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# SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2005

## Part I – Multiple-choice answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | **E** | **7.** | **B** | **13.** | **A** | **19.** | **C** | **25.** | **E** |
| **2.** | **C** | **8.** | **D** | **14.** | **E** | **20.** | **A** | **26.** | **D** |
| **3.** | **E** | **9.** | **A** | **15.** | **C** | **21.** | **A** | **27.** | **C** |
| **4.** | **A** | **10.** | **D** | **16.** | **D** | **22.** | **B** | **28.** | **E** |
| **5.** | **B** | **11.** | **D** | **17.** | **B** | **23.** | **C** | **29.** | **B** |
| **6.** | **E** | **12.** | **C** | **18.** | **B** | **24.** | **B** | **30.** | **C** |

## Part I- Multiple-choice solutions

### Question 1

$$f(x)=2x^2-5x+3$$

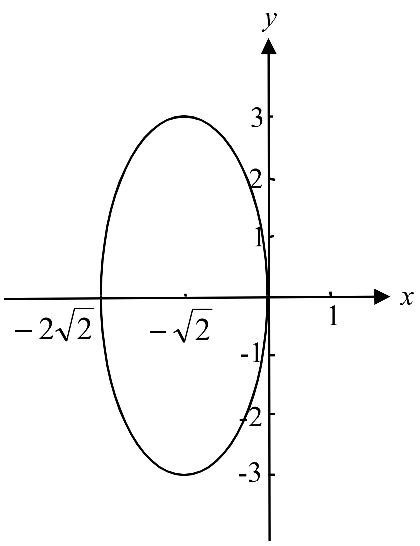
$$\frac{1}{f(x)}=\frac{1}{2x^2-5x+3}$$

$$=\frac{1}{(2x-3)(x-1)}$$

The graph of $y=\frac{1}{f(x)}$ has asymptotes at $x=\frac{3}{2}$ and at $x=1$.

The answer is E.

### Question 2




Four obvious tangents are located at

$x=-2\sqrt{2},\ y=3,\ x=0$ and $y=-3$.

The answer is C.

**Question 3**

The domain of $y = \text{Tan}^{-1}(2x)$ is $R$ and so the domain of $y = 2 + \text{Tan}^{-1}(2x)$ is also $R$.
The answer is E.

**Question 4**

$$y = \text{cosec}^2\left(\frac{x}{2}\right)$$
$$= \frac{1}{\sin^2\left(\frac{x}{2}\right)}$$

Asymptotes occur when the denominator of $\frac{1}{\sin^2\left(\frac{x}{2}\right)}$ is equal to zero.

Now, $\sin\left(\frac{x}{2}\right) = 0$ when $x = 0$ and then again at $x = \pm 2\pi$. However, we are interested in the graph from $x = -\pi$ to $x = \pi$. So the only asymptote occurs at $x = 0$.
The answer is A.

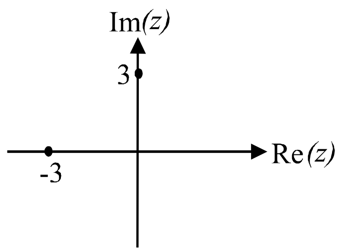
**Question 5**

$$\frac{1}{1+\bar{z}} = \frac{1}{1+(3-2i)}$$
$$= \frac{1}{4-2i}$$
$$= \frac{1}{4-2i} \times \frac{4+2i}{4+2i}$$
$$= \frac{4+2i}{20}$$
$$= \frac{1}{5} + \frac{1}{10}i$$

The answer is B.

**Question 6**

$z = 3i$
So $iz = -3$ and $|iz| = 3$
Also $\text{Arg}(-3) = \pi$
(Note that $-\pi < \text{Arg}\, z \le \pi$.)
The answer is E.



**Question 7**

$P(z)$ has real coefficients hence the roots of $P(z)=0$ occur in conjugate pairs.
So if $i$ is one root, then $-i$ is another.
So $(z-i)(z+i)=z^2+1$ is a factor

$$\begin{aligned}P(z)&=z^4+2z^3+3z^2+2z+2\\&=(z^2+1)(z^2+2z+2)\quad\text{(by inspection)}\\&=(z^2+1)((z^2+2z+1)+1)\\&=(z^2+1)((z+1)^2-i^2)\\&=(z^2+1)(z+1-i)(z+1+i)\end{aligned}$$

So all the roots are $\pm i,\ -1\pm i$.
The answer is B.

**Question 8**

$$z^3=8\text{cis}\left(\frac{3\pi}{2}\right)$$

Let $\quad z=r\text{cis}(\theta)$

so $\quad z^3=r^3\text{cis}(3\theta)\quad$ (De Moivres Theorem)

So $\quad r^3\text{cis}(3\theta)=8\text{cis}\left(\frac{3\pi}{2}\right)$

So $\quad r=2$ and $3\theta=\frac{3\pi}{2}+2\pi k,\quad k\in J$

$$3\theta=\frac{3\pi+4\pi k}{2}$$

$$\theta=\frac{3\pi+4\pi k}{6}$$

$k=0,\quad z=2\text{cis}\left(\frac{\pi}{2}\right)$

$k=1,\quad z=2\text{cis}\left(\frac{7\pi}{6}\right)$

$k=2,\quad z=2\text{cis}\left(\frac{11\pi}{6}\right)$

$k=3,\quad z=2\text{cis}\left(\frac{15\pi}{6}\right)$

$\quad\quad=2\text{cis}\left(\frac{\pi}{2}\right)$

The solutions begin to repeat. Check that your solutions are evenly spaced around a circle with radius 2.
The answer is D.

**Question 9**

The region is described by
$x + y \geq 4$
or $\text{Re}(z) + \text{Im}(z) \geq 4$
or $\text{Re}(\bar{z}) + \text{Im}(z) \geq 4$
since $z = x + yi,\ \bar{z} = x - yi$
so $\text{Re}(z) = \text{Re}(\bar{z})$.
The answer is A.

**Question 10**

$$\int \sin^4\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx$$

$$= \int u^4 \times 2\frac{du}{dx}\,dx \qquad u = \sin\left(\frac{x}{2}\right)$$

$$= 2\int u^4\,du \qquad \frac{du}{dx} = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$= 2\frac{u^5}{5} + c$$

$$= \frac{2}{5}\sin^5\left(\frac{x}{2}\right) + c$$

An antiderivative is $\frac{2}{5}\sin^5\left(\frac{x}{2}\right)$.
The answer is D.

**Question 11**

$$\int_0^3 2x\sqrt{1+x}\,dx$$

$u = 1 + x$ so $x = u - 1$

$$= 2\int_1^4 (u-1)\sqrt{u}\,\frac{du}{dx}\,dx \qquad \frac{du}{dx} = 1$$

$x = 3,\ u = 4$
$x = 0,\ u = 1$

$$= 2\int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

The answer is D.

**Question 12**

$$\int \frac{dx}{2x\sqrt{\log_e(x)}},\ x>0$$
$$=\frac{1}{2}\int \frac{1}{\sqrt{u}}\frac{du}{dx}dx$$
$$=\frac{1}{2}\int u^{-\frac{1}{2}}du$$
$$=\frac{1}{2}\times 2u^{\frac{1}{2}}+c$$
$$=\sqrt{\log_e(x)}+c$$

$$u=\log_e(x)$$
$$\frac{du}{dx}=\frac{1}{x}$$

The answer is C.

**Question 13**

$$\text{Area}=\int_0^{\frac{\pi}{6}}\cos^2(3x)dx$$
$$=\int_0^{\frac{\pi}{6}}\frac{1}{2}(1+\cos(6x))dx$$
$$=\frac{1}{2}\left[x+\frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{6}}$$
$$=\frac{1}{2}\left\{\left(\frac{\pi}{6}+\frac{1}{6}\sin(\pi)\right)-\left(0+\frac{1}{6}\sin(0)\right)\right\}$$
$$=\frac{1}{2}\left(\frac{\pi}{6}+0\right)$$
$$=\frac{\pi}{12}\text{ units}^2$$

since $\cos(2\theta)=2\cos^2(\theta)-1$

so $\cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$

The answer is A.

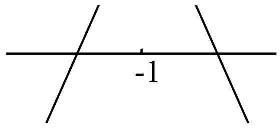
**Question 14**

Use a calculator to find this value. It is 5.8875 correct to 4 decimal places.
The answer is E.

**Question 15**

Stationary points occur when $f'(x) = 0$, that is at $x = -3, x = -1$ and $x = 2$.
So options A and B are incorrect.

Now $f'(x) > 0$ for $-3 < x < -1$ and $f'(x) < 0$ for $-1 < x < 2$ as indicated in the diagram so there is a local maximum at $x = -1$.
Options D and E are incorrect
The answer is C.




**Question 16**

$$\int_0^4 3\log_e(x^2+1)dx \approx \frac{2}{2}\left[3\log_e(1) + 6\log_e(5) + 3\log_e(17)\right]$$
$$= 6\log_e(5) + 3\log_e(17)$$
$$= \log_e(5)^6 + \log_e(17)^3$$
$$= \log_e\left(5^6 \times 17^3\right)$$

The answer is D.

**Question 17**

From the formulae sheet,

If $\frac{dy}{dx} = \frac{2x}{\sqrt{x^2+2}}$, $\quad x_0 = 1$ when $y_0 = 3$

then $x_1 = 1 \cdot 1$ and $y_1 = 3 + 0 \cdot 1 \times \frac{2 \times 1}{\sqrt{3}}$

$$= 3 + \frac{0.2}{\sqrt{3}}$$

so $x_2 = 1 \cdot 2$

and $y_2 = 3 + \frac{0 \cdot 2}{\sqrt{3}} + 0 \cdot 1 \times \frac{2 \times 1.1}{\sqrt{1.1^2 + 2}}$

$= 3.2383$ (to 4 decimal places)

The answer is B.

**Question 18**

$$V = 500\text{Sin}^{-1}\left(\frac{x}{10}\right)$$

$$\frac{dV}{dx} = \frac{500}{\sqrt{100 - x^2}}$$

Now $\quad \frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt} \qquad$ (Chain rule)

$$= \frac{\sqrt{100 - x^2}}{500} \times 10$$

$$= \frac{\sqrt{100 - x^2}}{50}$$

When $\quad x = 5,$

$$\frac{dx}{dt} = \frac{\sqrt{75}}{50}$$

$$= \frac{\sqrt{3}}{10} \text{ cm/sec}$$

The answer is B.

**Question 19**

Firstly, only options A and C have a magnitude of 2.

Secondly, $\left(\underset{\sim}{i} + \sqrt{2}\,\underset{\sim}{j} + \underset{\sim}{k}\right) \bullet \left(\underset{\sim}{i} + 2\,\underset{\sim}{j} - 2\sqrt{5}\,\underset{\sim}{k}\right)$

$= 1 + 2\sqrt{2} - 2\sqrt{5}$

$\neq 0$

and

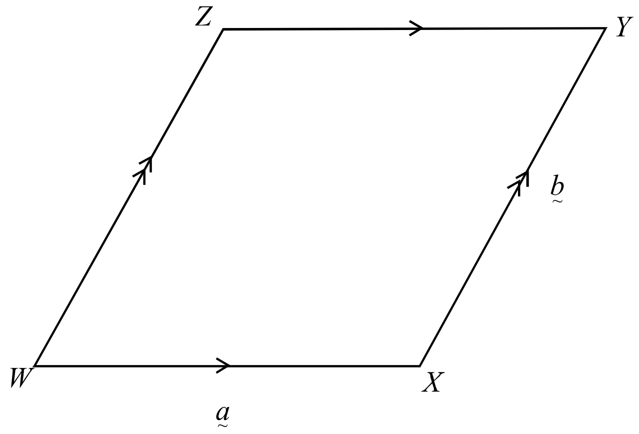
$\frac{2}{5}\left(2\,\underset{\sim}{i} + 4\,\underset{\sim}{j} + \sqrt{5}\,\underset{\sim}{k}\right) \bullet \left(\underset{\sim}{i} + 2\,\underset{\sim}{j} - 2\sqrt{5}\,\underset{\sim}{k}\right)$

$= \frac{2}{5} \times (2 + 8 - 10)$

$= 0$

The answer is C.

**Question 20**




Now, $\overrightarrow{WY} = \underset{\sim}{a} + \underset{\sim}{b}$ so C and D are incorrect.

Also, $|\overrightarrow{WY}| = |\underset{\sim}{a} + \underset{\sim}{b}|$ .

Note that $|\underset{\sim}{a}| + |\underset{\sim}{b}| > |\overrightarrow{WY}|$

The answer is A.

**Question 21**

The scalar resolute of $2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ in the direction of $\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ is given by

$$\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right) \bullet \frac{1}{\sqrt{1+1+9}}(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$$
$$= \frac{1}{\sqrt{11}}(2 - 1 - 3)$$
$$= -\frac{2}{\sqrt{11}}$$

The answer is A.

**Question 22**

The ball hits the ground when $v = 0$; that is at $t = 1$. The displacement of the ball in the first second is $\frac{1}{2} \times 2 \times 1 = 1\text{m}$.

The answer is B.

**Question 23**

$$\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j},\ \ 0 \le t \le \frac{\pi}{2}$$

$x = \sin(t) \qquad y = \cos(2t)$

$\qquad\qquad\qquad = 1 - 2\sin^2(t)$

So, $\qquad\qquad = 1 - 2x^2 \qquad 0 \le x \le 1$

The answer is C.

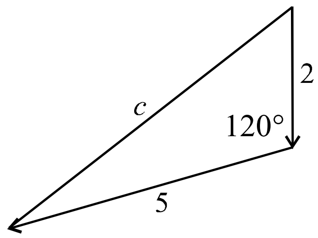
**Question 24**

$c^2 = 4 + 25 - 20\cos(120^\circ)$

$\quad = 29 + 20 \times \frac{1}{2}$

$c^2 = 39$

$c = \sqrt{39}$




The answer is B.

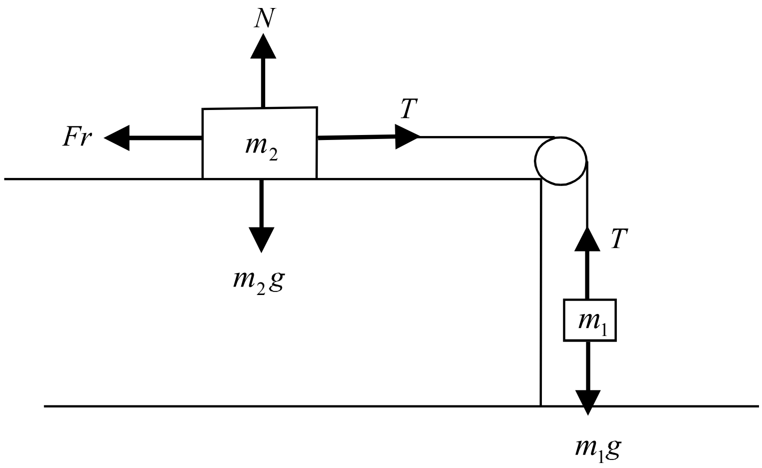
**Question 25**

Initial momentum is $3 \times 5 = 15$ kg m/s.
Momentum after 2 seconds is $3 \times 8 = 24$ kg m/s.
Change in momentum is $24 - 15 = 9$ kg m/s.
The answer is E.

**Question 26**

All the normal forces and gravitational forces are correct. All the diagrams that include *P* are possible. Option D is not possible because if there is no *P* force, then the tendency of the mass is to slip down the plane and hence the friction force opposes this and should be directed up the plane.
The answer is D.

**Question 27**




Since $m_2$ is at the point of moving, $Fr = \mu N$ .

So, $Fr = T$ and $T = m_1 g$

So, $Fr = m_1 g$

and $\mu N = m_1 g$

$$\mu = \frac{m_1 g}{N}$$

Since $N = m_2 g$,

$$\mu = \frac{m_1 g}{m_2 g}$$

$$= \frac{m_1}{m_2}$$

The answer is C.

**Question 28**

The amount of substance left to react is $S_0 - S$ .

So $\dfrac{dS}{dt} = k(S_0 - S)$.

The answer is E.

**Question 29**

Since the acceleration is constant we can use the constant acceleration formulae.
Now, $v = u + at$ gives us $v = t$ if $u = 0$ since $a = 1$ so option A is possible.
Using the same formula, option B is not possible.
Using $x = ut + \frac{1}{2}at^2$ gives us $2x = t^2$ if $u = 0$ since $a = 1$ so option C is possible.
Using $v^2 = u^2 + 2ax$ gives us $2x = v^2$ if $u = 0$ since $a = 1$ so option D is possible.
Using $x = ut + \frac{1}{2}at^2$ gives us $x = \frac{t^2 + 2t}{2}$ if $u = 1$ since $a = 1$ so option E is possible.
So B is not true.
The answer is B.

**Question 30**

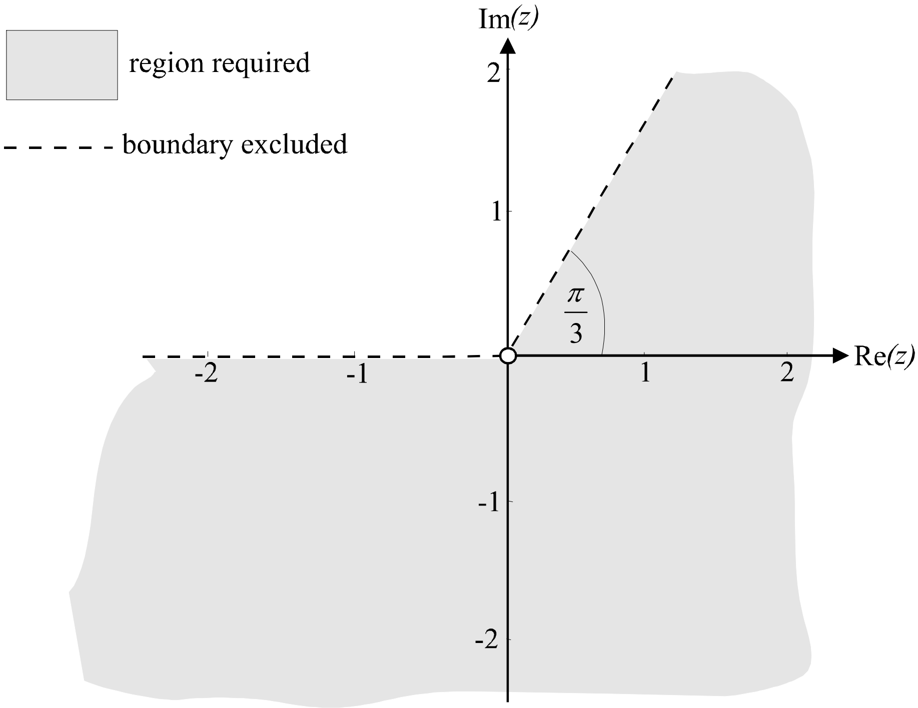
Between $t = 0$ and $t = 2$ the gradient of the *v/t* graph is positive and increasing from zero.
This rules out options B, D and E.
Between $t = 2$ and $t = 5$ the gradient of the *v/t* graph is zero and hence the acceleration is zero.
This rules out options A and E.
Between $t = 5$ and $t = 7$ the gradient is positive but decreasing to zero.
This rules out options B, D and E.
Only option C describes all these three features.
The answer is C.

**PART II**

**Question 1**

Now $\quad -\pi < \text{Arg}\, z \le \pi$

We require $\left\{z : \text{Arg}\, z < \dfrac{\pi}{3}\right\}$.




**(1 mark)** correct region
**(1 mark)** correct marking of boundaries

**Question 2**

$$z^6 = 729$$

$$z^6 - 729 = 0$$

$$(z^3 - 27)(z^3 + 27) = 0 \quad \textbf{(1 mark)}$$

$$(z-3)(z^2+3z+9)(z+3)(z^2-3z+9) = 0 \quad \textbf{(1 mark)}$$

$$(z-3)\left(\left(z^2+3z+\frac{9}{4}\right)-\frac{9}{4}+9\right)(z+3)\left(\left(z^2-3z+\frac{9}{4}\right)-\frac{9}{4}+9\right)=0$$

$$(z-3)\left(\left(z+\frac{3}{2}\right)^2-\frac{27i^2}{4}\right)(z+3)\left(\left(z-\frac{3}{2}\right)^2-\frac{27i^2}{4}\right)=0$$

$$(z-3)\left(z+\frac{3}{2}-\frac{\sqrt{27}i}{2}\right)\left(z+\frac{3}{2}+\frac{\sqrt{27}i}{2}\right)(z+3)\left(z-\frac{3}{2}-\frac{\sqrt{27}i}{2}\right)\left(z-\frac{3}{2}+\frac{\sqrt{27}i}{2}\right)=0$$

So, $\quad z = \pm 3,\ -\dfrac{3}{2} \pm \dfrac{3\sqrt{3}i}{2},\ \dfrac{3}{2} \pm \dfrac{3\sqrt{3}i}{2}$

**(1 mark)**

**Question 3**

$$\frac{(x-1)^2}{4}-\frac{(y+2)^2}{9}=1$$

The centre is at $(1,-2)$.

The asymptotes are $y+2=\pm\frac{3}{2}(x-1)$

$$y+2=\frac{3}{2}(x-1) \text{ or } y+2=-\frac{3}{2}(x-1)$$

$$y=\frac{3}{2}x-\frac{7}{2} \qquad y=-\frac{3}{2}x-\frac{1}{2}$$

<u>$x$-intercepts</u>

$y=0$ $\quad \frac{(x-1)^2}{4}-\frac{4}{9}=1$

$$\frac{(x-1)^2}{4}=\frac{13}{9}$$

$$(x-1)^2=\frac{52}{9}$$

$$x-1=\pm\sqrt{\frac{52}{9}}$$

$$x=1\pm\frac{2\sqrt{13}}{3}$$

$$(x=3\cdot 4 \text{ or } -1\cdot 4)$$

<u>$y$-intercepts</u>

$x=0$ $\quad \frac{1}{4}-\frac{(y+2)^2}{9}=1$

$$-\frac{(y+2)^2}{9}=\frac{3}{4}$$

$$-(y+2)^2=\frac{27}{4}$$

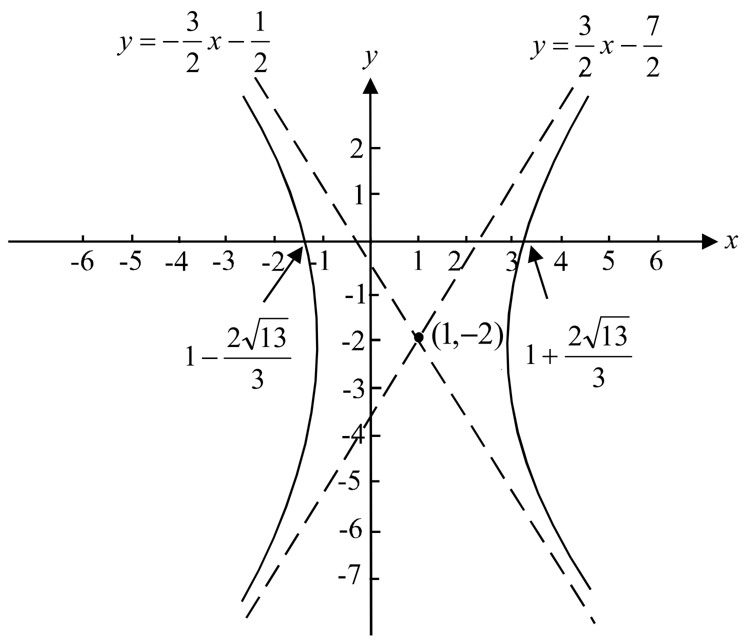
$$(y+2)^2=-\frac{27}{4}$$

$$y^2+4y+\frac{43}{4}=0$$

$$\Delta=16-4\times 1\times\frac{43}{4}$$

$$<0$$

There are no solutions and hence no $y$-intercepts.




**(1 mark)** – centre
**(1 mark)** – asymptotes
**(1 mark)** – intercepts
**(1 mark)** – shape of graph

**Question 4**

$$y = 4x^2 \text{Tan}^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = 8x\text{Tan}^{-1}\left(\frac{x}{a}\right) + 4x^2 \times \frac{a}{x^2 + a^2}$$

$$= 8x\text{Tan}^{-1}\left(\frac{x}{a}\right) + \frac{4ax^2}{x^2 + a^2}$$ **(1 mark)**

$$\text{LS} = \frac{dy}{dx} - \frac{2y}{x}$$

$$= 8x\text{Tan}^{-1}\left(\frac{x}{a}\right) + \frac{4ax^2}{x^2 + a^2} - \frac{8x^2\text{Tan}^{-1}\left(\frac{x}{a}\right)}{x}$$ **(1 mark)**

$$= \frac{4ax^2}{x^2 + a^2}$$

$$\text{RS} = 12 - \frac{108}{x^2 + 9}$$

$$= \frac{12(x^2 + 9) - 108}{x^2 + 9}$$

$$= \frac{12x^2}{x^2 + 9}$$

Now LS = RS

So $4a = 12$ and $a^2 = 9$

$a = 3$ $a = \pm 3$

So $a = 3$ (Note that $a = -3$ does not satisfy both criteria.)

**(1 mark)**

**Question 5**

a. $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(2\underset{\sim}{i} + 5\underset{\sim}{j}) + 4\underset{\sim}{i} + 3\underset{\sim}{j}$

$= 2\underset{\sim}{i} - 2\underset{\sim}{j}$ **(1 mark)**

b. $$\overrightarrow{AC} = \overrightarrow{AB} - \left(\overrightarrow{AB} \bullet \hat{\overrightarrow{OB}}\right)\hat{\overrightarrow{OB}}$$

$$= 2\underset{\sim}{i} - 2\underset{\sim}{j} - \left(\left(2\underset{\sim}{i} - 2\underset{\sim}{j}\right) \cdot \frac{1}{5}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)\right)\frac{1}{5}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$ **(1mark)**

$$= 2\underset{\sim}{i} - 2\underset{\sim}{j} - \frac{2}{25}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$

$$= \frac{1}{25}\left(42\underset{\sim}{i} - 56\underset{\sim}{j}\right)$$

**(1 mark)**

**c.** Find $\overrightarrow{OC}$.

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= 2\underset{\sim}{i} + 5\underset{\sim}{j} + \frac{1}{25}\left(42\underset{\sim}{i} - 56\underset{\sim}{j}\right)$$
$$= \frac{1}{25}\left(92\underset{\sim}{i} + 69\underset{\sim}{j}\right)$$ **(1 mark)**

$C$ is the point $\left(\frac{92}{25}, \frac{69}{25}\right)$. **(1 mark)**

**Question 6**

Let $\frac{1}{x(x-2)} \equiv \frac{A}{x} + \frac{B}{x-2}$

$$\frac{1}{x(x-2)} \equiv \frac{A(x-2)+Bx}{x(x-2)}$$

True iff $1 \equiv A(x-2) + Bx$

Put $x = 2$, $1 = 2B$ $B = \frac{1}{2}$

Put $x = 0$, $1 = -2A$ $A = -\frac{1}{2}$

So $\frac{1}{x(x-2)} = \frac{-1}{2x} + \frac{1}{2(x-2)}$ **(1 mark)**

If you have time, check your answer

i.e. $\frac{-1}{2x} + \frac{1}{2(x-2)} = \frac{-(x-2)+x}{2x(x-2)}$

$$= \frac{1}{x(x-2)}$$

So $\int_e^5 \frac{1}{x(x-2)}dx = \int_e^5 \left(\frac{-1}{2x} + \frac{1}{2(x-2)}\right)dx$

$$= \frac{1}{2}\int_e^5 \left(-\frac{1}{x} + \frac{1}{x-2}\right)dx$$ **(1 mark)**

$$= \frac{1}{2}\left[-\log_e(x) + \log_e(x-2)\right]_e^5$$
$$= \frac{1}{2}\left[\log_e\left(\frac{x-2}{x}\right)\right]_e^5$$
$$= \frac{1}{2}\left\{\log_e\left(\frac{3}{5}\right) - \log_e\left(\frac{e-2}{e}\right)\right\}$$
$$= \frac{1}{2}\log_e\left(\frac{3}{5} \times \frac{e}{e-2}\right)$$
$$= \frac{1}{2}\log_e\left(\frac{3e}{5(e-2)}\right)$$

**(1 mark)**
**Total 20 marks**