Victorian Certificate of Education
2005

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter
Words									

SPECIALIST MATHEMATICS

Written examination 2
(Analysis task)

Wednesday 2 November 2005

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 1.30 pm (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	60

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten), one approved graphics calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 13 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Working space

Question 1

A large tank initially contains 10 litres of contaminated water. Clean water that contains a **variable concentration** of a purifying chemical is pumped into the tank at a rate of 20 litres per minute.

After t minutes, the concentration of the chemical in the water being pumped into the tank is $\dfrac{2}{1+t^2}$ grams per litre. The solution in the tank is mixed and pumped out of the tank at a rate of 10 litres per minute.

a. If x grams is the amount of purifying chemical in the solution at time t minutes, give an expression for the concentration of the chemical in the tank, in grams per litre, at time t.

1 mark

b. Show that the rate of increase of chemical in the tank satisfies the differential equation

$$\frac{dx}{dt} + \frac{x}{1+t} = \frac{40}{1+t^2}$$

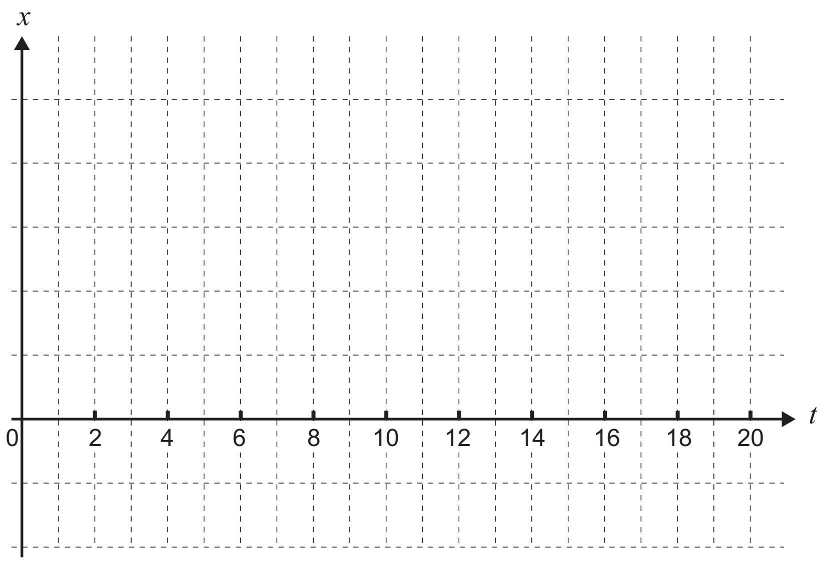
3 marks

c. **i.** If $x = \dfrac{40}{1+t}\operatorname{Tan}^{-1}(t) + \dfrac{20}{1+t}\log_e(1+t^2)$, show that $\dfrac{dx}{dt} = \dfrac{40}{1+t^2} - \dfrac{40\operatorname{Tan}^{-1}(t)}{(1+t)^2} - \dfrac{20\log_e(1+t^2)}{(1+t)^2}$.

ii. **Hence** verify that x satisfies the differential equation given in **part b**.

4 + 1 = 5 marks

d. Sketch the graph of $x = \dfrac{40}{1+t}\,\mathrm{Tan}^{-1}(t) + \dfrac{20}{1+t}\log_e(1+t^2)$ for $0 \le t \le 20$ on the axes below. Label any stationary points with their coordinates correct to two decimal places.



3 marks

e. **i.** The purifying chemical is only effective if there are at least 15 grams of it present in the tank. Find, correct to three decimal places, the value of t when the purifying chemical first becomes effective.

ii. **Hence** find for how long the purifying chemical is effective correct to two decimal places.

1 + 1 = 2 marks

Total 14 marks

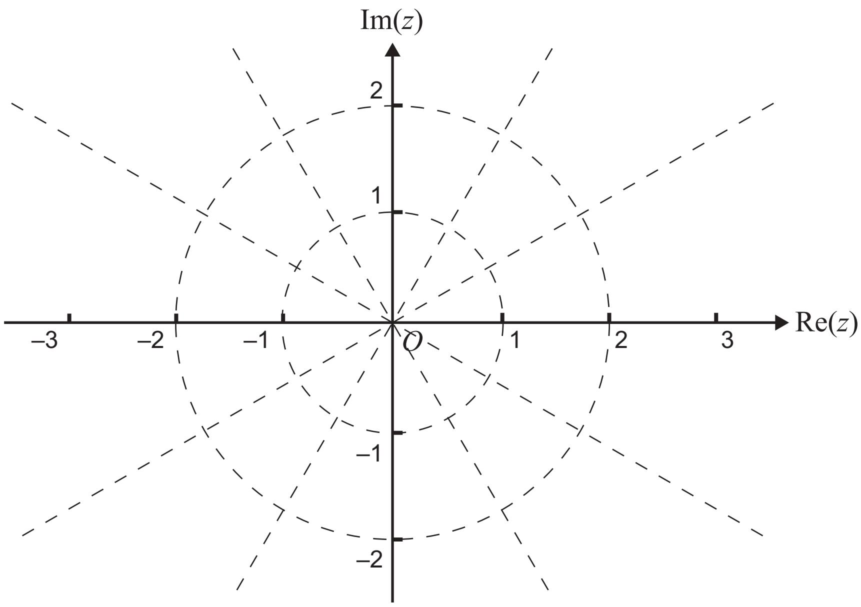
Question 2

Let $u = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

a. **i.** Express u in polar form.

ii. **Hence** show that $u^6 = 1$.

iii. Plot all roots of $z^6 - 1 = 0$ on the Argand diagram below, labelling u and w where $w = -u$.



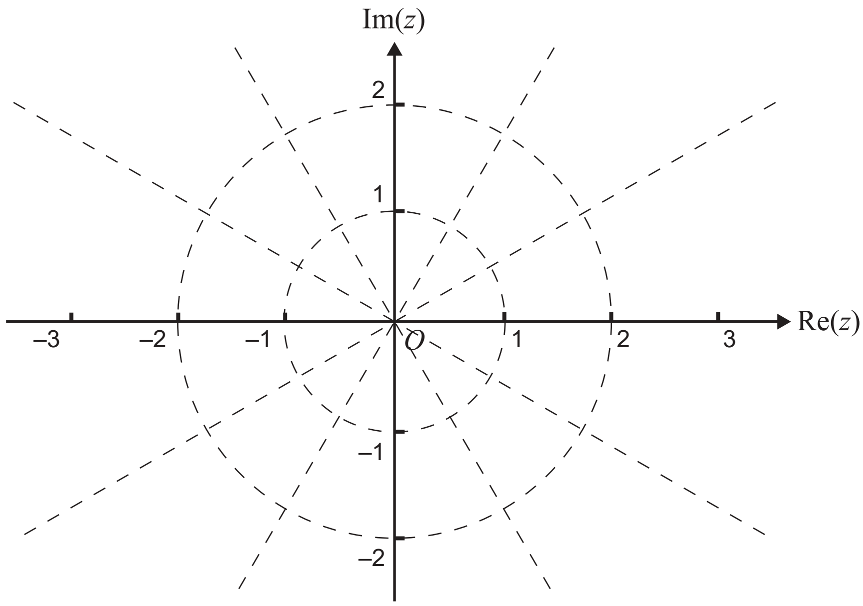
2 + 1 + 3 = 6 marks

b. Let $z \in C$. Shade the region $\{z : |z| \le 1\} \cap \{z : \text{Arg}(z) \le \text{Arg}(w)\}$ on the Argand diagram above.

3 marks

Question 2 – continued

c. **i.** Draw and label the subset of the complex plane given by $S = \{z : |z| = 1\}$ on the Argand diagram below.



ii. Draw and label the subset of the complex plane given by $T = \{z : |z - u| = |z + u|\}$ on the Argand diagram above.

iii. Find the coordinates of the points of intersection of S and T.

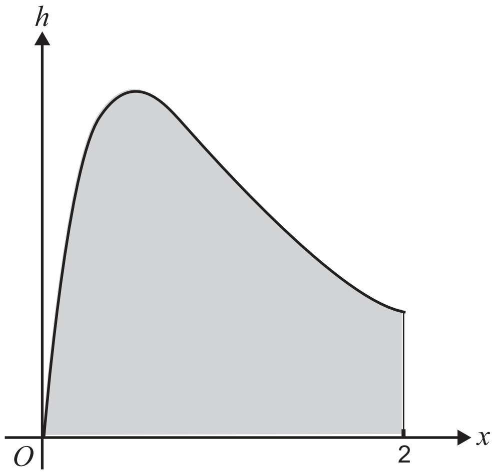
1 + 2 + 2 = 5 marks

Total 14 marks

Question 3

A council is planning to construct a fence at a park from pre-made panels. One side of each fence panel needs to be painted. To determine the amount of paint needed, the area of one side of each panel needs to be calculated. Each panel is 2 metres wide. An artist's sketch of a panel is given below.

Estimates of the height of one panel are made from the artist's sketch. Taking x metres as the distance from the left-hand end of a panel and h metres as an estimate of the height, these estimates are summarised in the table below.



x	h
0.0	0.00
0.5	1.60
1.0	1.25
1.5	0.85
2.0	0.55

a. Estimate the area of one side of a panel by approximating the shape of a panel using four trapeziums of equal width.

2 marks

A mathematician examines the artist's sketch and decides that the height of each panel can be modelled by the function

$$h(x) = \frac{10x}{(x^2+1)(3x+1)}, \quad x \in [0,2]$$

b. Given that $\frac{10x}{(x^2+1)(3x+1)}$ can be written in the form $\frac{x+A}{x^2+1} + \frac{B}{3x+1}$, find the values of A and B.

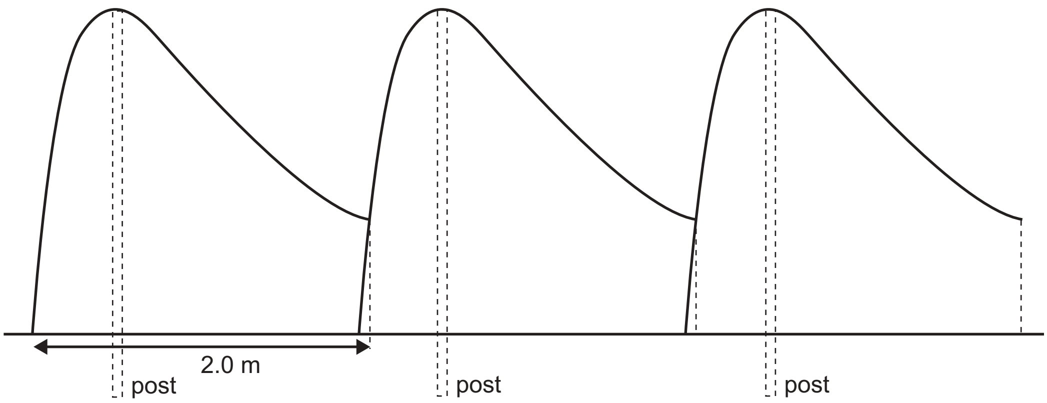
3 marks

c. Write down a definite integral which represents the area of one side of a pre-made panel according to the mathematician's model. **Hence** use calculus to find this area correct to two decimal places.

5 marks

The fence is to be constructed by overlapping the panels. The fence will be 100 metres long.

d. To build the fence, the pre-made panels are overlapped and secured to upright posts. An artist's sketch of three panels and posts of the fence is given below.



According to the mathematician's model, what is the minimum number of panels required?

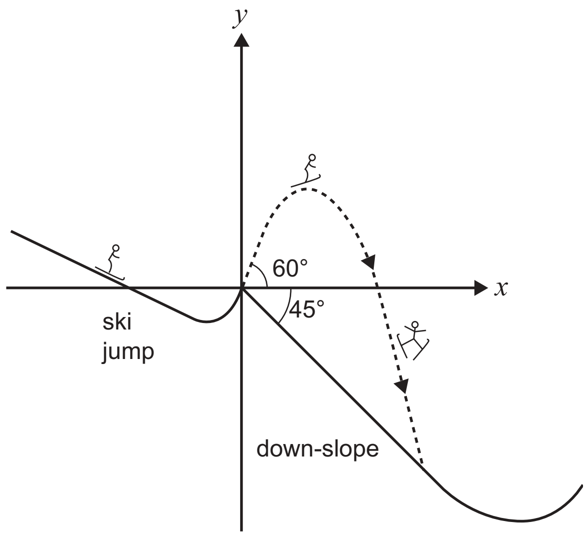
3 marks

Total 13 marks

Question 4

A skier accelerates down a slope and then skis up a short ski jump (see diagram below). The skier leaves the jump at a speed of 12 m/s and at an angle of 60° to the horizontal. The skier performs various gymnastic twists and lands on a straight line section of the 45° down-slope T seconds after leaving the jump.

Let the origin O of a cartesian coordinate system be at the point where the skier leaves the jump, with $\underset{\sim}{i}$ a unit vector in the positive x direction, and $\underset{\sim}{j}$ a unit vector in the positive y direction. Displacements are measured in metres, and time in seconds.



a. Show that the initial velocity of the skier when leaving the jump is $6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}$

1 mark

b. The acceleration of the skier, t seconds after leaving the ski jump, is given by

$$\ddot{\underset{\sim}{r}}(t) = -0.1t\,\underset{\sim}{i} - (g - 0.1t)\,\underset{\sim}{j},\ 0 \le t \le T$$

Show that the position vector of the skier, t seconds after leaving the jump, is given by

$$\underset{\sim}{r}(t) = \left(6t - \frac{1}{60}t^3\right)\underset{\sim}{i} + \left(6t\sqrt{3} - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)\underset{\sim}{j},\ 0 \le t \le T$$

3 marks

Question 4 – continued

c. Show that $T = \frac{12}{g}\left(\sqrt{3}+1\right)$.

3 marks

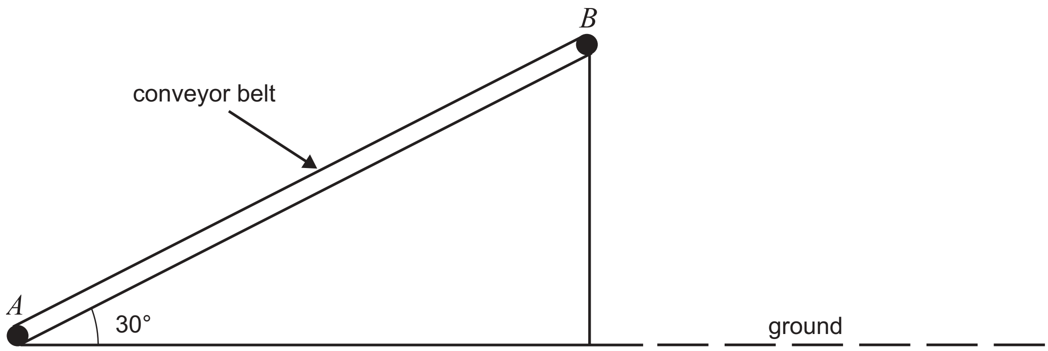
d. At what speed, in metres per second, does the skier land on the down-slope? Give your answer correct to one decimal place.

3 marks

Total 10 marks

Question 5

The diagram below shows a conveyor belt which loops around pulleys at *A* and *B*. *AB* is inclined at an angle of 30° to the horizontal. The conveyor belt is used to move packages from *A* to *B*.



A package of mass 5 kg is placed on the conveyor belt when it is stationary.

a. Given that the package does not slip, find the magnitude in newtons of the frictional force acting on it.

2 marks

b. If the conveyor belt accelerates at 0.8 m/s^2, the package moves up the incline with the same acceleration. If the acceleration is greater than 0.8 m/s^2, the package will slip on the conveyor belt.

Find, correct to two decimal places, the coefficient of friction μ between the package and the conveyor belt.

3 marks

c. A package of mass m kg is placed on the conveyor belt at A. The machinery that operates the conveyor belt breaks down before the package has been moved up the incline. A rope is attached to the package to drag it up the incline. When dragging the package, the rope is parallel to the conveyor belt and the tension in the rope is 160 N.

Find m, correct to one decimal place, if the package accelerates up the incline at 0.5 m/s^2.

4 marks

Total 9 marks

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$ $\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ $\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$ $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

function	Sin^{-1}	Cos^{-1}	Tan^{-1}
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $-\pi < \operatorname{Arg} z \le \pi$

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ $\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$ $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$ $$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$ $$\int \frac{1}{x}\,dx = \log_e(x) + c, \text{ for } x > 0$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$ $$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$ $$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$$ $$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$$

$$\frac{d}{dx}\left(\mathrm{Sin}^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$ $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c, a > 0$$

$$\frac{d}{dx}\left(\mathrm{Cos}^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$ $$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Cos}^{-1}\left(\frac{x}{a}\right) + c, a > 0$$

$$\frac{d}{dx}\left(\mathrm{Tan}^{-1}(x)\right) = \frac{1}{1+x^2}$$ $$\int \frac{a}{a^2+x^2}\,dx = \mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

mid-point rule: $\int_a^b f(x)\,dx \approx (b-a)f\left(\frac{a+b}{2}\right)$

trapezoidal rule: $\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\left(f(a)+f(b)\right)$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$ $s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET