2004 Mathematical Methods, Specialist Examination 2

Question 1

(a) (i) $v = t^2(3 - t)$

$= 3t^2 - t^3$

$a = \frac{dv}{dt}$

$= 6t - 3t^2$

(ii) maximum acceleration when $\frac{da}{dt} = 0$

$\frac{da}{dt} = 6 - 6t = 0$

$\therefore\ t = 1$

so maximum acceleration $= 6 \times 1 - 3 \times 1^2$

$= 3\ ms^{-2}$

(b) Total distance = area under the curve

$41 = 2\int_0^2 3t^2 - t^3\,dt = (T - 4)\,4$

$= 2\left[t^3 - \frac{t^4}{4}\right]_0^2 + 4T - 16$

$= 2\,(8 - 4) + 4T - 16$

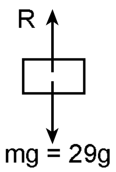
$= 8 + 4T - 16$

$= 4T - 8$

so $T = \frac{49}{4}$

$= 12.25$ s

(c)



$\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$R - 29g = 29a$

$R = 29a + 29g$

$= 29(a + g)$

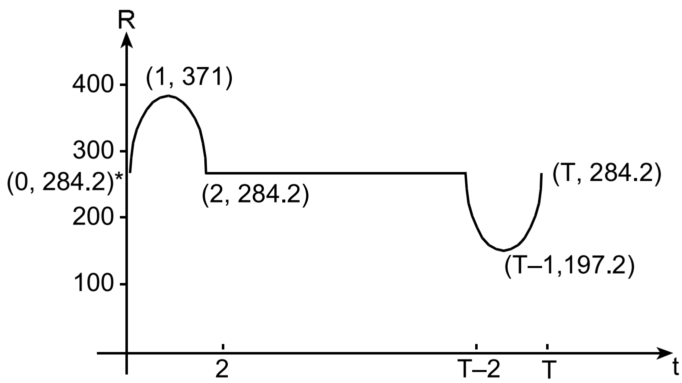
(d) maximum R when a is a maximum , i.e. $a = 3\ ms^{-2}$

so $R = 29(3 + 9.8)$

$= 371.2$

so $R \approx 371$ N (to nearest integer)

(e)



(f) (i) At $t = 1$, $|R_B| = |R_G| = 371.2$

Equation for velocity of boy $v_B = t^2\,(t - 3)$

$\therefore\ a_B = 3t^2 - 6t$

At $t = 1$, $a_B = -3\text{ms}^{-2}$

$R_B - m_B\,g = ma_B\,a$

$R_B = m_B(a + g) = 371.2$

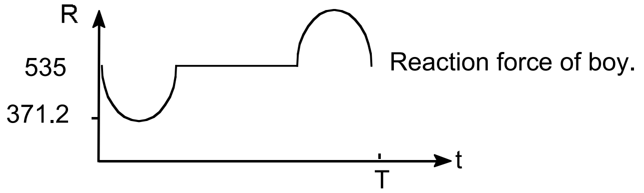
$\therefore\ m_B(-3 + 9.8) = 371.2$

$m_B = \frac{371.2}{6.8} = 54.6$ kg

so mass of boy is 54.6 kg

(ii) No

The boy's reaction force will always be greater than 371.2 and the girl's reaction force will have reached a maximum of 371.2



Question 2

(a) At P, $t = 0$

$\underset{\sim}{r}(0) = (2\sin(0))\underset{\sim}{i} + (2 + 0 - \frac{5}{3}\sin(0))\underset{\sim}{j}$

$= 0\underset{\sim}{i} + 2\underset{\sim}{j}$

so P is at (0,2)

(b) $\underset{\sim}{v}(t) = \dot{r}(t)$

$= \frac{4}{15}\cos(\frac{2t}{15})\underset{\sim}{i} \quad + (\frac{5}{3} - \frac{5}{9}\cos(\frac{1}{3}t))\underset{\sim}{j}\ ,\ 0 \le t \le \frac{15}{2}$

(c) let the angle be given by θ

$\tan\theta = \dfrac{(\frac{5}{3} - \frac{5}{9}\cos(\frac{1}{3}.\,0))}{\frac{4}{15}\cos(\frac{2}{15}.0)}$

$= \dfrac{\frac{10}{9}}{\frac{4}{15}}$

$= 4.1667$

So $\theta = 76.5^0$

(d) "swings" i.e, $\frac{dx}{dt} = 0$ (turning point in i direction, max i)

$\frac{dx}{dt} = \frac{4}{15}\cos(\frac{2}{15})t = 0$

$\cos(\frac{2}{15}t) = 0$

$\frac{2}{15}t = \frac{\pi}{2}$

$t = \frac{15\pi}{4}$

$= 11.78$ s

ie. $t = 11.8\ s$ (to nearest tenth sec)

(e) At J, $x = 1$ so $2 \sin (\frac{2}{15}t) = 1$

$$\sin (\tfrac{2}{15}t) = \tfrac{1}{2}$$

$$\tfrac{2}{15}t = \tfrac{\pi}{6}, \ \tfrac{5\pi}{6}$$

$$t = \tfrac{5\pi}{4}, \ \tfrac{25\pi}{4}$$

Substituting into $y = 2 + \frac{5}{3}t - \frac{5}{3} \sin (\frac{1}{3}t)$

If $t = \frac{5\pi}{4}$, $\quad y = 2 + \frac{25\pi}{12} - \frac{5}{3} - \sin (\frac{5\pi}{12})$

≈ 6.935 which occurs before ball swings

If $t = \frac{25\pi}{4}$, $\quad y = 2 + \frac{25\pi}{12} - \frac{5}{3} 5 \sin (\frac{5\pi}{12})$

$= 34.3$

So ball does not pass through *J*.

Question 3

(a) (i) $v = 8{,}000 \times 1 \times \text{Tan}^{-1} (1)$

$= 8{,}000 \times \frac{\pi}{4}$

$= 2{,}000\ \pi$ litres

(ii) $10\ 000 = 8000h\ \text{Tan}^{-1} (h)$

$\frac{10}{8} - h\ \text{Tan}^{-1} (h) = 0$

$h = 1.3429933$ (using calculator)

so $h = 1343$ mm

(b)

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$\frac{dv}{dh} = 8000\ \text{Tan}^{-1} (h) + 8000h \times \left(\frac{1}{1 + h^2}\right)$$

$$= 8000 \left(\text{Tan}^{-1} (h) + \frac{h}{1 + h^2}\right)$$

so $$\frac{dh}{dv} = \frac{1}{8000 \left(\text{Tan}^{-1}(h) + \frac{h}{1 + h^2}\right)}$$

$$\frac{dh}{dt} = \frac{1}{8000 \left(\text{Tan}^{-1}(h) + \frac{h}{1 + h^2}\right)} \times 2000$$

$$= \frac{1}{4 \left(\text{Tan}^{-1}(h) + \frac{h}{1 + h^2}\right)}$$

$$= \frac{1 + h^2}{4 ((1 + h^2)\ \text{Tan}^{-1}(h) + h)}$$

(i) $$\frac{dt}{dh} = \frac{4 ((1 + h^2)\ \text{Tan}^{-1} (h) + h)}{1 + h^2}$$

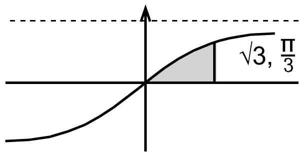
$$= 4 \left(\text{Tan}^{-1} (h) + \frac{h}{1 + h^2}\right)$$

so $t = \int 4 \left(\text{Tan}^{-1} (h) + \frac{h}{1 + h^2}\right) dh$

so $t = 4 \int_0^{\sqrt{3}} \text{Tan}^{-1} (h) + \frac{h}{1 + h^2} \ dh$

(ii) $t = 4\left(\int_0^{\sqrt{3}} \text{Tan}^{-1}(h)\,dh + \int_0^{\sqrt{3}} \frac{h}{1+h^2}\,dh\right)$

$\int_0^{\sqrt{3}} \text{Tan}^{-1}(h)\,dh$



$= \sqrt{3} \times \frac{\pi}{3} - \int_0^{\frac{\pi}{3}} \text{Tan}^{-1}(h)\,dh$

$= \frac{\pi}{\sqrt{3}} - \int_0^{\frac{\pi}{3}} \frac{\sin(h)}{\cos(h)}\,dh$

let $u = \cos(h)$ so $du = -\sin(h)\,dh$

$= \frac{\pi}{\sqrt{3}} + \int_0^{0.5} \frac{1}{u}\,du$

$= \frac{\pi}{\sqrt{3}} + [\log_e n]^{0.5}$,

$= \frac{\pi}{\sqrt{3}} + \log_e 0.5$

$\int_0^{\sqrt{3}} \frac{h}{1+h^2}\,dh$

let $w = 1 + h^2$

$dw = 2h\,dh$

$0.5w = h\,dh$

$= \frac{1}{2}\int_1^4 \frac{1}{w}\,dw$

$= [\frac{1}{2}\log_e w]_1^4$

$= \frac{1}{2}\log_e 4$

$= \log_e 2$

So $t = 4(\frac{\pi}{\sqrt{3}} + \log_e \frac{1}{2} + \log_e 2)$

$= 4(\frac{\pi}{\sqrt{3}} + \log_e 1)$

$= \frac{4\pi}{\sqrt{3}}$ minutes

Question 4

(a) (i) Since a, b, c are real numbers the complex conjugate theorem applies so the third root is $-1 + 2i$

(ii) $z^3 + az^2 + bz + c = (z-4)(z+1+2i)(z+1-2i)$

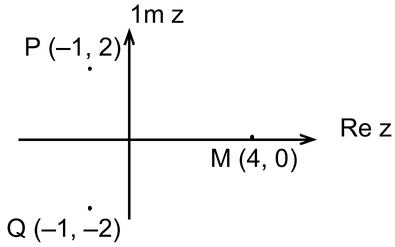
$= (z-4)((z+1)^2 - 4i^2)$

$= (z-4)(z^2 + 2z + 5)$

$= z^3 - 2z^2 - 3z - 20$

So $a = -2$, $b = -3$, $c = -20$

(b)



(c) $\overrightarrow{MQ} = \overrightarrow{MO} + \overrightarrow{OQ}$

$= -\overrightarrow{OM} + \overrightarrow{OQ}$

$= -(4\underset{\sim}{i} + 0\underset{\sim}{j}) + (-\underset{\sim}{i} - 2\underset{\sim}{j})$

$= -5\underset{\sim}{i} - 2\underset{\sim}{j}$

(d) **(i)** $\overrightarrow{DQ} = \overrightarrow{DO} + \overrightarrow{OQ}$

$= (-d\underset{\sim}{i}) + (-\underset{\sim}{i} - 2\underset{\sim}{j})$

$= (-1 - d)\underset{\sim}{i} - 2\underset{\sim}{j}$

(ii) Angle DQM = 90° since it is an angle in a semi circle

$\therefore \overrightarrow{DQ} \cdot \overrightarrow{QM} = 0$

$((-1 - d)\underset{\sim}{i} - 2\underset{\sim}{j}) \cdot (5i + 2j) = 0$

$-5 - 5d - 4 = 0$

$d = \frac{-9}{5}$

(e) Q is at $(-1, -2)$ so $-1 - 2i$

let $w = a + ib$

$(a + ib)(-1 - 2i) + (a - ib)(-1 + 2i) = 0 + 0i$

$-a + 2b - ib - 2ai - a + 2b + ib + 2ai = 0 + 0i$

$(-2a + 4b) + 0i + 0 + 0i$

so $a = 2b$

A possible w is $2 + i$

(f) T is a circle with centre at (b,k) and radius r

i.e. $(x - h)^2 + (y - k)^2 = r^2$

$r = |OC| = \sqrt{h^2 + k^2}$

and $r = |QC| = \sqrt{(h + 1)^2 + (k + 2)^2}$

so $h^2 + k^2 = (h + 1)^2 + (k + 2)^2$

$2h + 4k + 5 = 0$

let $h = 1 \therefore k = \frac{-7}{4}$ and $r = \sqrt{1 + \frac{49}{16}} = \frac{\sqrt{65}}{4}$

So a possible T is $\{ (x,y) : (x-1)^2 + (y + \frac{7}{4})^2 \le \frac{65}{16} \}$. Others are possible.

Question 5

(a) $1 - x^2 \neq 0 \quad \therefore \; x \neq \pm 1$

$1 - x^2 > 0 \quad$ so $x^2 < 1$
so domain $f = [0,1)$

(b) $x = 0.5$

$\therefore \; f(0.5) = 2(0.5)^{0.5} (1 - 0.5^2)^{0.25} + \frac{1}{(1-0.5^2)^{0.25}}$

i.e. D = 2.39

(c) $f'(x) = 2x^{0.5} - \frac{1}{4}(1 - x^2)^{-0.75}, - 2x + x^{-0.5}(1 - x^2)^{0.25} + \frac{-1}{4}(1 - x^2)^{-1.25} \bullet -2x$

$= - x^{1.5}(1 - x^2)^{-0.75} + x^{-0.5} (1 - x^2)^{0.25} + \frac{x}{2} (1 - x^2)^{-1.25}$

$f'(0.5) = 1.23557$

$\tan \theta = 1.23557$

so $\theta = 51.01°$

so angle slope makes with surface $= (90 - \theta)°$
$= 39°$ (to nearest degree)

(d) (i) $\left(f(x)\right)^2 = \left(2x^{0.5}(1 - x^2)^{0.25} + \frac{1}{(1-x^2)^{0.25}}\right)^2$

$= \left(2x^{0.5}(1 - x^2)^{0.25}\right)^2 + 2\left(2x^{0.5}(1-x^2)^{0.25}\right)\left(\frac{1}{(1-x^2)^{0.25}}\right) + \left(\frac{1}{(1-x^2)^{0.25}}\right)^2$

$= 4x\sqrt{1 - x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1-x^2}}$

So third term is $4\sqrt{x}$

(ii) $V = \pi \int y^2 \, dx$

$= \pi \int_0^{0.5} 4x\sqrt{1 - x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1-x^2}} \, dx$

$\int 4x\sqrt{1 - x^2} \, dx$

let $u = 1 - x^2$

$\frac{du}{dx} = -2x$

$-2 \, du = 4x \, dx$

$= -2 \ \sqrt{u}\ du$

$= -2\ .\ \frac{2}{3}\ u^{1.5}$

$= \frac{-4}{3}\ (1 - x^2\ )^{1.5}$

$\int 4\ \sqrt{x}\ dx$

$= 4.\ \frac{2}{3}\ x^{1.5}$

$= \frac{8}{3}\ x^{1.5}$

$\int \frac{1}{\sqrt{1 - x^2}}\ dx = \text{Sin}^{-1}\ (x)$

So $V = \pi\ [\frac{-4}{3}\ (1 - x^2\ )^{1.5} + \frac{8}{3} x^{1.5} + \text{Sin}^{-1}\ (x)\ ]_0^{0.5}$

$= \pi\ (1.9337)$

$= 6.07\ \text{m}^2$

$= 6.1\ \text{m}^3$ (to two significant figures)