Year 2005

VCE
Specialist Mathematics
Trial Examination 2

Suggested Solutions

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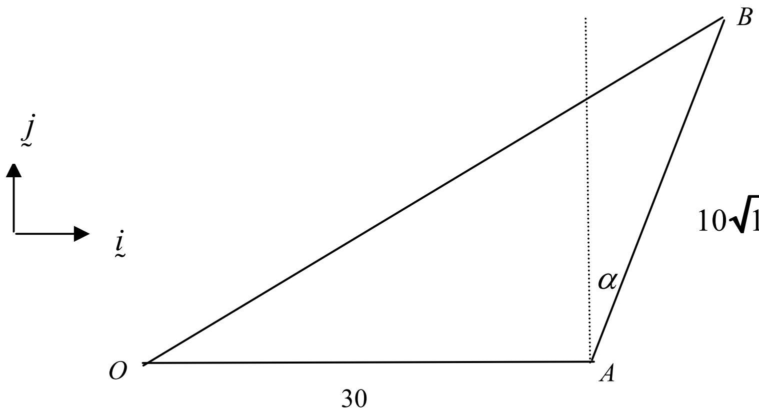
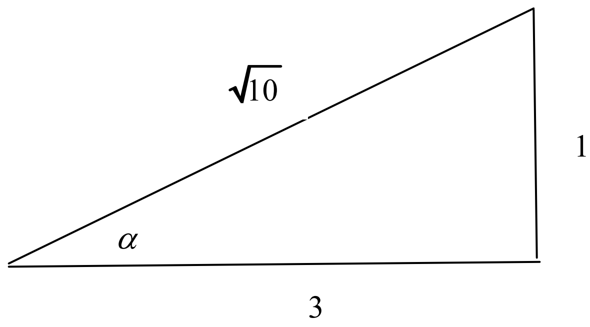
Question 1

a.

i.





From Pythagorus

$\tan(\alpha) = \frac{1}{3}$

$\cos(\alpha) = \frac{3}{\sqrt{10}}$

$\sin(\alpha) = \frac{1}{\sqrt{10}}$

$$\overrightarrow{OA} = 30\underset{\sim}{i} \qquad \overrightarrow{AB} = 10\sqrt{10}\sin(\alpha)\underset{\sim}{i} + 10\sqrt{10}\cos(\alpha)\underset{\sim}{j} = 10\underset{\sim}{i} + 30\underset{\sim}{j}$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 40\underset{\sim}{i} + 30\underset{\sim}{j}$$

1 mark

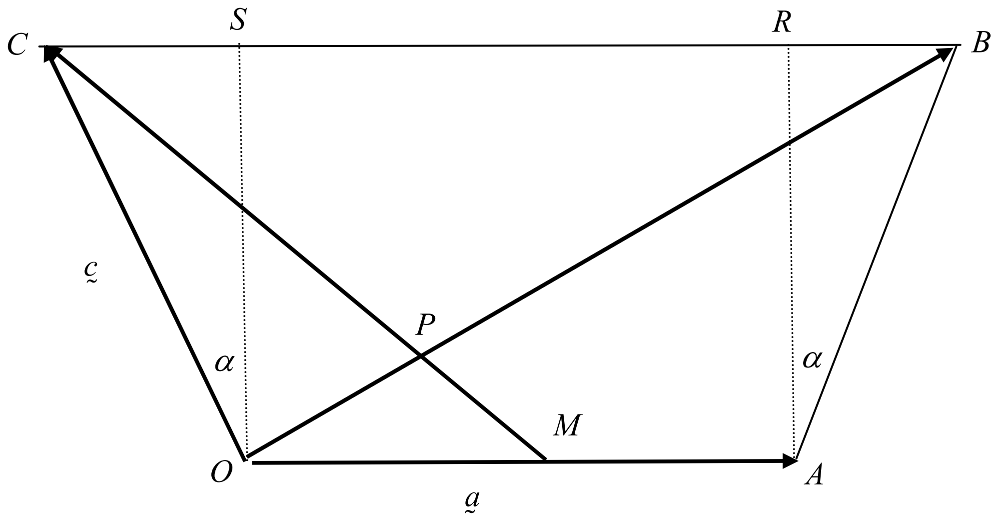
ii. Now $\left|\overrightarrow{OA}\right| = 30$ and $\left|\overrightarrow{OB}\right| = \sqrt{40^2 + 30^2} = 50$ and $\overrightarrow{OA}.\overrightarrow{OB} = 30 \text{ x } 40 = 1200$

$$\cos(\theta) = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|} = \frac{1200}{30 \text{ x } 50} = \frac{4}{5}$$

$$\theta = \text{Cos}^{-1}\left(\frac{4}{5}\right) = 36^0 52'$$

2 marks

b.



i. Now $\overrightarrow{OA} = \overrightarrow{SR} = \underset{\sim}{a}$ and $\overrightarrow{CS} = \overrightarrow{RB} = \frac{1}{3}\underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CS} + \overrightarrow{SR} + \overrightarrow{RB}$$

$$\overrightarrow{OB} = \underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a} + \underset{\sim}{a} + \frac{1}{3}\underset{\sim}{a}$$

$$\overrightarrow{OB} = \frac{1}{3}\left(5\underset{\sim}{a} + 3\underset{\sim}{c}\right)$$

1 mark

ii. Now $\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{MC} = \overrightarrow{MO} + \overrightarrow{OC} = -\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c}$ so

$$\overrightarrow{MP} = \frac{3}{13}\overrightarrow{MC}$$

$$\overrightarrow{MP} = \frac{3}{13}\left(-\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c}\right)$$

$$\overrightarrow{MP} = \frac{3}{26}\left(2\underset{\sim}{c} - \underset{\sim}{a}\right)$$

2 marks

iii. $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \frac{1}{2}\underset{\sim}{a} + \frac{3}{26}\left(2\underset{\sim}{c} - \underset{\sim}{a}\right)$

$$\overrightarrow{OP} = \left(\frac{1}{2} - \frac{3}{26}\right)\underset{\sim}{a} + \frac{3}{13}\underset{\sim}{c}$$

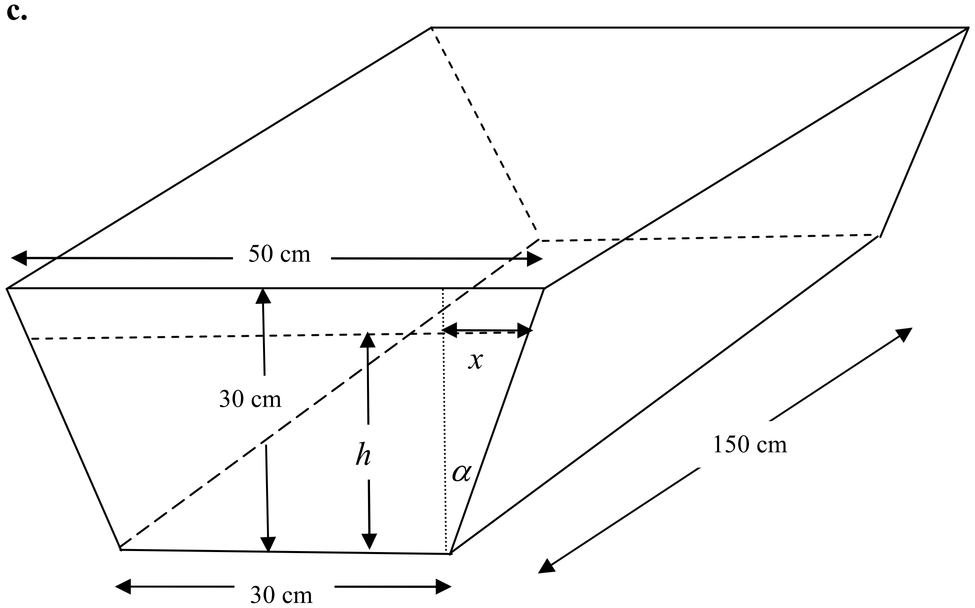
$$\overrightarrow{OP} = \frac{5}{13}\underset{\sim}{a} + \frac{3}{13}\underset{\sim}{c} = \frac{1}{13}\left(5\underset{\sim}{a} + \underset{\sim}{c}\right)$$

3 marks

$$\overrightarrow{OP} = \frac{3}{13}\overrightarrow{OB}$$

Hence the points O , P , B are collinear and $OP : OB = \frac{3}{13}$

c.



i. From the diagram, the cross-section and by similar triangles

$\tan(\alpha) = \frac{x}{h} = \frac{1}{3}$ so that $x = \frac{h}{3}$

The cross-sectional area (the area of the trapezium)

$A = \frac{1}{2}\left(30 + (2x + 30)\right)h = (30 + x)h$

$A = \left(30 + \frac{h}{3}\right)h$ the volume at height h is given by

$V = A\,h = \frac{1}{3}(90 + h)150h = 50h(90 + h) = 50\left(90h + h^2\right)$

Now $\frac{dV}{dh} = 50(90 + 2h)$ and we are given that $\frac{dV}{dt} = -80\sqrt{h}$

By the Chain rule $\frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt} = \frac{-80\sqrt{h}}{50(90 + 2h)} = \frac{-4\sqrt{h}}{5(45 + h)}$

2 marks

ii. Inverting gives $\dfrac{dt}{dh}=\dfrac{5(45+h)}{-4\sqrt{h}}=-\dfrac{5}{4}\left(45h^{-\frac{1}{2}}+h^{\frac{1}{2}}\right)$ integrating wrt h

$$t=-\frac{5}{4}\int_0^{25}\left(45h^{-\frac{1}{2}}+h^{\frac{1}{2}}\right)dh$$

$$t=-\frac{5}{4}\left[90h^{\frac{1}{2}}+\frac{2}{3}h^{\frac{3}{2}}\right]_0^{25}=-\frac{5}{4}\left[90\sqrt{25}+\frac{2}{3}(25)^{\frac{3}{2}}-0\right]$$

$$t=666\frac{2}{3}\ \text{minutes}$$

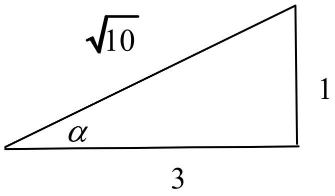
$$t=11\frac{1}{9}\ \text{hours}$$

3 marks

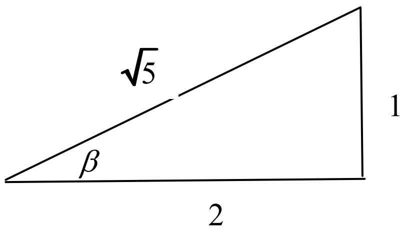
Question 2

Let $\alpha = \text{Tan}^{-1}\left(\frac{1}{3}\right)$ $\beta = \text{Tan}^{-1}\left(\frac{1}{2}\right)$ $u = 3+i$ and $v = 2+i$

From Pythagorus Theorem



$$\cos(\alpha) = \frac{3}{\sqrt{10}}$$
$$\sin(\alpha) = \frac{1}{\sqrt{10}}$$



$$\cos(\beta) = \frac{2}{\sqrt{5}}$$
$$\sin(\beta) = \frac{1}{\sqrt{5}}$$

a. $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$$\cos(\alpha+\beta) = \frac{3}{\sqrt{10}}.\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}}.\frac{1}{\sqrt{5}} = \frac{6-1}{\sqrt{50}} = \frac{5}{\sqrt{25.2}} = \frac{\sqrt{2}}{2}$$

2 marks

b. $v = 2+i$

$$v^2 = (2+i)^2 = 4+4i+i^2 = 3+4i$$

$$\text{Im}(v^2) = 4 \quad |v^2| = \sqrt{9+16} = \sqrt{25} = 5$$

$$\sin(2\beta) = \frac{\text{Im}(v^2)}{|v^2|} = \frac{4}{5}$$

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta) = 2\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}} = \frac{4}{5}$$

2 marks

c. $uv = (3+i)(2+i) = 6+2i+3i+i^2 = 5+5i$

$$\text{Arg}(uv) = \text{Tan}^{-1}\left(\frac{5}{5}\right) = \text{Tan}^{-1}(1) = \frac{\pi}{4} \quad \text{and}$$

$$\text{Arg}(uv) = Arg(u) + Arg(v) = \alpha+\beta \quad \text{so} \quad \alpha+\beta = \frac{\pi}{4}$$

2 marks

d. $u = 3 + i$ and $\overline{u} = 3 - i$ $\quad u + \overline{u} = 6$ $\quad u\overline{u} = 9 - i^2 = 10$

Since a and b are real numbers by the conjugate root theorem

$z^2 - 6z + 10$ is a factor

$z^3 + az^2 + bz + 20 = 0$

$(z^2 - 6z + 10)(z + 2)$

expanding gives coefficient of z^2: $a = 2 - 6 = -4$ $\quad z$: $b = 10 - 12 = -2$

$a = -4$ $b = -2$ the roots are $z = 3 \pm i$ and $z = -2$

2 marks

e. $Q(z) = z^3 - (2 + i)z^2 + 5z - 10 - 5i = 0$

$Q(2 + i) = (2 + i)^3 - (2 + i)(2 + i)^2 + 5(2 + i) - 10 - 5i$

$Q(2 + i) = (2 + i)^3 - (2 + i)^3 + 10 + 5i - 10 - 5i = 0$ shown

So $z = 2 + i$ is a root $(z - 2 - i)$ is a factor

$Q(z) = z^3 - (2 + i)z^2 + 5z - 10 - 5i = 0$

$Q(z) = (z - 2 - i)(z^2 + 5) = (z - 2 - i)(z + \sqrt{5}i)(z - \sqrt{5}i) = 0$

the roots are $z = 2 + i$ and $z = \pm\sqrt{5}\,i$

2 marks

Question 3

a. $u = 10$, $v = 5$ and $a = -2.5$ $t = ?$ $s = ?$

using constant acceleration formulae

$v = u + at$

$5 = 10 - 2.5t$

$2.5t = 5$

$t = 2$ sec

$s = \left(\dfrac{u+v}{2}\right)t$

$s = \left(\dfrac{5+10}{2}\right)2$

$s = 15$ metres

2 marks

b.i. from Newton's Law $800\ddot{x} = -40v^2$

$\ddot{x} = v\dfrac{dv}{dx} = -\dfrac{v^2}{20}$

$\dfrac{dv}{dx} = -\dfrac{v}{20}$

1 mark

ii. $\displaystyle\int \frac{dv}{v} = -\frac{1}{20}\int dx$

$\log_e(v) = -\dfrac{x}{20} + C_1$

to find C_1 use $v = 10$ when $x = 0$

$C_1 = \log_e(10)$

$\log_e(v) = -\dfrac{x}{20} + \log_e(10)$

$\log_e(v) - \log_e(10) = -\dfrac{x}{20}$

$\log_e\left(\dfrac{v}{10}\right) = -\dfrac{x}{20}$

$x = -20\log_e\left(\dfrac{v}{10}\right)$ Now when $v = 5$

$x = -20\log_e\left(\dfrac{5}{10}\right) = 20\log_e(2)$

3 marks

iii. $\ddot{x} = \dfrac{dv}{dt} = -\dfrac{v^2}{20}$

$\displaystyle\int \frac{dv}{v^2} = -\frac{1}{20}\int dt = \frac{-t}{20} + C_2$

$-\dfrac{1}{v} = \dfrac{-t}{20} + C_2$

to find C_2 use $v = 10$ when $t = 0$

$C_2 = -\dfrac{1}{10}$

$-\dfrac{1}{v} = -\dfrac{t}{20} - \dfrac{1}{10} = \dfrac{-(t+2)}{20}$

Now when $v = 5$

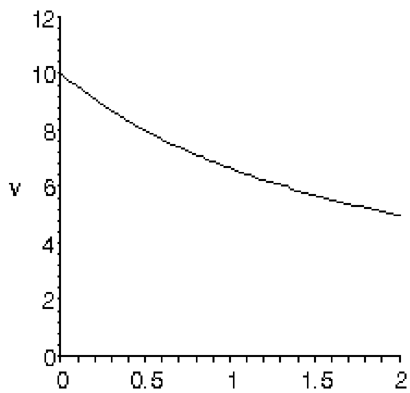
$5 = \dfrac{20}{t+2}$

$t + 2 = \dfrac{20}{5} = 4$

$t = 2$ sec

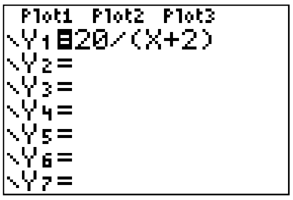
3 marks

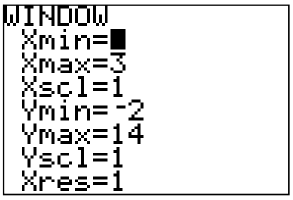
iv. $v = v(t) = \dfrac{20}{t+2}$ for $0 \le t \le 2$

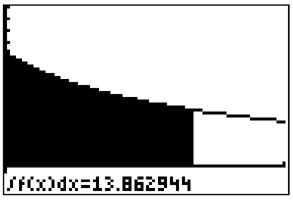


As a check $x = 20\log_e(2) \approx 13.863$

2 marks


Plot1 Plot2 Plot3
\Y1=20/(X+2)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=


WINDOW
Xmin=
Xmax=3
Xscl=1
Ymin=-2
Ymax=14
Yscl=1
Xres=1


∫f(x)dx=13.862944

Question 4

a. $\underset{\sim}{r}(t) = 2\cot(t)\underset{\sim}{i} + (1 - \cos(2t))\underset{\sim}{j}$ for $t \geq 0$

$x(t) = 2\cot(t)$ to eliminate t

$y(t) = 1 - \cos(2t) = 1 - (1 - 2\sin^2(t)) = 2\sin^2(t)$

$x^2 + 4 = 4\cot^2(t) + 4 = 4\operatorname{cosec}^2(t) = \dfrac{4}{\sin^2(t)} = \dfrac{4}{\frac{1}{2}y}$

so $y = \dfrac{8}{x^2 + 4}$

2 marks

b i. $f(x) = \dfrac{8}{x^2 + 4} = 8(x^2 + 4)^{-1}$ for stationary points

$f'(x) = -\dfrac{16x}{(x^2 + 4)^2} = 0$ when $x = 0$ so $f'(0) = 0$ $f(0) = 2$

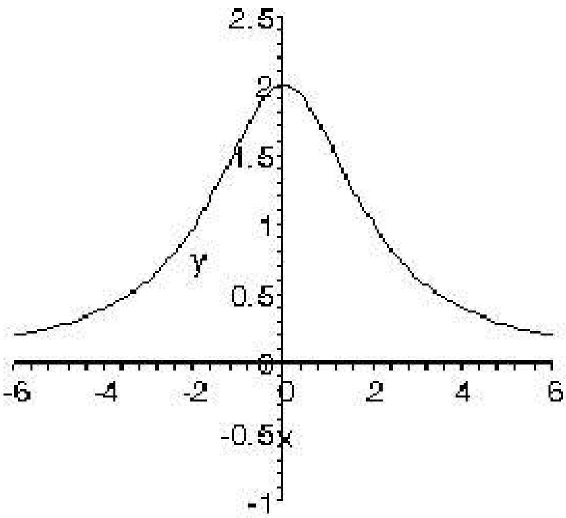
$f''(x) = -\dfrac{16(x^2 + 4)^2 - 4x(x^2 + 4)16x}{(x^2 + 4)^4} = \dfrac{16(3x^2 - 4)}{(x^2 + 4)^3} = 0$ $f''(0) = -1 < 0$

When $3x^2 - 4 = 0$ $x = \pm\dfrac{2}{\sqrt{3}} = \pm\dfrac{2\sqrt{3}}{3}$ $f\left(\dfrac{2\sqrt{3}}{3}\right) = \dfrac{3}{2}$ $f\left(-\dfrac{2\sqrt{3}}{3}\right) = \dfrac{3}{2}$

$(0,2)$ is a maximum $\left(\dfrac{2\sqrt{3}}{3}, \dfrac{3}{2}\right)$ and $\left(-\dfrac{2\sqrt{3}}{3}, \dfrac{3}{2}\right)$ are inflexion points

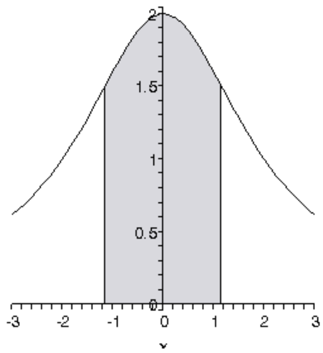
4 marks

ii.



1 mark

iii. The area of the door way is

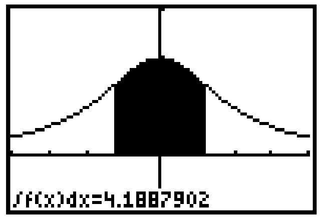


$$\int_{-\frac{2\sqrt{3}}{3}}^{\frac{2\sqrt{3}}{3}} \frac{8}{x^2+4}\,dx = 2\int_{0}^{\frac{2\sqrt{3}}{3}} \frac{8}{x^2+4}\,dx$$

$$= \left[8\,\mathrm{Tan}^{-1}\left(\frac{x}{2}\right)\right]_0^{\frac{2\sqrt{3}}{3}}$$

$$= 8\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{3}}{3}\right) - 8\,\mathrm{Tan}^{-1}(0)$$

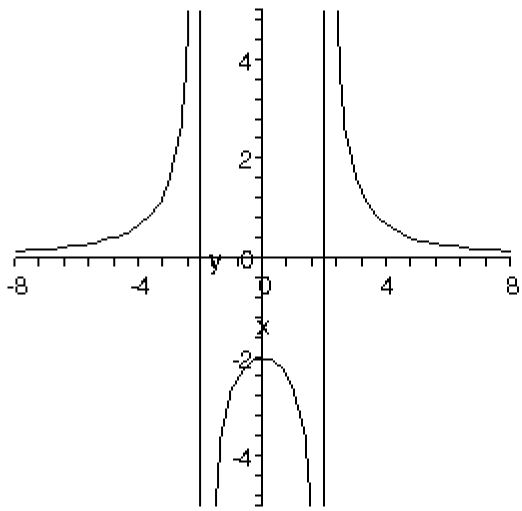
$$= \frac{4\pi}{3}$$



As a check on the TI-83

3 marks

c. i. the domain is $R \setminus \{-2, 2\}$ and the graph has a maximum turning point at $(0, -2)$ range is $(-\infty, -2] \cup (0, \infty)$ the graph has vertical asymptotes at $x = \pm 2$ and a horizontal asymptote at $y = 0$ (the x-axis)



2 marks

ii. The volume formed is given by $V = \int_6^8 \left[g(x)\right]^2 dx$

by partial fractions

$$g(x) = \frac{8}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2)+B(x+2)}{x^2-4} = \frac{x(A+B)+2(B-A)}{x^2-4}$$

$A+B=0$ and $B-A=4$ so that $A=-2$ and $B=2$

$$g(x) = \frac{8}{x^2-4} = \frac{2}{x-2} - \frac{2}{x+2}$$

$$V = \pi \int_6^8 \left(\frac{4}{(x-2)^2} - \frac{8}{(x-2)(x+2)} + \frac{4}{(x+2)^2} \right) dx$$

$$V = \pi \int_6^8 \left(\frac{4}{(x-2)^2} - \frac{2}{x-2} + \frac{2}{x+2} + \frac{4}{(x+2)^2} \right) dx$$

$$V = \pi \left[\frac{-4}{x-2} + 2\log_e\left(\frac{x+2}{x-2}\right) - \frac{4}{x+2} \right]_6^8$$

$$V = \pi \left[-\frac{4}{6} + 2\log_e\left(\frac{10}{6}\right) - \frac{4}{10} + \frac{4}{4} - 2\log_e\left(\frac{8}{4}\right) + \frac{4}{8} \right]$$

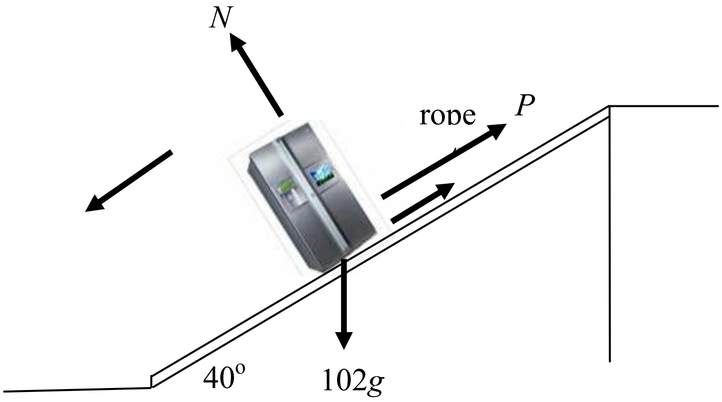
$$V = \pi \left(\frac{13}{30} + 2\log_e\left(\frac{5}{6}\right) \right)$$

4 marks

Question 5

a.

i.



1 mark

ii. Given that $\mu = 0.25$ find P

The frictional force is up, since the motion is down

Resolving the forces, using Newtons 2nd law of motion

perpendicular to the plane (1) $N - 102g\cos(40^o) = 0$

parallel to the plane (2) $102g\sin(40^o) - \mu N - P = 0$

from (1) $N = 102g\cos(40^o)$

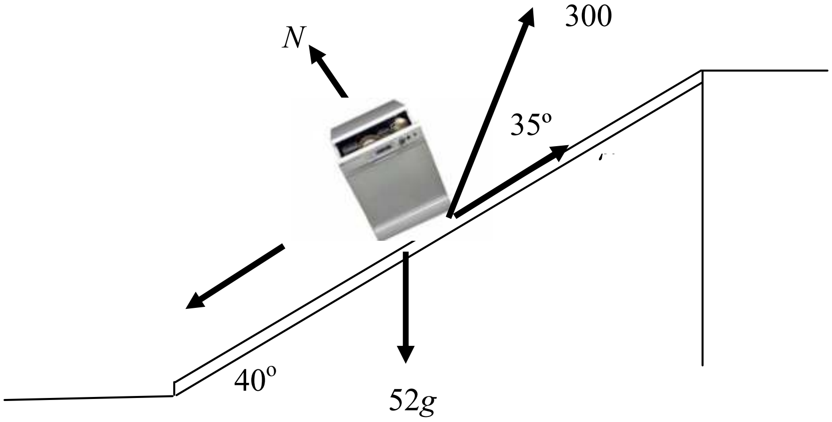
from (2) $P = 102g\sin(40^o) - \mu N = 102g\sin(40^o) - 0.25 \text{ x } 102g\cos(40^o)$

$P = 102g(\sin(40^o) - 0.25\cos(40^o)) = 451.10$ newtons

3 marks

b.

i.



1 mark

ii. The frictional force is up, since the motion is down

Resolving the forces, using Newtons 2nd law of motion

Perpendicular to the plane (1) $N + 300\sin\left(35^o\right) - 52g\cos\left(40^o\right) = 0$

parallel to the plane (2) $52g\sin\left(40^o\right) - \mu N - 300\cos\left(35^o\right) = 52\text{x}0.5$

from (1) $N = 52g\cos\left(40^o\right) - 300\sin\left(35^o\right) = 218.303$

from (2) $\mu N = 52g\sin\left(40^o\right) - 300\cos\left(35^o\right) - 52\text{x}0.5 = 55.819$ so that

$$\mu = \frac{55.819}{218.303} = 0.256$$

4 marks

End of 2005 Specialist Mathematics Trial Examination 2 Solutions

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