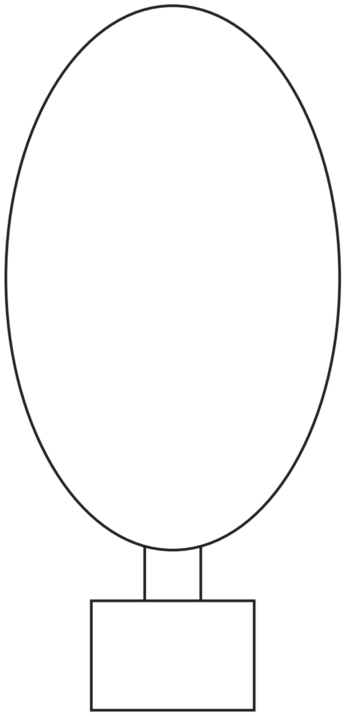
Question 1

A hot air balloon is made in the shape of a football, which can be modelled by rotating the curve $4x^2 + y^2 - 8y = 0$ about the y-axis.



a Express the equation $4x^2 + y^2 - 8y = 0$ in the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

(2 marks)

b **i** Looking from a distance, the hot air balloon appears 2-dimensional and in the shape of an ellipse. Find the area of the sky the balloon appears to cover, giving your answer to two decimal places.

(3 marks)

b **ii** Find, in cubic units, the exact volume of air enclosed by the balloon.

(3 marks)

When the hot air balloon is at a height of 75 metres and ascending at 3 m/s, a tennis ball is dropped over the side of the passenger basket. Assume $g = 9.8 \text{ m/s}^2$ and give all answers to 3 decimal places. Neglect air resistance.

c **i** Find the time taken from when it was released for the tennis ball to reach the ground.

(2 marks)

c **ii** Find the speed of the tennis ball when it strikes the ground.

(2 marks)

Total 12 marks

Question 2

a Find the solutions over C of $z^2 - 4z + 6 = 0$, expressing the solutions in:

i exact Cartesian form

ii Polar form, giving the angle in degrees to 2 decimal places

(1 + 2 = 3 marks)

b Two of the solutions of $z^3 + az^2 + bz + 6 = 0$, where $a, b \in R$, are the solutions of the quadratic equation given in part **a**.

i Find the values of a and b.

ii Find the other solution.

(4 + 1 = 5 marks)

Total 8 marks

Question 3

On the island of Sedgwickus, just off the South Australian coast, a flock of 10 giant flightless birds, Dromaius Novaehollandiae, commonly known as emus, escaped into the wild on January 1st, 2000. There are no native emus on Sedgwickus, and no attempt was made to recapture the birds. An emu only lays one clutch of eggs each year, but lays multiple eggs. The growth rate of the population of emus on Sedgwickus can be modelled by the differential equation $\frac{dN}{dt} = kN$, where N is the number of emus present at time t years.

a Show that the solution to this differential equation can be expressed in the form $N = 10e^{kt}$.

(3 marks)

b By January 1st, 2002, the flock of emus has increased to 70 birds. Show that $k = \frac{1}{2}\ln 7$.

(1 mark)

c If the flock of emus continues to increase in numbers, what does this model predict the size of the flock to be after 5 years?

(1 mark)

d According to this model of the population growth of the emus, how long, to the nearest year, will it take for the emu population to exceed 10 000 birds?

(2 marks)

The model of unrestrained exponential population growth is unrealistic, as the island of Sedgwickus can only support 6 000 emus. A modified exponential growth model is proposed, where the rate of population growth is jointly proportional to the number of emus present, N, and the difference between the upper limit and the number present, according to:

$$\frac{dN}{dt} = kN(6000 - N)$$

e **i** Show that a solution to this differential equation can be expressed as: $N = \dfrac{6000Ae^{6000kt}}{1 + Ae^{6000kt}}$

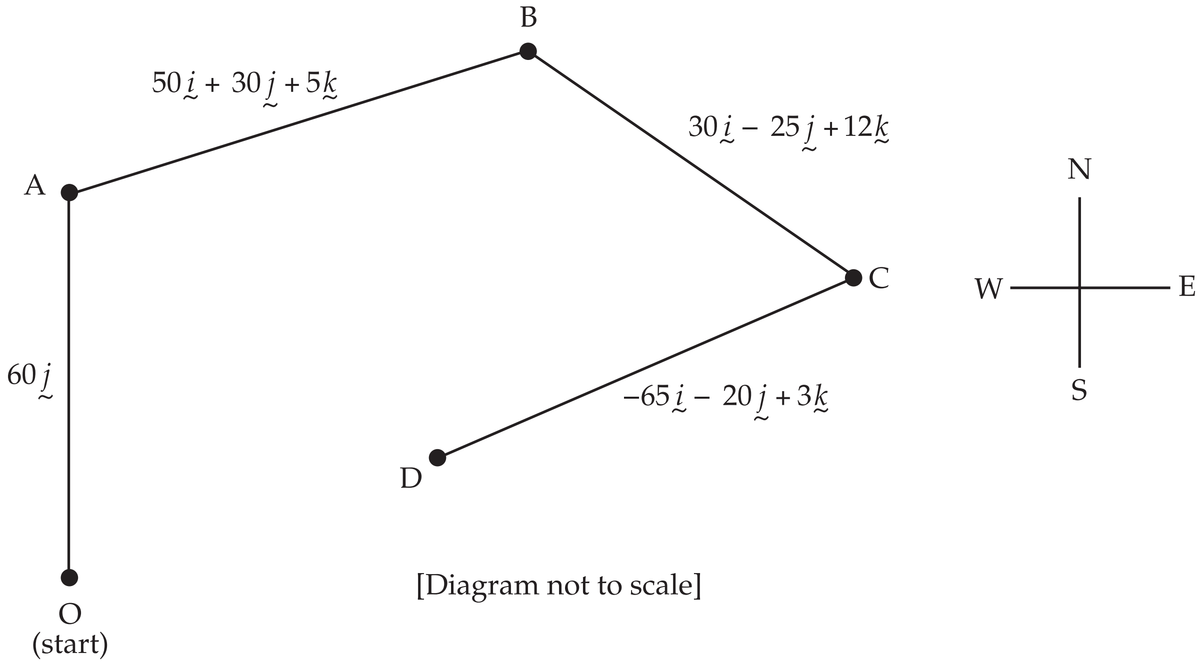
ii Hence, or otherwise, find the exact value of A.

(4 + 1 = 5 marks)

Total 12 marks

Question 4

In order to improve his navigational skills Les Lostalot has taken up the sport of orienteering. His first course is a beginners course which has four checkpoints, represented on the diagram below as A, B, C, D. For each checkpoint, the position vector of the checkpoint relative to the previous checkpoint is given, i.e. $\overrightarrow{AB} = 50\underset{\sim}{i} - 25\underset{\sim}{j} + 12\underset{\sim}{k}$.



[Diagram not to scale]

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the East and North directions respectively, and let 1 unit = 1 metre.

a Find the position vectors of A, B, C, D relative to O.

(2 marks)

b **i** Find to the nearest metre the distance between checkpoint B and checkpoint C.

ii Find, to the nearest degree, the acute angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(1 + 2 = 3 marks)

c When Les reaches checkpoint D, he finds he is on a hill overlooking the start (O). At what angle of depression to the nearest degree, must Les look down to see O?

(3 marks)

d Whilst he was at checkpoint B, Les removed his hat to scratch his head. It wasn't until he was back at O that Les realised his hat was back at B and that he would have to repeat part of the course in order to retrieve his hat. He remembered when he got back to A, that the path from A to B was very rocky. An alternative path to B which avoided the rocky section, was to leave A and walk parallel to $\overrightarrow{BC}$, then cut across perpendicularly to checkpoint B. If Les follows this alternative path when leaving checkpoint A, how far, to the nearest metre, would he walk before turning perpendicularly towards B?

(2 marks)

Total 10 marks

Question 5

Let $f : D \to R$ where $f(x) = 4\text{Cos}^{-1}\left(\frac{1}{\sqrt{x}}\right)$.

a State D, the implied domain of f.

(1 mark)

b Show that $f'(x) = \frac{2}{x\sqrt{x-1}}$.

(3 marks)

c Hence, show that $\int_2^4 \frac{1}{x\sqrt{x-1}}\,dx = \frac{\pi}{6}$.

(2 marks)

d **i** Use the mid-point rule with two equal intervals to find, correct to three decimal places, an estimate of $\int_2^4 \frac{1}{x\sqrt{x-1}}\,dx$.

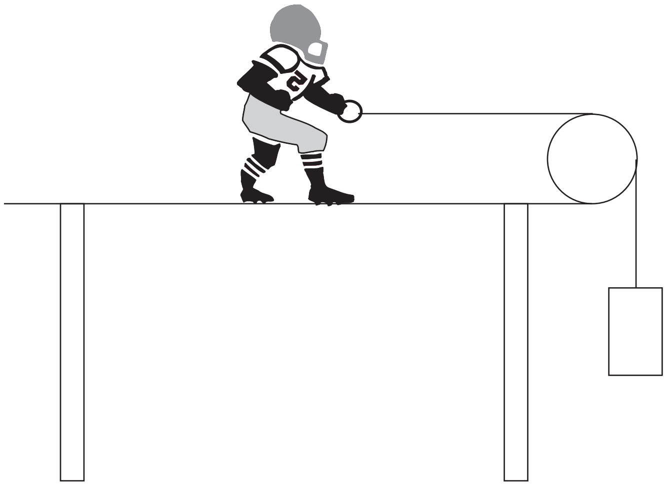
ii Comment on the accuracy of the estimate.

(2 + 1 = 3 marks)

Total 9 marks

Question 6

Lisa has just opened a packet of Special P breakfast cereal, which includes a novelty toy in each packet. One of these toys is a plastic man named Arnie Schwartzasplatter whose plastic components come apart if you drop him on the floor. A plastic ring is attached to Arnie, through which is tied an inelastic string which passes over a smooth pulley to an attached mass of 9 grams, as shown below. Arnie himself has a mass of 25 grams.



The objective with this toy, is to have Arnie waddle to the end of the table, topple over the pulley, splatter on the floor and fall to pieces. Arnie can then be put back together.

a Lisa places Arnie on a flat wooden table. The co-efficient of friction between Arnie and the table is 0.20. Assume $g = 9.8 \text{ m/s}^2$. Find, correct to 3 decimal places;

i Arnie's acceleration along the table.

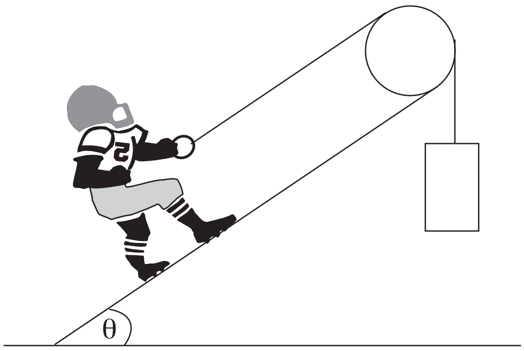
ii The tension in the string.

(3 marks)

b Lisa decides to place Arnie on her laminated desk. She places him 0.5 m from the edge of the desk and measures with a stopwatch that it takes Arnie 2 seconds from when he is released to reach the edge of the table. Calculate, correct to 3 decimal places, the co-efficient of friction between Arnie and Lisa's desktop.

(3 marks)

c Lisa now places Arnie on a sloping board, with the pulley attached to the high end, as shown below.



This time Lisa has doubled the dangling mass to 18 grams. The co-efficient of friction between Arnie and the board is 0.40. Lisa raises the board until Arnie is stationary, but on the point of moving up the slope. Find the angle, θ, the board makes with the horizontal. Give your answer to two decimal places.

(3 marks)

Total 9 marks