THE HEFFERNAN GROUP

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SPECIALIST MATHEMATICS

TRIAL EXAMINATION 2

(ANALYSIS TASK)

2005

Reading Time: 15 minutes
Writing time: 90 minutes

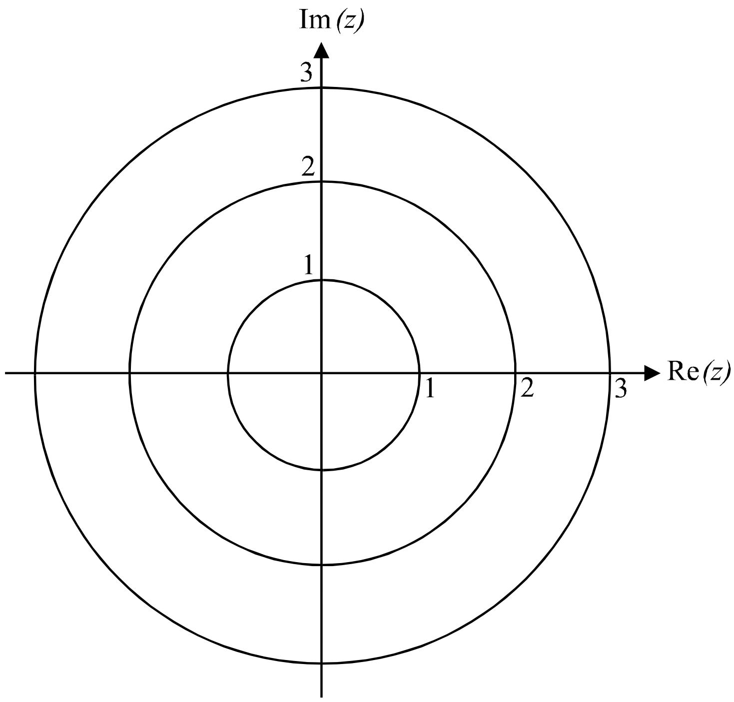
Instructions to students

This exam consists of 5 questions.
All questions should be answered.
There is a total of 60 marks available.
The marks allocated to each of the five questions are indicated throughout.
Students may bring up to two A4 pages of pre-written notes into the exam.
The acceleration due to gravity should be taken to have magnitude g m/s^2 where $g = 9.8$
Formula sheets can be found on pages 15-17 of this exam.

Question 1

a. Let $u = 2\text{cis}\left(\frac{\pi}{6}\right)$.

i. Indicate the complex number u on the Argand diagram below.



1 mark

ii. Given that u is a root of the equation $z^3 = a$ where $a \in C$, find the other two roots in polar form.

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2 marks

iii. Hence find a.

2 marks

b. **i.** Find $\overline{u}$ in Cartesian form.

1 mark

ii. Show algebraically that if $z = x + iy,\ x, y \in R$, and $|z - u| = |z - \overline{u}|$ then $z = x$.

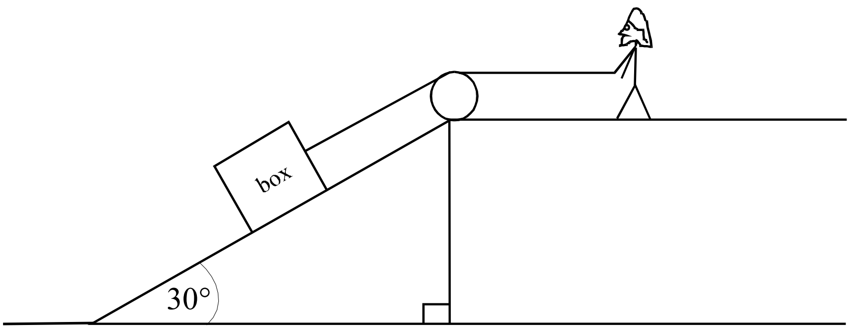
2 marks

iii. Using a diagram, explain why it is that if $|z - u| = |z - \overline{u}|$ then $z = x$.

1 mark
Total 9 marks

Question 2

A box of mass 10kg is being hauled up a 5 metre long ramp that is at an angle of 30° to the horizontal. At the top of the ramp is a horizontal floor.
The box is attached to one end of a rope that passes over a smooth pulley and is held at the other end by a man.
The man exerts a constant pulling force of 200 N on the rope. The coefficient of friction between the box and the ramp is $\sqrt{3}$.



a. Find the acceleration of the box up the ramp.

4 marks

b. Given that the box started from rest at the lower end of the ramp, how long does it take for the box to be hauled halfway up the ramp? Express your answer as an exact value.

2 marks

c. When the box is at this point, halfway up the ramp, the man lets go of the rope and the box becomes stationary.

Draw a diagram showing all the forces acting on the box and explain whether or not the box is at the point of slipping down the ramp.

3 marks

d. If the coefficient of friction between the ramp and the box was $\frac{1}{2}$, the box would slide down the ramp once stationary after the man let go of the rope. How far would the box travel in this situation in the first second after coming to rest? Express your answer to the nearest centimetre.

3 marks
Total 12 marks

Question 3

The rate of change in the temperature *T*, in degrees *C*, in a living room, *t* hours after a heater is switched on is given by

$$\frac{dT}{dt} = \sqrt{400 - T^2}, \ t \geq 0$$

The temperature in the living room at the time that the heater was turned on was $\left(10\sqrt{3}\right)^\circ$ (approximately $17 \cdot 32^\circ$).

a. Find the function $T(t)$.

4 marks

b. Find the values of *t* for which the function $T(t)$ is defined.

2 marks

c. By considering the sign of the function $\frac{dT}{dt}$, explain why the maximum temperature occurs at $t = \frac{\pi}{6}$.

2 marks
Total 8 marks

Question 4

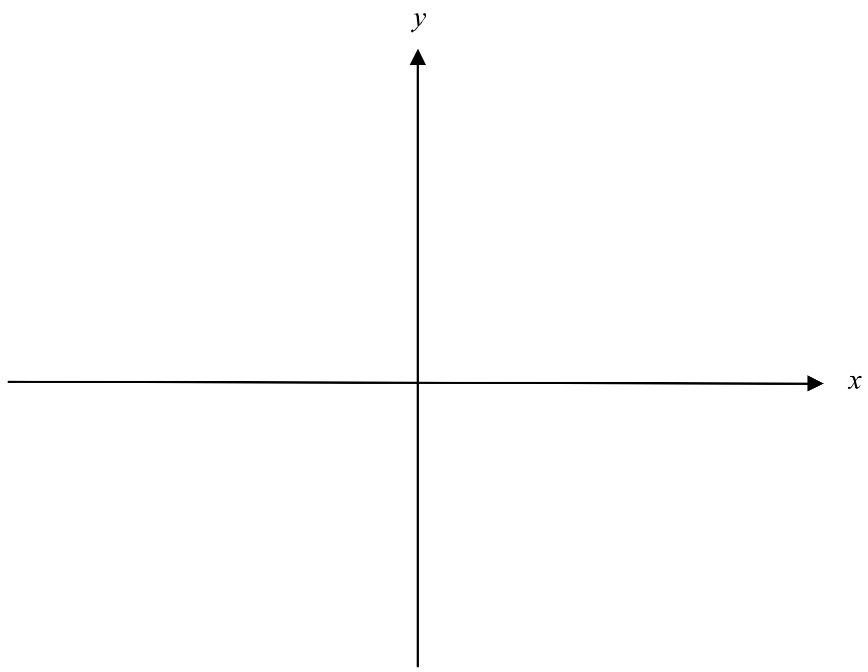
Consider the function

$$f : D \to R \text{ where } f(x) = \frac{e^x}{\sqrt{1+e^x}} + \sqrt{1+e^x}$$

a. Given that D is the maximal domain of f, find D.

1 mark

b. Sketch the graph of $y = f(x)$ on the set of axes below clearly showing any important features.



3 marks

c. On the same set of axes sketch the graph of the inverse function $y = f^{-1}(x)$.

1 mark

d. Use calculus to show that the graph of $y = f(x)$ has no stationary points.

4 marks

e. **i.** Show that $y^2 = \dfrac{e^{2x}}{1+e^x} + 3e^x + 1$.

1 mark

ii. If $\dfrac{e^{2x}}{1+e^x} = a + \dfrac{b}{1+e^x}$, find the terms that a and b represent.

1 mark

f. The area enclosed by the function *f*, the positive *x* and *y*-axes and the line $x = 1$ is rotated around the *x*-axis to create a solid of revolution.
Find the exact volume of this solid of revolution.

5 marks
Total 16 marks

Question 5

A boy drops a feather from the roof of a shed. The boy hopes that the feather will drop on the head of his sister who is lying on the ground directly below the point where he releases the feather.
Instead, the feather is taken by the breeze and floats to the ground away from the boy's sister. The position vector $\underset{\sim}{r}(t)$ of the feather at time *t* seconds after the boy releases it is given by

$$\underset{\sim}{r} = \sin\left(\frac{t}{10}\right)\underset{\sim}{i} + \left(1 - \cos\left(\frac{t}{10}\right)\right)\underset{\sim}{j} - \log_e\left(\frac{t+1}{20}\right)\underset{\sim}{k}, \quad t \geq 0$$

where $\underset{\sim}{i}$ is a unit vector in the east direction, $\underset{\sim}{j}$ is a unit vector in the north direction and $\underset{\sim}{k}$ is a unit vector directed vertically up.

The position of the sister's head can be taken as *O*, the origin of the coordinate system. All distances are measured in metres.

a. Find the height above the ground of the point where the feather was released. Express your answer to the nearest metre.

2 marks

b. Show that it takes 19 seconds for the feather to reach the ground.

1 mark

c. What is the distance of the feather from the boy's sister 10 seconds after he releases it? Express your answer in metres correct to 2 decimal places.

2 marks

d. Find when the feather is directly north-east of the boy's sister for $t > 0$. Express your answer in seconds correct to one decimal place.

2 marks

e. Find an expression for the velocity of the feather.

1 mark

f. Find the exact speed of the feather at $t = 9$ seconds.

2 marks

g. At what angle does the path of the feather make with the north direction as it reaches the ground. Express your answer in degrees and minutes to the nearest minute.

3 marks

h. What is the furthest distance east of the origin that the feather reaches?

2 marks
Total 15 marks

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2=a^2+b^2-2ab\cos C$

Coordinate geometry

ellipse:	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
hyperbola:	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=\sec^2(x)$ $\qquad$ $\cot^2(x)+1=\operatorname{cosec}^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ $\qquad$ $\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$ $\qquad$ $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ $\qquad$ $\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

$\sin(2x)=2\sin(x)\cos(x)$ $\qquad$ $\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\text{cis}\,\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < \text{Arg}\, z \le \pi$

$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\qquad \displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$

$\dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\qquad \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$ $\qquad \displaystyle\int \frac{1}{x}\,dx = \log_e(x) + c, \text{ for } x > 0$

$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\qquad \displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\qquad \displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\qquad \displaystyle\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}\left(\text{Sin}^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\dfrac{d}{dx}\left(\text{Cos}^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \text{Cos}^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\dfrac{d}{dx}\left(\text{Tan}^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\qquad \displaystyle\int \frac{a}{a^2+x^2}\,dx = \text{Tan}^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

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mid-point rule: $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)dx \approx \frac{1}{2}(b-a)(f(a)+f(b))$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$,
then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\quad s = ut + \frac{1}{2}at^2$

$v^2 = u^2 + 2as$ $\quad s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$\left|\underset{\sim}{r}\right| = \sqrt{x^2 + y^2 + z^2} = r$ $\quad \underset{\sim}{r_1} . \underset{\sim}{r_2} = r_1 r_2 \cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

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