Year 2004

VCE

Specialist Mathematics

Trial Examination 2

Suggested Solutions

Kilbaha Multimedia Publishing ABN 47 065 111 373
PO Box 2227
Kew Vic 3101
Australia
Tel: 03 9817 5374
Fax: 03 9817 4334
chemas@chemas.com
www.chemas.com

Question 1

i. differentiating using the Product Rule

$$\frac{d}{dx}\left[x\ \text{Cos}^{-1}(x)\right] = \text{Cos}^{-1}(x) + x\frac{d}{dx}\left(\text{Cos}^{-1}(x)\right) = \text{Cos}^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$$

$$\text{so} \int\left(\text{Cos}^{-1}(x) - \frac{x}{\sqrt{1-x^2}}\right)dx = x\ \text{Cos}^{-1}(x)$$

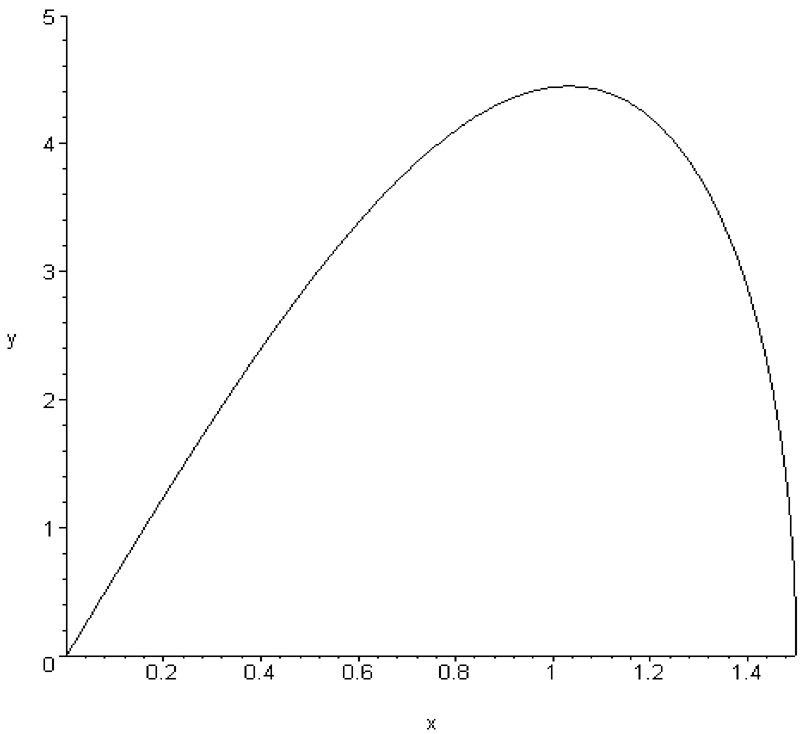
$$\int \text{Cos}^{-1}(x)dx = x\,\text{Cos}^{-1}(x) + \int\frac{x}{\sqrt{1-x^2}}dx$$

Now consider $\int\frac{x}{\sqrt{1-x^2}}dx$ let $u = 1 - x^2$ so that $\frac{du}{dx} = -2x$

$$\int\frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = -u^{\frac{1}{2}} = -\sqrt{1-x^2}$$

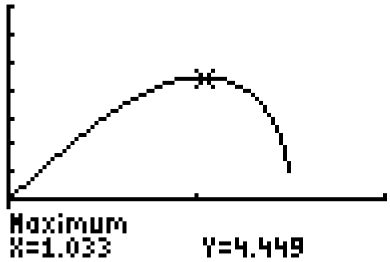
so $\int \text{Cos}^{-1}(x)\,dx = x\ \text{Cos}^{-1}(x) - \sqrt{1-x^2}$ as required.

ii.



we require that $\left|\frac{4x^2}{9}\right| \le 1$ or $0 \le \frac{4x^2}{9} \le 1$

$$0 \le x^2 \le \frac{9}{4} \quad 0 \le x \le \frac{3}{2} \quad \text{so} \quad b = \frac{3}{2}$$

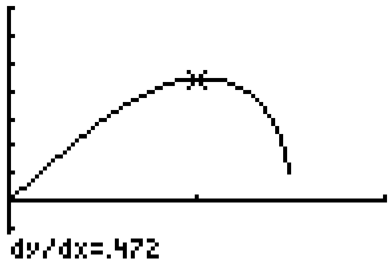


iii. max (1.033 , 4.449) so the height is 4.449 m

iv. $\frac{dy}{dt} = -2 \qquad \left.\frac{dy}{dx}\right|_{x=1} = 0.472$

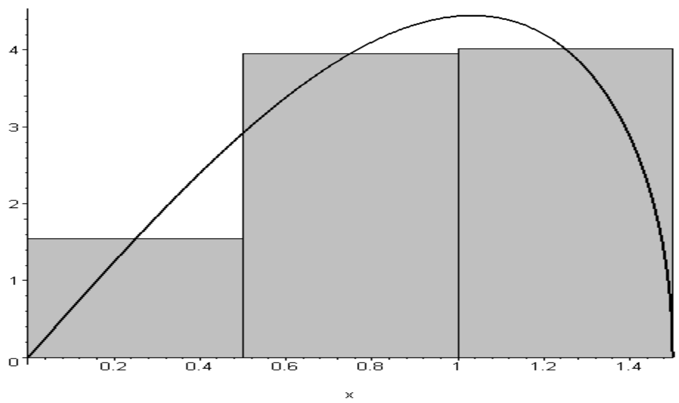
$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = -\frac{2}{0.472}$$

$$= -4.239 \text{ m/s}$$



v. $f(x) = 4x \operatorname{Cos}^{-1}\left(\frac{4x^2}{9}\right) \qquad a = 0 \quad b = \frac{3}{2} \quad n = 3 \quad h = \frac{b-a}{n} = \frac{1}{2}$

x	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{4}$
$f(x)$	$\operatorname{Cos}^{-1}\left(\frac{1}{36}\right)$	$3\operatorname{Cos}^{-1}\left(\frac{1}{4}\right)$	$5\operatorname{Cos}^{-1}\left(\frac{25}{36}\right)$



$$M = \frac{1}{2}\left(\operatorname{Cos}^{-1}\left(\frac{1}{36}\right) + 3\operatorname{Cos}^{-1}\left(\frac{1}{4}\right) + 5\operatorname{Cos}^{-1}\left(\frac{25}{36}\right)\right) = 4.757$$

vi. $A = \int_0^{\frac{3}{2}} 4x\text{Cos}^{-1}\left(\frac{4x^2}{9}\right)dx$ let $u = \frac{4x^2}{9}$ $\frac{du}{dx} = \frac{8x}{9}$ so $4x\,dx = \frac{9}{2}du$

change terminals

when $x = \frac{3}{2}$ $u = 1$

and when $x = 0$ $u = 0$

$A = \frac{9}{2}\int_0^1 \text{Cos}^{-1}(u)du$ from **i.**

$A = \frac{9}{2}\left[u\,\text{Cos}^{-1}(u) - \sqrt{1-u^2}\right]_0^1$

$A = \frac{9}{2}\left[\left(1\,\text{Cos}^{-1}\,1 - 0\right) - \left(0 - \sqrt{1}\right)\right]$

$A = \frac{9}{2} = 4.5\,\text{m}^2$ (exactly)

as a check only

Question 2

a.

i $T = \{z : 3\text{Re}(z) - 4\,\text{Im}(z) = 25\}$

Let $z = x + iy$ $\text{Re}(z) = x$ and $\text{Im}(z) = y$

So T is the line $3x - 4y = 25$ or $y = \frac{3x}{4} - \frac{25}{4}$

This line has a gradient of $m = \frac{3}{4}$ and intersects the real axis (x-axis) at $\left(8\frac{1}{3}, 0\right)$

and the imaginary axis (y-axis) at $\left(0, -6\frac{1}{4}\right)$

Now $U = \{z : |z| = |z - 6 + 8i|\}$ Let $z = x + iy$

$|x + iy| = |(x - 6) + i(y + 8)|$

$\sqrt{x^2 + y^2} = \sqrt{(x-6)^2 + (y+8)^2}$ squaring both sides

$x^2 + y^2 = (x-6)^2 + (y+8)^2$ expanding

$\cancel{x^2} + \cancel{y^2} = \cancel{x^2} - 12x + 36 + \cancel{y^2} + 16y + 64$

$12x - 16y = 100$

$3x - 4y = 25$

so $T = U$

ii. $S = \{z : |z - 3 + 4i| = 5\}$ Let $z = x + iy$

$|(x - 3) + i(y + 4)| = 5$

$\sqrt{(x-3)^2 + (y+4)^2} = 5$

$(x-3)^2 + (y+4)^2 = 25$

S is the circle with centre $(3, -4)$ and radius 5

$R=\{z:(z-3+4i)(\bar{z}-3-4i)=25\}$
$\quad=\{z:(z-c)(\bar{z}-\bar{c})=25\}$

with $c=3-4i$ so that $\bar{c}=3+4i$ and $c\bar{c}=9-16i^2=25$
Now $(z-c)(\bar{z}-\bar{c})=25$ expanding becomes
$z\bar{z}-z\bar{c}-\bar{z}c+c\bar{c}=25$ with $z=x+iy$ $\bar{z}=x-iy$ and $z\bar{z}=x^2+y^2$

$x^2+y^2-(x+iy)(3+4i)-(x-iy)(3-4i)+25=25$

$x^2+y^2-[3x+3iy+4ix+4i^2y]-[3x-3iy-4ix+4i^2y]=0$

$x^2+y^2-6x+8y=0$ completing the squares

$x^2-6x+9+y^2+8y+16=25$

$(x-3)^2+(y+4)^4=25$ is the circle with centre $(3,-4)$ and radius 5

so $S=R$

iii. Let $z_A=7-i$ substituting z_A into $S=\{z:|z-3+4i|=5\}$
$|(7-i)-(3-4i)|=|4+3i|=5$ so z_A lies on S

$z_A=7-i$ substituting z_A into $T=\{z:3\mathrm{Re}(z)-4\,\mathrm{Im}(z)=25\}$

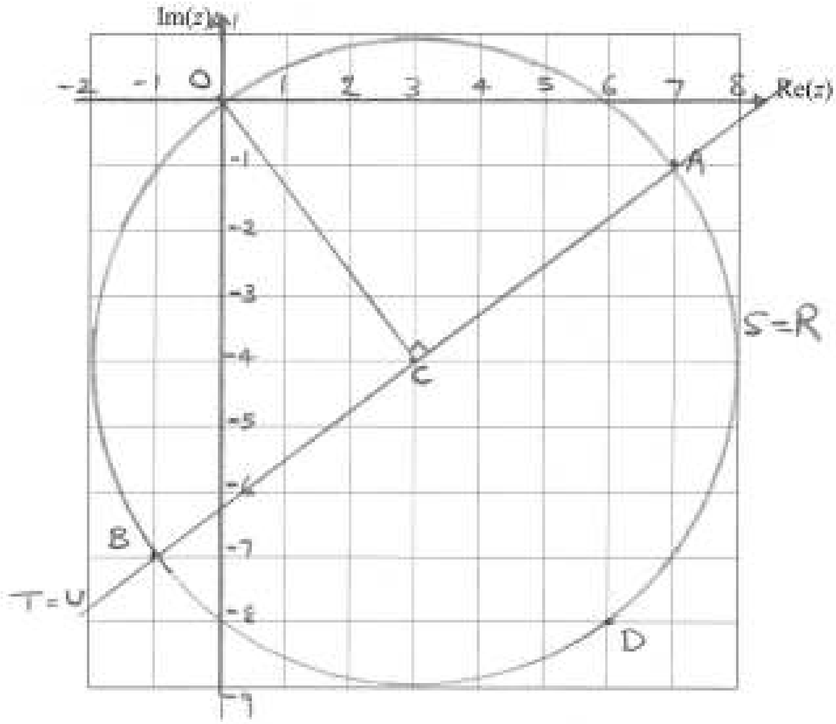
$3\times 7-4\times{}^{-}1=21+4=25$ so z_A lies on T

Let $z_B=-1-7i$ substituting z_B into $S=\{z:|z-3+4i|=5\}$
$|(-1-7i)-(3-4i)|=|-4-3i|=5$ so z_B lies on S

$z_B=-1-7i$ substituting z_B into $T=\{z:3\mathrm{Re}(z)-4\,\mathrm{Im}(z)=25\}$

$3\times{}^{-}1-4\times{}^{-}7=-3+28=25$ so z_B lies on T

iv.



v. $O(0,0)$ $A(7,-1)$ $B(-1,-7)$ $C(3,-4)$

$\overrightarrow{OA} = 7\underset{\sim}{i} - \underset{\sim}{j}$ $\overrightarrow{OB} = -\underset{\sim}{i} - 7\underset{\sim}{j}$ $\overrightarrow{OC} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -4\underset{\sim}{i} - 3\underset{\sim}{j}$

$\overrightarrow{AC}.\overrightarrow{OC} = -12 + 12 = 0$ so $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AC}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -8\underset{\sim}{i} - 6\underset{\sim}{j}$

$= 2(-4\underset{\sim}{i} - 3\underset{\sim}{j})$

$\overrightarrow{AB} = 2\overrightarrow{AC}$

so A, B, C are collinear

Now $\left|\overrightarrow{AB}\right| = 10$ it is the diameter of the circle

and $\left|\overrightarrow{AC}\right| = 5$ it is the radius of the circle.

vi. It is the set of points equidistant from both O and D

b. $\underset{\sim}{r}(t) = \left(3 + 5\cos(2t)\right)\underset{\sim}{i} + \left(-4 + 5\sin(2t)\right)\underset{\sim}{j}$ $\quad t \geq 0$

i. $x = 3 + 5\cos(2t)$ $\quad y = -4 + 5\sin(2t)$

$\cos(2t) = \dfrac{x-3}{5}$ $\quad \sin(2t) = \dfrac{y+4}{5}$

$\sin^2(2t) + \cos^2(2t) = 1$

$$\frac{(y+4)^2}{25} + \frac{(x-3)^2}{25} = 1$$

$\Rightarrow (x-3)^2 + (y+4)^2 = 25$

so P moves on S

ii. $\dot{\underset{\sim}{r}}(t) = -10\sin(2t)\underset{\sim}{i} + 10\cos(2t)\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{100\sin^2(2t) + 100\cos^2(2t)}$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{100\left(\sin^2(2t) + \cos^2(2t)\right)}$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{100}$

$|\dot{\underset{\sim}{r}}(t)| = 10$

momentum $\quad \underset{\sim}{p} = m|\dot{\underset{\sim}{r}}(t)|$

$= 2 \times 10$

$= 20 \text{ kg m/s}$

Question 3

a. no air resistance

i. $s = -100 \quad a = -9.8 \quad u = 0 \quad t = ?$

$s = ut + \frac{1}{2}at^2$

$-100 = 0 - 4.9t^2$

$t = \sqrt{\frac{100}{4.9}} = 4.52 \text{ sec}$

ii. $v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times {}^{-}9.8 \times {}^{-}100$

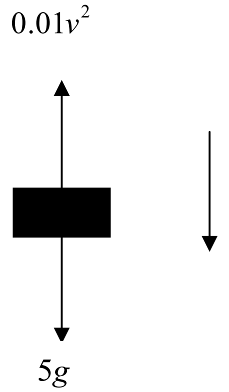
$= 1960$

$v = \pm\sqrt{1960}$

downward speed is 44.27 m/s

b. with air resistance

i. by Newton's Second Law of Motion



$5a = 5g - 0.01v^2$ using $a = v\frac{dv}{dx}$

$v\frac{dv}{dx} = \frac{49 - 0.01v^2}{5}$ shown

ii. $v\frac{dv}{dx} = \frac{49 - 0.01v^2}{5}$

$\frac{dv}{dx} = \frac{49 - 0.01v^2}{5v}$ inverting both sides

$\frac{dx}{dv} = \frac{5v}{49 - 0.01v^2}$ integrating with respect to v

$x = \int \frac{5v\,dv}{49 - 0.01v^2} = -\frac{5}{0.02}\log_e\left(49 - 0.01v^2\right) + C$

but when $x = 0 \ v = 0 \quad \Rightarrow C = \frac{5}{0.02}\log_e 49$

$x = \frac{5}{0.02}\log_e 49 - \frac{5}{0.02}\log_e\left(49 - 0.01v^2\right) = 250\log_e\left(\frac{49}{49 - 0.01v^2}\right)$

iii. Now the hammer hits the ground when $x = 100$

$$100 = 250\log_e\left(\frac{49}{49 - 0.01v^2}\right)$$

$$e^{\frac{10}{25}} = \frac{49}{49 - 0.01v^2}$$

$$49 - 0.01v^2 = 49e^{-\frac{10}{25}}$$

$$0.01v^2 = 49 - 49e^{-\frac{10}{25}}$$

$$v = \sqrt{4900\left(1 - e^{-\frac{10}{25}}\right)} = 40.1924$$

$$v = 40.19 \text{ m/s}$$

iv. $a = \dfrac{dv}{dt} = \dfrac{49 - 0.01v^2}{5}$ $\qquad \dfrac{dt}{dv} = \dfrac{5}{49 - 0.01v^2}$

$$t = \int \frac{5\ dv}{49 - 0.01v^2}$$

by partial fractions $\qquad \dfrac{5}{49 - 0.01v^2} = \dfrac{A}{7 + 0.1v} + \dfrac{B}{7 - 0.1v}$

$$\frac{5}{49 - 0.01v^2} = \frac{A(7 - 0.1v) + B(7 + 0.1v)}{49 - 0.01v^2} = \frac{7(A + B) + 0.1v(B - A)}{49 - 0.01v^2}$$

so that $7(A + B) = 5$ and $A - B = 0$ so $A = B = \dfrac{5}{14}$

$$t = \frac{5}{14}\int\left(\frac{1}{7 - 0.1v} + \frac{1}{7 + 0.1v}\right)dv$$

$$t = \frac{5}{14}\left[\frac{1}{0.1}\left[-\log_e(7 - 0.1v) + \log_e(7 + 0.1v)\right]\right] + C$$

but when $t = 0 \quad v = 0 \ \Rightarrow \ C = 0$

$$t = \frac{50}{14}\log_e\left(\frac{7 + 0.1v}{7 - 0.1v}\right)$$

$$t = \frac{25}{7}\log_e\left(\frac{70 + v}{70 - v}\right)$$

v. when $T = ?$ when $v = 40.1924$

$$t = \frac{25}{7}\log_e\left(\frac{70+v}{70-v}\right)$$

$$t = \frac{25}{7}\log_e\left(\frac{70+40.1924}{70-40.1924}\right)$$

$t = 4.67$ sec (correct to two decimal places)

vi. now transposing $t = \frac{25}{7}\log_e\left(\frac{70+v}{70-v}\right)$ to make v the subject

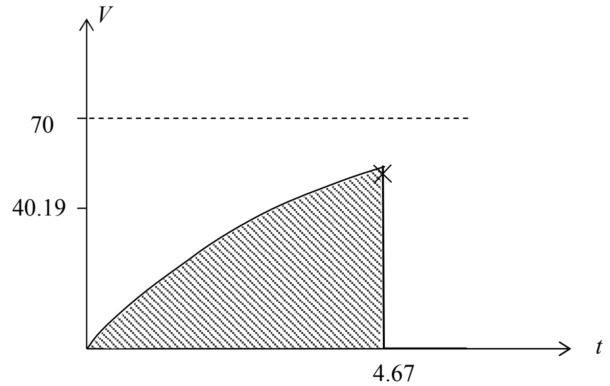
$$e^{-0.28\,t} = \frac{70-v}{70+v} \qquad (70+v)\,e^{-0.28\,t} = 70 - v$$

$$70e^{-0.28t} + ve^{-0.28\,t} = 70 - v \qquad v + ve^{-0.28\,t} = 70 - 70e^{-0.28t}$$

$$v\left(1+e^{-0.28t}\right) = 70\left(1-e^{-0.28\,t}\right)$$

$$v = v(t) = \frac{70\left(1-e^{-0.28t}\right)}{1+e^{-0.28t}} \qquad 0 \le t \le T$$

vii.



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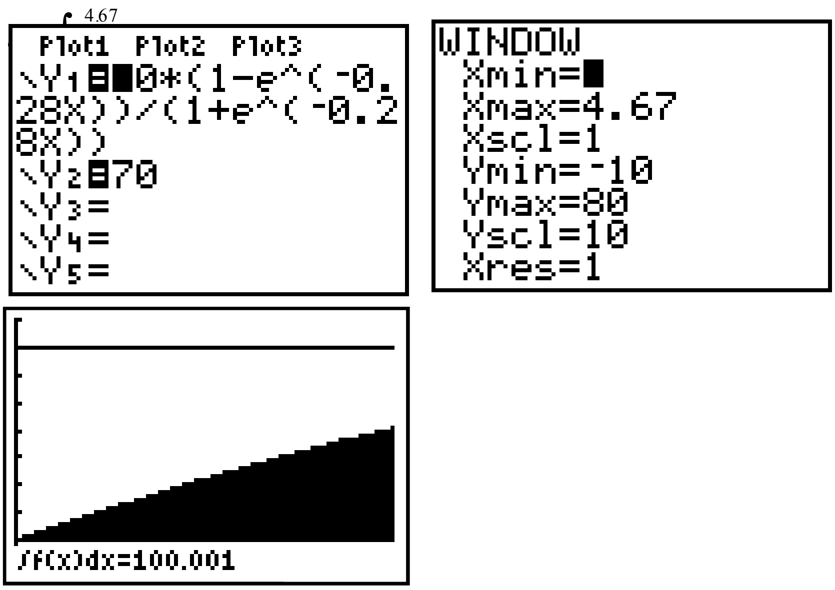


Question 4

i. $\underset{\sim}{r} \cdot \underset{\sim}{k} = 0 \;\Rightarrow\; 4 + 4\sin(\pi t) = 0$ or $4\sin(\pi t) = -4$

$\sin \pi t = -1$

$\pi t = \dfrac{3\pi}{2}$

$t = \dfrac{3}{2}$

ii. $\dot{\underset{\sim}{r}}(t) = 10\underset{\sim}{i} + 90\underset{\sim}{j} + 4\pi\cos(\pi t)\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(0) = 10\underset{\sim}{i} + 90\underset{\sim}{j} + 4\pi\underset{\sim}{k}$

$\left|\dot{\underset{\sim}{r}}(0)\right| = \sqrt{10^2 + 90^2 + 16\pi^2}$

$\left|\dot{\underset{\sim}{r}}(0)\right| = 91.42$ m/s

now the angle α at which it is hit is given by $\tan\alpha = \dfrac{4\pi}{\sqrt{10^2 + 90^2}}$

$\alpha = \text{Tan}^{-1}\left(\dfrac{4\pi}{\sqrt{10^2 + 90^2}}\right) = 7.9°$

so $\alpha = 8°$ to the nearest degree.

iii. $\underset{\sim}{r}\left(\dfrac{3}{2}\right) = 15\underset{\sim}{i} + 135\underset{\sim}{j}$

$\left|\underset{\sim}{r}\left(\dfrac{3}{2}\right)\right| = \sqrt{15^2 + 135^2}$

$\left|\underset{\sim}{r}\left(\dfrac{3}{2}\right)\right| = 136$ m

iv. at max height $4\sin\pi t + 4 = 8$ when

$\sin(\pi t) = 1$

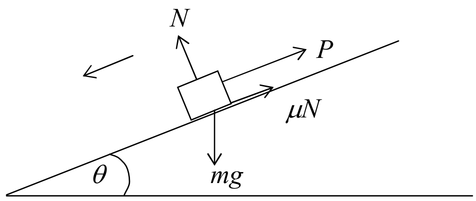
$\pi t = \dfrac{\pi}{2}$

$t = \dfrac{1}{2}$

Now $\underset{\sim}{r}\left(\dfrac{1}{2}\right) = 5\underset{\sim}{i} + 45\underset{\sim}{j} + 8\underset{\sim}{k}$

Question 5

i.



ii. resolving perpendicular to the plane $N - mg\cos\theta = 0$

so that $N = mg\cos\theta$ (1)

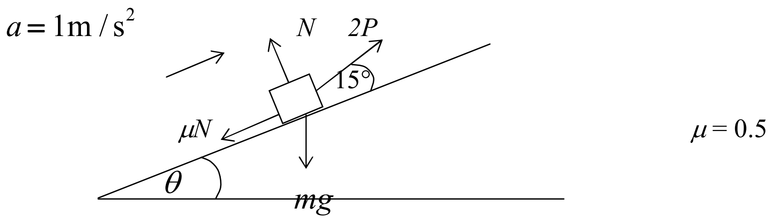
resolving up and parallel to the plane $P + \mu N - mg\sin\theta = 0$

$P = mg\sin\theta - \mu N$

$P = mg\sin\theta - \mu mg\cos\theta$ from (1) and $\mu = 0.5$

$P = mg(\sin\theta - 0.5\cos\theta)$

iii.



iv. resolving perpendicular to the plane

$N + 2P\sin 15° - mg\cos\theta = 0$ (3)

resolving up and parallel to the plane, using Newton's Second Law of Motion

$2P\cos 15° - \mu N - mg\sin\theta = ma$ (4)

from (3) $N = mg\cos\theta - 2P\sin 15°$ into (4)

$2P\cos 15° - \mu(mg\cos\theta - 2P\sin 15°) - mg\sin\theta = ma$

$2P(\cos 15° + \mu\sin 15°) - mg(\sin\theta + \mu\cos\theta) = ma$

but from **i.**

$P = mg(\sin\theta - 0.5\cos\theta)$ and $\mu = 0.5$ $a = 1$

$2mg(\sin\theta - 0.5\cos\theta)(\cos 15° + 0.5\sin 15°) - mg(\sin\theta + 0.5\cos\theta) = 1m$

$2g(\sin\theta - 0.5\cos\theta)(\cos 15° + 0.5\sin 15°) - g(\sin\theta + 0.5\cos\theta) = 1$

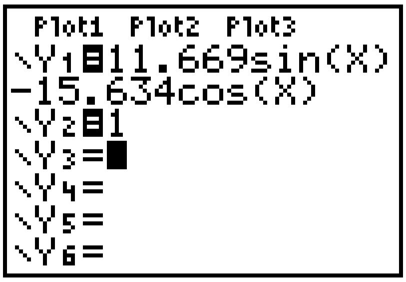
$21.4686(\sin\theta - 0.5\cos\theta) - 9.8(\sin\theta + 0.5\cos\theta) = 1$

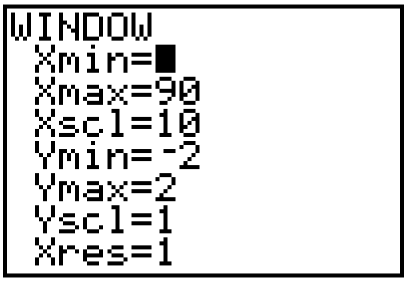
$(21.4686 - 9.8)\sin\theta - 0.5(21.4686 + 9.8)\cos\theta = 1$

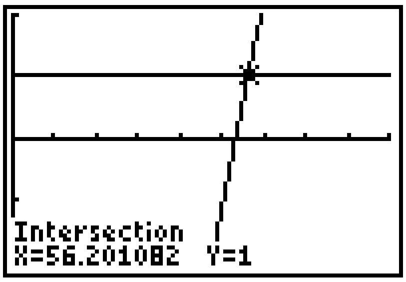
$11.669\sin\theta - 15.634\cos\theta = 1$

v. $11.669\sin\theta - 15.634\cos\theta = 1$

solving this equation on the TI-83 with the calculator in the DEGREES mode.







$\theta = 56.2$
$\theta = 56°$

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KILBAHA MULTIMEDIA PUBLISHING	**TEL: (03) 9817 5374**
PO BOX 2227	**FAX: (03) 9817 4334**
KEW VIC 3101	**chemas@chemas.com**
AUSTRALIA	**www.chemas.com**