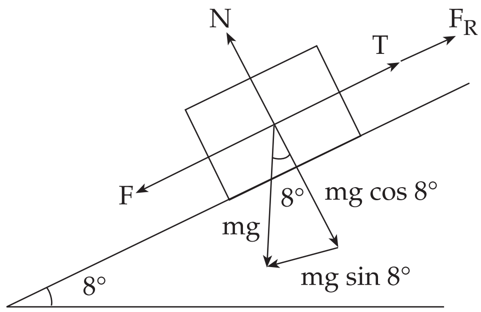
2003 Specialist Mathematics Written Examination 2 (analysis task) Suggested Answers and Solutions

Question 1

a

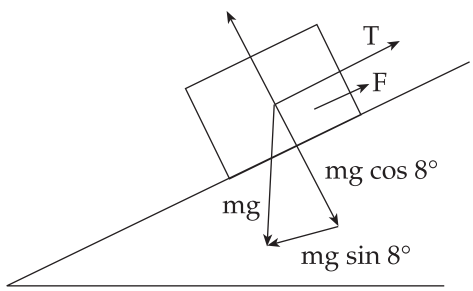


Resultant force: $F_R = ma$
$= 1200 \times 0.25$
$= 300$

$F_R = T - F - mg\sin 8^\circ$
$F = \mu N$
$= 0.09 \times 1200 \times 9.8\cos 8^\circ$

$T = 300 + 1200 \times 9.8\,(0.09\cos 8^\circ + \sin 8^\circ)$
$= 2985\text{N}$

b



$F + T - mg\sin 8^\circ = 0$

$T = mg\sin 8^\circ - F$

$= 1200 \times 9.8\sin 8^\circ - F$

$= 1636.68 - F$

c **i** $T = 1636.68 - F$

$T = 1636.68 - \mu N$

$= 1636.68 - 0.09 \times 1200 \times 9.8\cos 8^\circ$

$= 589$

c **ii** $T = 1636.68 - 0.15 \times 1200 \times 9.8\cos 8^\circ$

$= -110$

In this instance T = 0 because friction can only act to oppose motion.

Question 2

a
$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$
$$= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} \times \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$
$$= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

b $u = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i$

$v = \bar{u} = \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i$

$v - u = 0 - 2\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)i$

$\approx 0 - 0.518i$

$Arg(v-u) = -\frac{\pi}{2}$

c $u = cis\left(\frac{\pi}{12}\right)$

$v = cis\left(-\frac{\pi}{12}\right)$

$\frac{v}{u} = \frac{1}{1}cis\left(-\frac{\pi}{12} - \frac{\pi}{12}\right)$

$= cis\left(-\frac{2\pi}{12}\right)$

$= cis\left(-\frac{\pi}{6}\right)$

$Arg\left(\frac{v}{u}\right) = -\frac{\pi}{6}$

d $(z-u)(z-v) = 0$

$z^2 - uz - vz + uv = 0$

$z^2 - (u+v)z + uv = 0$

But $z^2 + az + b = 0$

so $b = uv = 1^2 cis\left(\frac{\pi}{12} - \frac{\pi}{12}\right)$

$= 1cis0$

$= 1(\cos 0 + i\sin 0)$

$= 1(1 + 0i)$

$= 1$

$u = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i$

$v = \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i$

$-a = u + v = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4} + 0$

$= 2 \times \frac{\sqrt{6}+\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{2}$

$a = \frac{-\sqrt{6}-\sqrt{2}}{2}$

Question 3

a $\overrightarrow{OC} = \underset{\sim}{c} = -2\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{b} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OA} = \underset{\sim}{a} = 4\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -\underset{\sim}{b} + \underset{\sim}{c} = -4\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\underset{\sim}{a} + \underset{\sim}{b} = -2\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{OC} = \overrightarrow{AB}$ and $\overrightarrow{OA} = \overrightarrow{CB}$

Therefore opposite sides are parallel and equal in length.

$\left|\overrightarrow{AB}\right| = \sqrt{4+25+4} = \sqrt{33}$

$\left|\overrightarrow{BC}\right| = \sqrt{16+1+16} = \sqrt{33}$

∴ All sides are of equal length and opposite sides are parallel.

∴ Base OABC is a rhombus.

b $\overrightarrow{OC} = \underset{\sim}{c} = -2\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{OA} = \underset{\sim}{a} = 4\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

Let $\theta = \angle AOC$

$\cos\theta = \frac{\underset{\sim}{a}.\underset{\sim}{c}}{|\underset{\sim}{a}||\underset{\sim}{c}|}$

$= \frac{(-8+5-8)}{\sqrt{33}\times\sqrt{33}}$

$= \frac{-11}{33}$

$\theta = Cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$

c i Let $\underset{\sim}{e} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$

If $\underset{\sim}{e}$ is a unit vector

$p^2 + q^2 + r^2 = 1$ 1

Given $\underset{\sim}{e}$ is perpendicular to $\overrightarrow{OA}$

We know $\underset{\sim}{e}.\overrightarrow{OA} = 0$

$\Rightarrow 4p + q + 4r = 0$ 2

Given $\underset{\sim}{e}$ is perpendicular to $\overrightarrow{OC}$

We know $\underset{\sim}{e}.\overrightarrow{OC} = 0$

$\Rightarrow -2p + 5q - 2r = 0$ 3

Solving 2 and 3 simultaneously

$11q = 0$

$\Rightarrow q = 0$ (As required)

$\therefore p^2 + r^2 = 1$ 4

and $4p + 4r = 0$ 5

From 5 $p = -r$

Substituting $p = -r$ into 4

$p^2 + p^2 = 1$

$\Rightarrow p^2 = \frac{1}{2}$

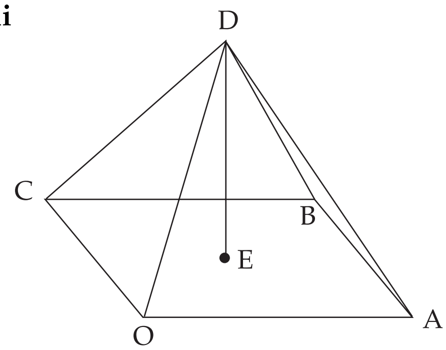
$p = \pm\frac{1}{\sqrt{2}}$

Given p>0

$\therefore p = \frac{1}{\sqrt{2}}$

and $r = -\frac{1}{\sqrt{2}}$

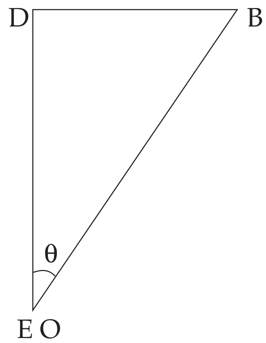
c ii



We know that $\overline{ED}$ is perpendicular to the base and thus is perpendicular to $\overrightarrow{OC}$ and $\overrightarrow{OA}$.

This suggests that $\overline{ED}$ is parallel to $\underset{\sim}{e}$ from previous question.

Where $\underset{\sim}{e} = \frac{1}{\sqrt{2}}\left(\underset{\sim}{i} - \underset{\sim}{k}\right)$



$\left|\overrightarrow{DE}\right| = \left|\overrightarrow{OD}\right|\cos\theta$

we know that $\underset{\sim}{e}$ is an unit vector

$\therefore \left|\overrightarrow{DE}\right| = \left|\overrightarrow{OD}\right|\left|\underset{\sim}{e}\right|\cos\theta$

$\Rightarrow \left|\overrightarrow{DE}\right| = \overrightarrow{OD}.\tilde{e}$

$\overrightarrow{OD}.\underset{\sim}{e} = \left(3\underset{\sim}{i} + 4\underset{\sim}{j} - \frac{\underset{\sim}{k}}{3}\right).\frac{1}{\sqrt{2}}\left(\underset{\sim}{i} - \underset{\sim}{k}\right)$

$= \frac{3}{\sqrt{2}} + \frac{1}{3\sqrt{2}}$

$= \frac{9+1}{3\sqrt{2}}$

$= \frac{10}{3\sqrt{2}} = \frac{10\sqrt{2}}{6}$

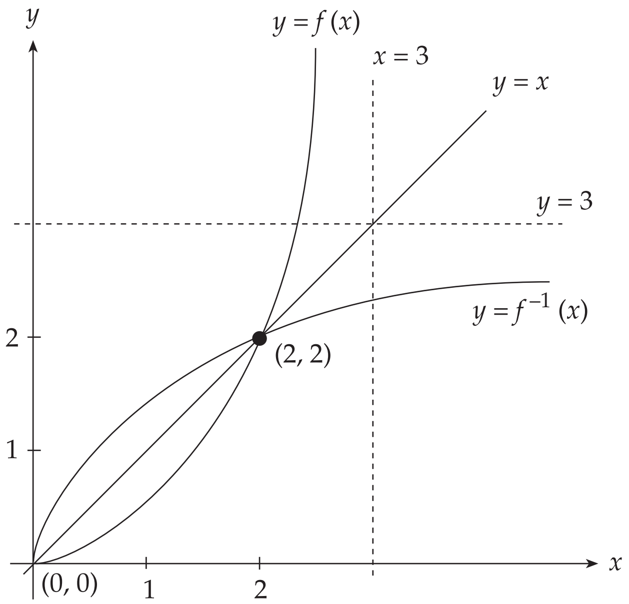
$= \frac{5\sqrt{2}}{3}$

Question 4

a $f(z) = -2 + 2\sec\left(\frac{\pi}{3}\right)$

$= -2 + 4 = 2$

b



c $y = -2 + 2\sec\left(\frac{\pi x}{6}\right)$

To find rule of inverse function, interchange x for y, and solve for y.

$x = -2 + 2\sec\left(\frac{\pi y}{6}\right)$

$\Rightarrow x + 2 = \dfrac{2}{\cos\left(\frac{\pi y}{6}\right)}$

$\Rightarrow \cos\left(\frac{\pi y}{6}\right) = \dfrac{2}{x+2}$

$\Rightarrow \frac{\pi y}{6} = Cos^{-1}\left(\frac{2}{x+2}\right)$

$\Rightarrow y = \frac{6}{\pi} Cos^{-1}\left(\frac{2}{x+2}\right)$

$\therefore a = \frac{6}{\pi}$

d $A = \int_0^2 \frac{\pi}{6} Cos^{-1}\left(\frac{2}{x+2}\right) + 2 - 2\sec\left(\frac{\pi x}{6}\right) dx$

$= 1.939$ (using Graphics calculator)

e i Let $y = \log_e(u)$

where $u = \dfrac{1 + \sin kx}{\cos kx}$

$$\frac{du}{dx} = \frac{k\cos^2 kx + (1 + \sin kx)k\sin kx}{\cos^2 kx}$$

$$= \frac{k\left(\cos^2 kx + \sin kx + \sin^2 kx\right)}{\cos^2 kx}$$

$$= \frac{k(1 + \sin kx)}{\cos^2 kx}$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{\cos kx}{1 + \sin kx} \times \frac{k(1 + \sin kx)}{\cos^2 kx}$$

$= k\sec(kx)$ (as required)

e ii $A = \int_0^2 \frac{\pi}{6} Cos^{-1}\left(\frac{2}{x+2}\right) + 2 - 2\sec\left(\frac{\pi x}{6}\right) dx$

Due to symmetry about $y = x$

This can be re-written as:

$$A = 2\int_0^2 x - \left(-2 + 2\sec\left(\frac{\pi x}{6}\right)\right) dx$$

$$= 2\int_0^2 x + 2 - 2\sec\left(\frac{\pi x}{6}\right) dx$$

$$= 2\int_0^2 x + 2 - 2 \times \frac{6}{\pi} \times \frac{\pi}{6}\sec\left(\frac{\pi x}{6}\right) dx$$

$$= 2\left[\frac{x^2}{2} + 2x - \frac{12}{\pi}\log_e\left(\frac{1 + \sin\left(\frac{\pi x}{6}\right)}{\cos\left(\frac{\pi x}{6}\right)}\right)\right]_0^2$$

$$= 2\left[\left(2 + 4 - \frac{12}{\pi}\log_e\left(\frac{\frac{2+\sqrt{3}}{2}}{\frac{1}{2}}\right)\right) - \left(0 + 0 - \frac{12}{\pi}\log_e(1)\right)\right]$$

$$= 2\left[6 - \frac{12}{\pi}\log_e\left(2 + \sqrt{3}\right)\right]$$

Question 5

a $\frac{dy}{dt} = a(100 - y)$

$\frac{dt}{dy} = -1\frac{-1}{a(100-y)}$

$t + c = -\frac{1}{a}\log_e(100 - y)$

let $-ca = d$

$-at + d = \log_e(100 - y)$

$e^{-at+d} = 100 - y$

$e^d = A$

$Ae^{-at} = 100 - y$

$y = 100 - Ae^{-at}$

We know that at $t = 0$, $y = 5$

$5 = 100 - A$

$\Rightarrow A = 95$

$\therefore y = 100 - 95e^{-at}$ (as required)

b $y = 10 + Ae^{-b(t-T)}$ 1

$\frac{dy}{dt} = -bAe^{-b(t-T)}$

From 1 we know that

$Ae^{-b(t-T)} = y - 10$

$\therefore \frac{dy}{dt} = -b(y - 10)$

To evaluate A, use the initial conditions, that is: at $t = T$, $y = 48$

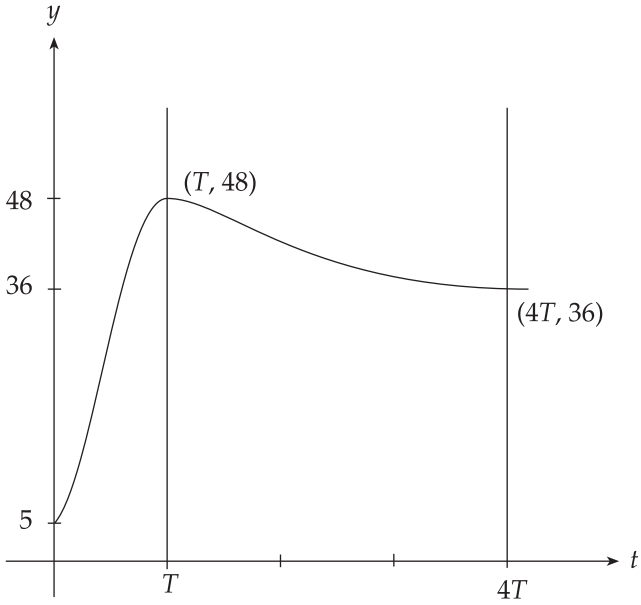
$y = 10 + Ae^{-b(t-T)}$

$\therefore 48 = 10 + Ae^0$

$A = 38$

$\therefore y = 10 + 38Ae^{-b(t-T)}$

c



d $y = 100 - 95e^{-at}$

at $t = T$, $y = 48$

$48 = 100 - 95e^{-aT}$

$-52 = -95e^{-aT}$

$\frac{52}{95} = e^{-aT}$

$-aT = \log_e\left(\frac{52}{95}\right)$ 1

$y = 10 + 38e^{-b(t-T)}$

at $t = 4T$, $y = 36$

$36 = 10 + 38e^{-3bT}$

$26 = 38e^{-3bT}$

$-3bT = \log_e\left(\frac{26}{38}\right)$ 2

$1 \div 2$

$$\frac{aT}{3bT} = \frac{\log_e\left(\frac{52}{95}\right)}{\log_e\left(\frac{26}{38}\right)}$$

$$\frac{a}{b} = 3\frac{\log_e\left(\frac{52}{95}\right)}{\log_e\left(\frac{26}{38}\right)}$$

$= 4.76$ (correct to 3 significant figures).

