# SPECIALIST MATHS EXAM 1 SOLUTIONS

## Exam 1 Part I.

**Question 1** **A**

General equation hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

From graph $a = 2$

Asymptote equations: $y = \pm\frac{b}{a}x$

From graph asymptotes are $y = \pm 2x$

$\Rightarrow \quad 2 = \frac{b}{2}, \; b = 4$

Equation hyperbola: $\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$

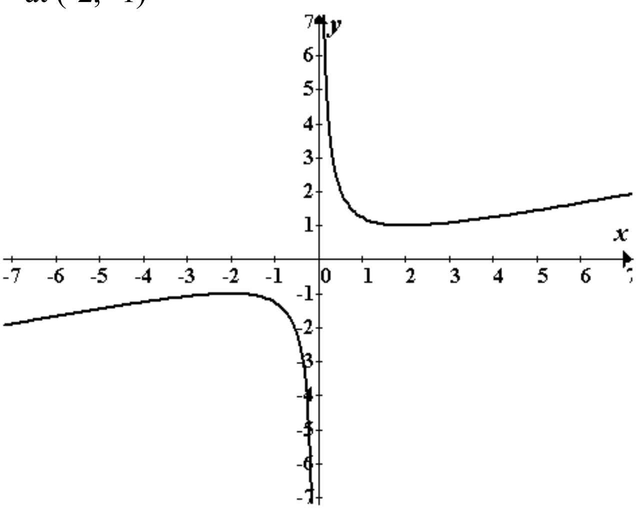
$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{16} = 1$

$\Rightarrow \quad 4x^2 - y^2 = 16$

**Question 2** **C**

Sketch $y = \frac{(x^2+4)}{(4x)}$

Graph shows there is a maximum turning point at (-2, -1)




**Question 3** **E**

$f(x)$ has asymptotes at $x = -1, \quad 2$

$\Rightarrow \frac{1}{f(x)} = 0$ at $x = -1, \quad 2$

$\Rightarrow \frac{1}{f(-x)} = 0$ at $x = -2, \quad 1$

$f(x) > 0$ for $x < -1 \cup x > 2$ and

$f(x) < 0$ for $-1 < x < 2$

$\Rightarrow \quad \frac{1}{f(-x)} > 0$ for $x < -2 \cup x > 1$

and $\frac{1}{f(-x)} < 0$ for $-2 < x < 1$

**Question 4** **C**

$\tan x = a$ and $x \in \left[\pi, \frac{3\pi}{2}\right]$

($3^{rd}$ quadrant: $\cos x$ is negative)

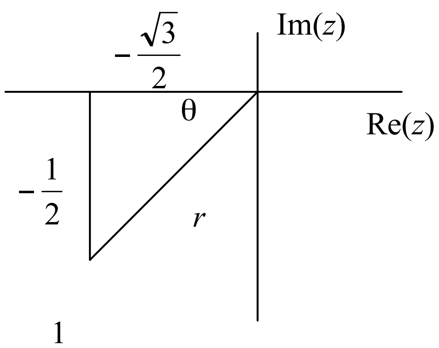
$\sec^2 x = \tan^2 x + 1$

$\sec^2 x = a^2 + 1$

$\cos^2 x = \frac{1}{a^2+1}$

$\cos x = -\frac{1}{\sqrt{a^2+1}}$, since cos x is negative in the third quadrant.

**Question 5** **A**




$$\tan\theta = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$
$$= -\frac{1}{2} \times -\frac{2}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}$$
$$\theta = \frac{\pi}{6}$$

Hence Arg $z = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

**Question 6** **A**

General complex form of a circle is

$|z - (\text{centre})| = \text{radius}$

$|z - (2 - 3i)| = 4$ a circle, centre (2, − 3), radius 4

OR Let $z = x + yi$

$$|(x + yi) - 2 + 3i| = 4$$
$$|(x - 2) + (y + 3)i| = 4$$
$$\sqrt{(x-2)^2 + (y+3)^2} = 4$$
$$(x-2)^2 + (y+3)^2 = 16$$

circle, centre (2, − 3) radius 4

**Question 7** **C**

$$\bar{z} = c + di$$
$$\bar{z} - z = (c + di) - (c - di) = 2di$$

Imaginary part is $2d$ NOT $2di$

**Question 8** **E**

If $z = 2 - i$ is a root, then $(z - 2 + i)$ is a factor

Polynomial division:

$$z - 1 - i$$
$$z - 2 + i \;\big)\; z^2 - 3z + (3+i)$$
$$-\; z^2 - 2z + iz$$
$$-z - iz + (3+i)$$
$$-\; -z + 2 - i$$
$$-iz - 2 + (3 + i) + i$$
$$-iz + 2i - i^2$$
$$0$$

Hence $z - 1 - i$ is a factor,

So $z = 1 + i$ is a root.

**Question 9** **E**

$$\tan\left(3x - \frac{\pi}{2}\right) = -\frac{1}{\sqrt{3}}$$
$$3x - \frac{\pi}{2} = n\pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad n = 0, 1, 2, 3, 4, 5$$
$$3x = n\pi - \frac{\pi}{6} + \frac{\pi}{2}$$
$$3x = n\pi + \frac{\pi}{3}$$
$$x = \frac{3n\pi + \pi}{9}$$
$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$$
$$\Rightarrow \quad x = \frac{13\pi}{9}$$

**Question 10** **B**

$$\frac{1}{\sin x} \times \frac{\cos 2x}{1} = \frac{1}{\sin x} \times \frac{\cos 2x}{\sin 2x} \times \frac{\sin 2x}{1}$$
$$= \frac{1}{\sin x} \times \cot 2x \times \frac{2\sin x \cos x}{1}$$
$$= \cot 2x \times \frac{2\cos x}{1}$$
$$= 2\cos x \cot 2x$$

**Question 11** **D**

Let $f(x) = \text{Cos}\, x$

Range $f(x)$ is [-1, 1]

$\Rightarrow$ Domain: $\text{Tan}^{-1}(f(x))$ is [-1, 1]

Range: $\text{Tan}^{-1}(f(x))$ is $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

**Question 12** **A**

$$\frac{dy}{dx} = \frac{2\cos 2x}{\sin 2x}$$

Applying the quotient rule

$$\frac{d^2y}{d^2x} = \frac{2(-2\sin 2x)\sin 2x - 2\cos 2x(2\cos 2x)}{(\sin 2x)^2}$$

$$= \frac{-4\sin^2 2x - 4\cos^2 2x}{(\sin 2x)^2}$$

$$= \frac{-4(\sin^2 2x + \cos^2 2x)}{(\sin 2x)^2}$$

$$= \frac{-4}{(\sin 2x)^2}$$

$$= -4\operatorname{cos ec}^2 2x$$

**Question 13** **B**

By partial fractions

$$\frac{1}{(1-3x)(1+3x)} = \frac{1}{2(1-3x)} + \frac{1}{2(1+3x)}$$

$$\int \frac{1}{1-9x^2}dx = \int \frac{1}{2(1+3x)} + \frac{1}{2(1-3x)}dx$$

$$= \frac{1}{6}\int \frac{3}{(1+3x)} - \frac{-3}{(1-3x)}dx$$

$$= \frac{1}{6}\left[\log_e(1+3x) - \log_e(1-3x)\right]$$

$$= \frac{1}{6}\log_e\left(\frac{1+3x}{1-3x}\right)$$

**Question 14** **B**

Stationary points at $x = -3, \quad -1, \quad 2$

$x = -3$, gradient changes positive, zero, negative

$\Rightarrow$ max turning pt

$x = -1$ gradient changes negative, zero, positive

$\Rightarrow$ min turning pt

$x = 2$ gradient changes positive, zero, positive

$\Rightarrow$ inflexion

**Question 15** **C**

$$\int_0^m \sin^2 x dx = \int_0^m \frac{1}{2}(1-\cos 2x)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^m$$

$$= \frac{m}{2} - \frac{\sin 2m}{4}$$

**Question 16** **D**

$$f'(x) = \frac{dy}{dx} = \frac{x^2}{x-1}$$

Using $f(x+h) \approx hf'(x) + f(x)$

When $x = 2, \quad f(x) = 4$

$\Rightarrow f'(2) = \frac{dy}{dx} = \frac{2^2}{2-1} = 4, \qquad h = 0.1$

$f(2+0.1) \approx 0.1f'(2) + f(2)$

$f(2.1) \approx 0.1 \times 4 + 4 = 4.4$

$f(2.1+0.1) \approx 0.1f'(2.1) + f(2.1)$ and

$f'(2.1) = \frac{2.1^2}{2.1-1} = 4.0091$

$f'(2.2) \approx 0.1 \times 4.0091 + 4.4 = 4.8009$

**Question 17** **D**

Volume of water in tank after $t$ minutes is $100 + 3t$ litres

Concentration of sugar after $t$ minutes is

$\frac{Q}{100+3t}$ kg/litre

Rate of inflow per minute is $8 \times 0.5$

$= 4$ kg/minute

Rate of outflow per minute is

$5 \times \frac{Q}{100+3t} = \frac{5Q}{100+3t}$ kg/minute

$\frac{dQ}{dt}$ = rate of inflow – rate of outflow

$= 4 - \frac{5Q}{100+3t}$ kg/minute

**Question 18** **A**

Vector resolute of ***a*** parallel to ***b*** is given by

$$(\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

$$|\boldsymbol{b}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$\hat{\boldsymbol{b}} = \frac{1}{\sqrt{6}}(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{a}\cdot\hat{\boldsymbol{b}} = \frac{1}{\sqrt{6}}(2\times 1 - 3\times -2 + 1\times 1)$$

$$= \frac{9}{\sqrt{6}}$$

$$(\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = \frac{9}{\sqrt{6}}\times\frac{1}{\sqrt{6}}(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$$

$$= \frac{3}{2}(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$$

**Question 19** **D**

$$\vec{OA} = \vec{OB} + \vec{BA}$$

$$= \vec{OB} - \vec{AB}$$

$$= (2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) - (\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$$

$$= \boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$$

$$|\vec{OA}| = \sqrt{1^2 + 6^2 + (-2)^2} = \sqrt{41}$$

**Question 20** **B**

$$\boldsymbol{r}'(t) = 2\sin(3t)\boldsymbol{i} - \boldsymbol{j}$$

$$\boldsymbol{r}(t) = \int (2\sin(3t)\,\boldsymbol{i} - \boldsymbol{j})dt$$

$$\boldsymbol{r}(t) = -\frac{2}{3}\cos(3t)\boldsymbol{i} - t\,\boldsymbol{j} + \boldsymbol{c}$$

Since when $t = 0$, $\boldsymbol{r}(t) = -\frac{2}{3}\boldsymbol{i} + \boldsymbol{j}$

$$-\frac{2}{3}\boldsymbol{i} + \boldsymbol{j} = -\frac{2}{3}\boldsymbol{i} + \boldsymbol{c}$$

$$\boldsymbol{c} = \boldsymbol{j}$$

Hence $\boldsymbol{r}(t) = -\frac{2}{3}\cos(3t)\boldsymbol{i} - t\,\boldsymbol{j} + \boldsymbol{j}$

$$\boldsymbol{r}(t) = -\frac{2}{3}\cos(3t)\boldsymbol{i} + (1 - t)\boldsymbol{j}$$

**Question 21** **D**

$x = \sin t$, $y = \cos 2t = 1 - 2\sin^2 t$

$$y = 1 - 2x^2$$

since $0 \le t \le \pi$, $0 \le \sin t \le 1$

hence, since $x = \sin t$, $0 \le x \le 1$

**Question 22** **C**

$$v = \frac{dx}{dt} = 3.2 - \cos\left(\frac{t}{4}\right)$$

Since $-1 \le \cos\left(\frac{t}{4}\right) \le 1$,

$v_{max} = 3.2 + 1 = 4.2$ m/s

**Question 23** **C**

$u = 12$ m/s, $a = -3$ m/s$^2$, $v = 0$ m/s

$$v^2 - u^2 = 2as$$

$$0 - 12^2 = 2(-3)s$$

$$-144 = -6s$$

$s = 24$ metres

**Question 24** **E**

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 4 - e^{2x}$$

$$\frac{1}{2}v^2 = \int (4 - e^{2x})dx$$

$$\frac{1}{2}v^2 = 4x - \frac{1}{2}e^{2x} + c$$

$x = 0$, $v = -2$, hence

$$\frac{1}{2}(-2)^2 = 4(0) - \frac{1}{2}e^{2\times 0} + c$$

$$2 = 0 - \frac{1}{2} + c$$

$$c = \frac{5}{2}$$

Hence $v^2 = 8x - e^{2x} + 5$

$$v = \pm\sqrt{8x - e^{2x} + 5}$$

$$v = -\sqrt{8x - e^{2x} + 5},$$

since $x = 0$, $v = -2$

**Question 25** **D**

Considering upwards as positive:

$u = 18$ m/s, $a = -9.8$ m/s$^2$, $s = -520$ m

$$s = ut + \frac{1}{2}at^2$$

$$-520 = 18t - 4.9t^2$$

$$49t^2 - 180t - 5200 = 0$$

Quadratic formula gives $t = 12.3$ and $-8.6$

Since $t \geq 0$, $t = 12.3$ seconds

Alternatively, consider upward and downward motion separately:

Upward motion (with upwards positive)

$u = 18$ m/s, $v = 0$, $a = -9.8$ m/s$^2$

$$v^2 - u^2 = 2as$$

$$s = \frac{v^2 - u^2}{2a}$$

$$= \frac{-18^2}{-19.6}$$

$= 16.53$ metres (further distance the stone rises)

$$v = u + at$$

$$t_1 = \frac{v - u}{a}$$

$$= \frac{-18}{-9.8}$$

$= 1.84$ seconds to reach maximum height

Downward motion (with downwards positive)

$u = 0$ m/s, $s = 520 + 16.53 = 536.53$m,

$a = 9.8$ m/s$^2$

$$s = ut + \frac{1}{2}at^2$$

$$t^2 = \frac{2s}{a} = 109.50$$

$t_2 = 10.46$ seconds to fall from the peak to the ground.

$t_1 + t_2 = 12.3$ seconds

**Question 26** **B**

Since body is in equilibrium

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$$

Therefore $\boldsymbol{F}_3 = -(\boldsymbol{F}_1 + \boldsymbol{F}_2)$

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 = (2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) + (2\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$$

$$= 4\boldsymbol{i} - 3\boldsymbol{k}$$

Hence $\boldsymbol{F}_3 = -4\boldsymbol{i} + 3\boldsymbol{k}$

**Question 27** **C**

Points A, B and C will be collinear if $\vec{AB}$ and $\vec{BC}$ are parallel and have a point in common, ie B.

Hence $\vec{AB} = k\,\vec{BC}$

**Question 28** **D**

Since lift is accelerating upwards, the resultant force is upwards, hence

$$N - mg = ma \quad \Rightarrow \quad N = ma + mg$$

$$N = 55(3) + 55(9.8) \;= 704 \text{ Newtons}$$

**Question 29** **E**

Assuming the 3kg mass moves down,

2 kg mass: $T - 2g = 2a$ (I)

3 kg mass: $3g - T = 3a$ (II)

(I) + (II) $g = 5a$

$$a = \frac{g}{5}$$

From (I), T = 2a + 2g

$$= 2\left(\frac{g}{5}\right) + 2g$$

$$= \frac{2g}{5} + \frac{10g}{5}$$

$$= \frac{12g}{5}$$

**Question 30** **C**

3 kg mass: $T - 3g\sin 60 = 3a$ (I)

5kg mass $5g - T = 5a$ (II)

Adding equations (I) and (II)

$$5g - 3g\frac{\sqrt{3}}{2} = 8a$$

$$\frac{g(10 - 3\sqrt{3})}{2} = 8a$$

$$a = \frac{g(10 - 3\sqrt{3})}{16}$$

# EXAM 1 Part II

**Question 1**

Let $u = 2x + 1 \Rightarrow 2x = u - 1$

$\Rightarrow \quad \frac{du}{dx} = 2 \qquad x = \frac{u-1}{2}$ **M1**

$$\int x\sqrt{2x+1}\;dx = \frac{1}{2}\int\left(\frac{u-1}{2}\right)u^{\frac{1}{2}}\;\frac{du}{dx}dx$$

$$= \frac{1}{4}\int(u-1)\,u^{\frac{1}{2}}\;du$$

$$= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\;du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + \text{c}$$

$c$ is a constant **M1**

$$= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + \text{c}$$

$$= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + \text{c}$$ **A1**

**Question 2**

$y = (\cos x)^{-1}$

Applying the chain rule

$\frac{dy}{dx} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}$ **A1**

At $x = \frac{\pi}{4}$, $\quad \frac{dy}{dx} = \frac{\sin(\frac{\pi}{4})}{\cos^2(\frac{\pi}{4})}$

$$= \frac{\frac{\sqrt{2}}{2}}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{2}$$

$\Rightarrow \quad a = \sqrt{2}$ **A1**

Equation of tangent $\quad y = \sqrt{2}x + b$

At $\left(\frac{\pi}{4}, \sec\frac{\pi}{4}\right)$, $\sec\frac{\pi}{4} = \sqrt{2} \times \frac{\pi}{4} + b$ **M1**

$b = \sqrt{2} - \sqrt{2} \times \frac{\pi}{4} \;= \sqrt{2}(1 - \frac{\pi}{4})$ **A1**

$\therefore$ Equation of tangent is

$$y = \sqrt{2}x + \sqrt{2}(1 - \frac{\pi}{4})$$

## Question 3

$$\frac{dx}{dy} = \frac{1}{2y-1}$$

$$x = \frac{1}{2}\int \frac{2}{2y-1}\,dy + c$$

$$x = \frac{1}{2}\log_e(2y-1) + c \qquad \textbf{A1}$$

$$y(0) = 1 \;\Rightarrow\; 0 = \frac{1}{2}\log_e(2-1) + c$$

$$\therefore \quad c = 0 \qquad \textbf{A1}$$

$$x = \frac{1}{2}\log_e(2y-1)$$

$$e^{2x} = 2y - 1$$

$$y = \frac{1}{2}(e^{2x}+1) \qquad \textbf{A1}$$

## Question 4

**a. i.** $\overrightarrow{OM} = p\overrightarrow{OB}$

$= p\left(\overrightarrow{OB} + \overrightarrow{BC}\right)$ **M1**

$= p(\boldsymbol{a} + \boldsymbol{c})$

**ii.** $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM}$ **M1**

$= \overrightarrow{OC} + q\overrightarrow{CA}$

$= \boldsymbol{c} + q(\boldsymbol{a} - \boldsymbol{c})$ **M1**

$= q\boldsymbol{a} + (1-q)\boldsymbol{c}$

**b.** Equating the two equations for $\overrightarrow{OM}$

$p(\boldsymbol{a} + \boldsymbol{c}) = q\boldsymbol{a} + (1-q)\boldsymbol{c}$

$p\boldsymbol{a} + p\boldsymbol{c} - q\boldsymbol{a} - \boldsymbol{c} + q\boldsymbol{c} = \boldsymbol{0}$

$(p-q)\boldsymbol{a} + (p+q-1)\boldsymbol{c} = \boldsymbol{0}$ **M1**

Since $\boldsymbol{a}$ and $\boldsymbol{c}$ are adjacent sides of a parallelogram, they cannot be parallel.

Hence the above expression can only be true if:

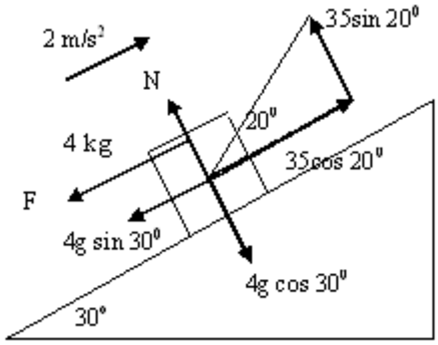
$p - q = 0$ and $p + q - 1 = 0$

Hence $p = q$ and $2p - 1 = 0$ **A1**

$\Rightarrow \quad p = q = ½$

Therefore, from $\overrightarrow{OM} = p\overrightarrow{OB}$, the diagonals of a parallelogram bisect each other.

## Question 5




**a.** From $R = ma$

(force up plane) − (force down plane) = $ma$

$35\cos 20 - (F + 4g\sin 30) = 4(2)$ **M1**

$F = 35\cos 20 - 4g\sin 30 - 8$

$= 5.29$ newtons

Considering forces perpendicular to the plane, where the forces balance:

$N + 35\sin 20 = 4g\cos 30$

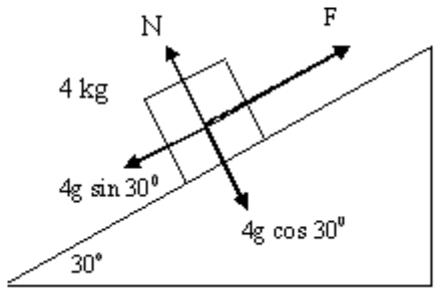
$N = 4g\cos 30 - 35\sin 20$

$= 21.98$ **A1**

$F = \mu N$

$\mu = \frac{F}{N} = \frac{5.29}{21.98} = 0.24$ **A1**

**b.**




$N = 4g\cos 30 = 4g\frac{\sqrt{3}}{2}$

$F = \mu N$, where from part **a.** $\mu = 0.24$

$F = 0.24 \times 4g\frac{\sqrt{3}}{2}$ **M1**

$4g\sin 30 - F = ma$

$4g\sin 30 - 0.24 \times 2g\sqrt{3} = 4a$

$\Rightarrow \quad a = 2.86$ m/s$^2$ **A1**