x > 4$

$\Rightarrow \{x : x > \frac{4}{5}\} = \left(\frac{4}{5}, \infty\right)$

ii.

Quotient Rule

Let $u = x$

$\Rightarrow \frac{du}{dx} = 1$

Let $v = \sqrt{5x - 4} = (5x - 4)^{\frac{1}{2}}$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2} \times 5(5x - 4)^{-\frac{1}{2}} = \frac{5}{2\sqrt{5x - 4}}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{5x-4} - \frac{5x}{2\sqrt{5x-4}}}{5x - 4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{5x-4}\left[\frac{2(5x-4) - 5x}{2\sqrt{5x-4}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(5x - 8)(5x - 4)^{-\frac{3}{2}}$$

iii.

For stationary points

$\frac{dy}{dx} = \frac{1}{2}(5x - 8)(5x - 4)^{-\frac{3}{2}} = 0$

$\Rightarrow x = \frac{8}{5}$

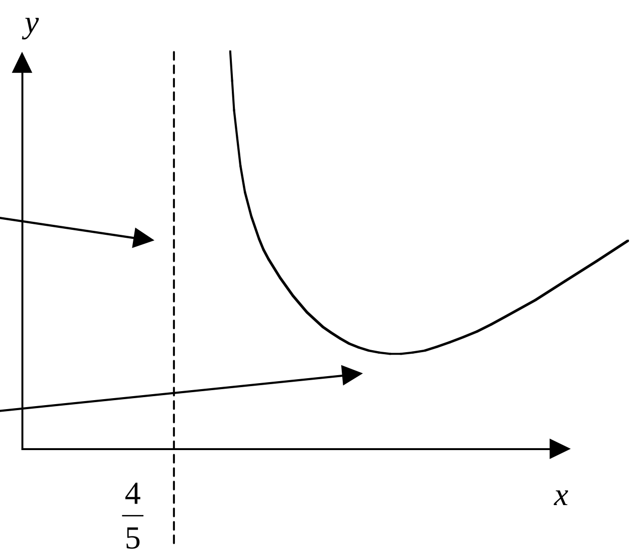
$\Rightarrow x = \frac{4}{5}$ is a vertical asymptote

$y = \frac{x}{\sqrt{5x - 4}}$

$\Rightarrow f(\frac{8}{5}) = \frac{8}{10} = \frac{4}{5}$

Turning point is $(\frac{8}{5}, \frac{4}{5})$. Minimum

Range $(\frac{4}{5}, \infty)$



Question 6

iv.

At $x = 4, f(4) = 1$

$P(4,1)$

$$f'(x) = \frac{1}{2}(5x-8)(5x-4)^{-\frac{3}{2}}$$

$$f'(4) = \frac{12}{2 \times 16^{\frac{3}{2}}} = \frac{3}{32} = \text{gradient}$$

Tangent

$$y - 1 = \frac{3}{32}(x-4)$$

$$\Rightarrow y = \frac{3x}{32} + \frac{5}{8}$$

v.

$$A_T = \int_1^8 \frac{x}{\sqrt{5x-4}} dx = 7.413 \quad (\text{PRGM TI} - 83)$$

$a = 1,\ b = 8,\ n = 7$

Vi

$$A_T = \int_1^8 \frac{x}{\sqrt{5x-4}} dx = \int_1^8 x(5x-4)^{-\frac{1}{2}} dx$$

Let $u = 5x - 4, \Rightarrow \frac{du}{dx} = 5$

$\Rightarrow 5x = u + 4$

$\Rightarrow x = \frac{1}{5}(u+4)$

When $x = 8, u = 36$

When $x = 1, u = 1$

$$A = \frac{1}{5}\int_1^{36} (u+4) \times u^{-\frac{1}{2}} \times \frac{1}{5} du$$

$$\Rightarrow A = \frac{1}{25}\int_1^{36} (u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}) du$$

$$\Rightarrow A = \frac{1}{25}\left[\frac{2}{3}u^{\frac{3}{2}} + 8u^{\frac{1}{2}}\right]_1^{36}$$

$$\Rightarrow A = \frac{1}{25}\left[(\frac{2}{3} \times 36^{\frac{3}{2}} + 8\sqrt{36}) - (\frac{2}{3} \times 1 + 8)\right]$$

$$\Rightarrow A = 7\tfrac{1}{3}$$

Check $\text{fnInt}(Y_1, X, 1, 8) = 7.333$

End of suggested solutions 2003 Specialist Mathematics Trial Examination 2

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